GOCE DELCEV UNIVERSITY - STIP
FACULTY OF COMPUTER SCIENCE

ISSN 2545-479X print
ISSN 2545-4803 on line

BALKAN JOURNAL OF APPLIED MATHEMATICS AND INFORMATICS (BJAMI)

BALKAN JOURNAL
OF APPLIED MATHEMATICS AND INFORMATICS

YEAR 2018

VOLUME I, Number 2

GOCE DELCEV UNIVERSITY - STIP, REPUBLIC OF MACEDONIA
FACULTY OF COMPUTER SCIENCE

ISSN 2545-479X print
ISSN 2545-4803 on line

BALKAN JOURNAL OF APPLIED MATHEMATICS AND INFORMATICS

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OF APPLIED MATHEMATICS AND INFORMATICS

(BJAMI)

YEAR 2018 VOLUME I, Number 2

AIMS AND SCOPE:
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Address of the editorial office
Goce Delcev University – Stip
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Krste Misirkov 10-A
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BALKAN JOURNAL
OF APPLIED MATHEMATICS AND INFORMATICS (BJAMI), Vol 1

ISSN 2545-479X print
ISSN 2545-4803 on line
Vol. 1, No. 2, Year 2018

EDITORIAL BOARD

CONTENT

UDC: 517.588:519.644

OPERATIONAL REPRESENTATIONS AND GENERATING FUNCTIONS OF CERTAIN QUADRUPLE HYPERGEOMETRIC SERIES

Maged G. Bin-Saad, Jihad A. Younis
Department of Mathematics, Aden University,
Aden, Khormaksar, P.O.Box 6014, Yemen
E-mail: mgbinsaad@yahoo.com
E-mail: jihadalsaqqaf@gmail.com

Abstract

We aim in this work to establish new operational representations for the hypergeometric functions of four variables $X_6^{(4)}, X_7^{(4)}, X_8^{(4)}, X_9^{(4)}, X_{10}^{(4)}$. By means of these operational representations, a number of generation functions involving these hypergeometric functions are then found.

MSC 2010: 33C20, 33C65.

Keywords: Formal operators, operational representations, generating functions, quadruple functions.

1. Introduction

The subject of operational calculus has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering [4, 11, 12]. One of the most interesting developments on the use of operational calculus is the finding of symbolic representations for hypergeometric series and polynomials that play an important role in the investigation of various useful properties of the hypergeometric series and polynomials (for example [1, 2, 5-10]). Recently, Bin-Saad et al. [3] have introduced five new quadruple hypergeometric functions whose names are $X_6^{(4)}, X_7^{(4)}, X_8^{(4)}, X_9^{(4)}, X_{10}^{(4)}$ to investigate their five Laplace integral representations that include the confluent hypergeometric functions ${}_0F_1$, ${}_1F_1$, the Humbert functions Φ_2, Φ_3 and Ψ_2 in their kernels, we recall these quadruple hypergeometric functions are defined by

$$X_6^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_2,a_2;c_1,c_1,c_2,c_2;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+q}(a_2)_{q+n+2p}}{(c_1)_{m+n}(c_2)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.1}$$

$$X_7^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+2q+n+p}(a_2)_n(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.2}$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+2q+n+p}(a_2)_n(a_3)_p}{(c_1)_{m+n}(c_2)_p(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.3}$$

$$X_9^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_2,c_1,c_1,c_3;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+2q+n+p}(a_2)_n(a_3)_p}{(c_1)_{n+p}(c_2)_m(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.4}$$

$$X_{10}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_2,c_3,c_4;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.5}$$

where each series $X^{(4)}(.)$ contains twelve parameters (eight a's and four c's). The 1st, 2nd, 3rd and 4th parameters in $X^{(4)}(.)$ are connected with the integers m, n, p and q, respectively. Each repeated parameter in the series $X^{(4)}(.)$ points out a term with double parameters. For example, $X^{(4)}(a_1,a_1,a_2,a_2,a_3,a_3,a_4,a_5)$ shows that $(a_1)_{m+n}(a_2)_{p+q}(a_3)_{m+n}(a_4)_p(a_5)_q$. Similarly, for $X^{(4)}(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3)$, $(a_1)_{2m+2n+p}(a_2)_{p+q}(a_3)_q$ and

$X^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_2,a_2)$ points out the term $(a_1)_{2m+n+p}\ (a_2)_{n+p+2q}$. Similarly, we can write various combinations of indices, for more details see [3].

We have organized the rest of this paper in the following way: Section 2 establishes operational representations of the multinomial-type for the quadruple series $X_6^{(4)}, X_7^{(4)}, X_8^{(4)}, X_9^{(4)}, X_{10}^{(4)}$. In Section 3, we aim to establish operational representations of exponential and multinomial type. The aim of Section 4 is to use the operational representations obtained in Section 3 to derive a number of generating functions for the quadruple series $X_6^{(4)}, X_7^{(4)}, X_8^{(4)}, X_9^{(4)}, X_{10}^{(4)}$.

2. Operational representations of the multinomial-type

Here we will deal with operational definitions ruled by the operators D_x and D_x^{-1} where D_x denotes the derivative operator and D_x^{-1} defines the inverse of the derivative. It is evident that D_x^{-1} is essentially an integral operator and the lower limit has been assumed to be zero.

The following two formulas are well-known consequences of the derivative operator D_x and the integral operator D_x^{-1} (see, Ross [12]):

$$D_x^m x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+1)} x^{\lambda-m}\ , \tag{2.1}$$

$$D_x^{-m} x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+m+1)} x^{\lambda+m}\ , \tag{2.2}$$

$$m \in N \cup \{0\},\ \lambda \in C\ (complex\ number) \setminus \{-1,-2,\ldots\},$$

where $\Gamma(\lambda)$ is usual Euler's Gamma function given by

$$\Gamma(\lambda) = \frac{\Gamma(\lambda+1)}{\lambda}, \quad (\lambda \neq 0,-1,-2,\ldots).$$

Based on the operational relations (2.1) and (2.2) the action of the set of operators

$$\left\{ D_\alpha^m\ D_\beta^{-m} \right\},\ (m=0,1,2,3,\ldots)\ ,$$

on the power

$$\alpha^{a+m-1}\ \beta^{b-1}\ ,\ \left(\mathrm{Re}(a)>0, \mathrm{Re}(b)>0\right)\ ,$$

is given by [1]

$$D_\alpha^m\ D_\beta^{-m}\left\{\alpha^{a+m-1}\ \beta^{b-1}\right\} = \alpha^{a-1}\ \beta^{b+m-1}\frac{(a)_m}{(b)_m}\ . \tag{2.3}$$

It is easily verified that

$$\left(D_\alpha \beta^{-1} D_\beta^{-1}\alpha\right)^m\left\{\alpha^{a-1}\beta^{c-1}\right\} = D_\alpha^m \beta^{-m} D_\beta^{-m}\alpha^m\left\{\alpha^{a-1}\beta^{c-1}\right\} = \alpha^{a-1}\beta^{c-1}\frac{(a)_m}{(c)_m},(m=0,1,2,\ldots)\ . \tag{2.4}$$

Moreover, the multinomial expansion of algebra

$$\left(1-x_1-x_2\cdots-x_n\right)^{-a}=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}}{m_1!\ldots m_n!}x_1^{m_1}\ldots x_n^{m_n}\,, \tag{2.5}$$

has its analogue the operator multinomial expansion

$$\begin{aligned}&\left(1-\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-\cdots-\left[D_{\alpha_n}\beta_n^{-1}D_{\beta_n}^{-1}\alpha_n\right]\right)^{-a}\\&\qquad=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}}{m_1!..m_n!}\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]^{m_1}\ldots\left[D_{\alpha_n}\beta_n^{-1}D_{\beta_n}^{-1}\alpha_n\right]^{m_n}.\end{aligned} \tag{2.6}$$

Now, by means of the operators in (2.1) to (2.6), we aim in this section to establish the following operational representations:

$$\begin{aligned}&\left(1-x\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\left[D_{\alpha_1}(\beta')^{-1}D_{\beta'}^{-1}\alpha_1\right]\right)^{-b_1}\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\left[D_{\alpha_2}(\beta'')^{-1}D_{\beta''}^{-1}\alpha_2\right]\right)^{-b_2}\\&\cdot\left(1-z\left[D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_2\right]\left[D_{\alpha_2}(\beta''')^{-1}D_{\beta'''}^{-1}\alpha_2\right]\right)^{-b_3}\left(1-u\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\left[D_{\alpha_2}(\beta'''')^{-1}D_{\beta''''}^{-1}\alpha_2\right]\right)^{-b_4}\\&\cdot\left\{\alpha_1^{a_1-1}\alpha_2^{a_2-1}(\beta')^{b_1-1}(\beta'')^{b_2-1}(\beta''')^{b_3-1}(\beta'''')^{b_4-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\}\\&\quad=\alpha_1^{a_1-1}\alpha_2^{a_2-1}(\beta')^{b_1-1}(\beta'')^{b_2-1}(\beta''')^{b_3-1}(\beta'''')^{b_4-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\cdot X_6^{(4)}\left(a_1,a_1,a_2,a_1,a_1,a_2,a_2,a_2;c_1,c_1,c_2,c_2;x,y,z,u\right);\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\left(1-x\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\left[D_{\alpha_1}\beta^{-1}D_{\beta}^{-1}\alpha_1\right]-u\left[D_{\alpha_1}\beta_4^{-1}D_{\beta_4}^{-1}\alpha_1\right]\left[D_{\alpha_1}\beta^{-1}D_{\beta}^{-1}\alpha_1\right]\right)^{-b}\\&\cdot\left(1-y\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)^{-a_2}\left(1-z\left[D_{\alpha_1}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\right]\right)^{-a_3}\left\{\alpha_1^{a_1-1}\beta^{b-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\right\}\\&\quad=\alpha_1^{a_1-1}\beta^{b-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\cdot X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x,y,z,u\right);\end{aligned} \tag{2.8}$$

$$\begin{aligned}&\left(1-x\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\left[D_{\alpha_1}(\beta')^{-1}D_{\beta'}^{-1}\alpha_1\right]\right)^{-b_1}\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)^{-a_2}\\&\cdot\left(1-z\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)^{-a_3}\left(1-u\left[D_{\alpha_1}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\right]\left[D_{\alpha_1}(\beta'')^{-1}D_{\beta''}^{-1}\alpha_1\right]\right)^{-b_2}\left\{\alpha_1^{a_1-1}(\beta')^{b_1-1}(\beta'')^{b_2-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&\quad=\alpha_1^{a_1-1}(\beta')^{b_1-1}(\beta'')^{b_2-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\cdot X_8^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_2;x,y,z,u\right);\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\left(1-x\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\left[D_{\alpha_1}(\beta')^{-1}D_{\beta'}^{-1}\alpha_1\right]\right)^{-b_1}\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)^{-a_2}\\&\cdot\left(1-z\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)^{-a_3}\left(1-u\left[D_{\alpha_1}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\right]\left[D_{\alpha_1}(\beta'')^{-1}D_{\beta''}^{-1}\alpha_1\right]\right)^{-b_2}\left\{\alpha_1^{a_1-1}(\beta')^{b_1-1}(\beta'')^{b_2-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&\quad=\alpha_1^{a_1-1}(\beta')^{b_1-1}(\beta'')^{b_2-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\cdot X_9^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_2,c_1,c_1,c_3;x,y,z,u\right);\end{aligned} \tag{2.10}$$

$$\begin{aligned}&\left(1-x\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\left[D_{\alpha_1}\beta^{-1}D_{\beta}^{-1}\alpha_1\right]-z\left[D_{\alpha_1}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\right]\left[D_{\alpha_3}\beta^{-1}D_{\beta}^{-1}\alpha_3\right]\right)^{-b}\\&\cdot\left(1-y\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]-u\left[D_{\alpha_3}\beta_4^{-1}D_{\beta_4}^{-1}\alpha_3\right]\right)^{-a_2}\left\{\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta^{b-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\right\}\\&\quad=\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta^{b-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\cdot X_{10}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_2,c_3,c_4;x,y,z,u\right);\end{aligned} \tag{2.11}$$

Proof of the operational representations of the multinomial-type

To prove (2.7), let us denote, for convenience, the left-hand side of assertion (2.7) by δ . Then, as a consequence of the binomial theorem, it can be easily seen that:

$$\begin{aligned}\delta=\sum_{m,n,p,q=0}^{\infty}&\frac{(b_1)_m(b_2)_n\,(b_3)_p(b_4)_q\,x^my^nz^pu^q\,(\beta')^{-m}(\beta'')^{-n}(\beta''')^{-p}(\beta'''')^{-q}\,\beta_1^{-(m+n)}\beta_2^{-(p+q)}}{m!n!\,p!q!}D_{\alpha_1}^{2m+n+q}D_{\alpha_2}^{n+q+2p}\\&\cdot D_{\beta'}^{-m}D_{\beta''}^{-n}D_{\beta'''}^{-p}D_{\beta''''}^{-q}D_{\beta_1}^{-(m+n)}D_{\beta_2}^{-(p+q)}\cdot\left\{\alpha_1^{a_1+2m+n+q-1}\alpha_2^{n+q+2p-1}(\beta')^{b_1-1}(\beta'')^{b_2-1}(\beta''')^{b_3-1}(\beta'''')^{b_4-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\}.\end{aligned}$$

Upon using (2.3) and considering the definition (1.1), we are finally led to the right-hand side of the assertion (2.7). The proofs of the assertions (2.8) to (2.11) run parallel to that of the assertion (2.7) thus we skip the details.

3. Operational representations of the exponential and the multinomial type

We recall that the exponential expansion is defined by (see [13])

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \tag{3.1}$$

making use of the expansion formula above and the technique illustrated in section 2, we can easily derive the following operational representations of exponential and multinomial type for $X_i^{(4)}$ ($i = 6,7,8,9,10$)

$$\begin{aligned}&\left(1-x\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\left[D_{\alpha_1}\beta^{-1}D_{\beta}^{-1}\alpha_1\right]-z\left[D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_2\right]\left[D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\alpha_2\right]\right)^{-b}\\&\cdot\exp\left(y\left[D_{\alpha_1}D_{\alpha_2}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\alpha_2\right]+u\left[D_{\alpha_1}D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\alpha_2\right]\right)\\&\cdot\left\{\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{b-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\}\\&\qquad=\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{b-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\cdot X_6^{(4)}\left(a_1,a_1,a_2,a_1,a_1,a_2,a_2,a_2;c_1,c_1,c_2,c_2;x,y,z,u\right);\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+u\left[D_{\alpha_1}^2\beta_4^{-1}D_{\beta_4}^{-1}\alpha_1^2\right]\right)\cdot\left(1-y\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)^{-a_2}\left(1-z\left[D_{\alpha_1}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\right]\right)^{-a_3}\\&\cdot\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\right\}\\&\qquad=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\cdot X_7^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x,y,z,u\right);\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]\right)\cdot\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)^{-a_2}\left(1-z\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)^{-a_3}\\&\cdot\exp\left(u\left[D_{\alpha_1}^2\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1^2\right]\right)\cdot\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&\qquad=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\cdot X_8^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_2;x,y,z,u\right);\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\exp\left(x\left[D_{\alpha_1}^2\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1^2\right]\right)\cdot\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)^{-a_2}\left(1-z\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)^{-a_3}\\&\cdot\exp\left(u\left[D_{\alpha_1}^2\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1^2\right]\right)\cdot\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&\qquad=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\cdot X_9^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_2,c_1,c_1,c_3;x,y,z,u\right);\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+z\left[D_{\alpha_1}D_{\alpha_3}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\alpha_3\right]\right)\cdot\left(1-y\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]-u\left[D_{\alpha_3}\beta_4^{-1}D_{\beta_4}^{-1}\alpha_3\right]\right)^{-a_2}\\&\cdot\left\{\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\right\}\\&\qquad=\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\cdot X_{10}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_2,c_3,c_4;x,y,z,u\right);\end{aligned} \tag{3.6}$$

Proof of the operational representations of exponential and multinomial type

To prove formula (3.6), we proceed as follows: Let us denote the left hand side of (3.6) by δ . Then in view of the exponential expansion (3.1) and the multinomial expansion (2.5) one gets :

$$\begin{aligned}\delta=\sum_{m,n,p,q=0}^{\infty}&\frac{(a_2)_{n+q}\,x^m y^n z^p u^q\ \beta_1^{-m}\beta_2^{-n}\beta_3^{-p}\beta_4^{-q}}{m!n!p!q!}D_{\alpha_1}^{2m+n+p}D_{\alpha_3}^{p+q}D_{\beta_1}^{-m}D_{\beta_2}^{-n}D_{\beta_3}^{-p}D_{\beta_4}^{-q}\\&\cdot\left\{\alpha_1^{a_1+2m+n+p-1}\alpha_3^{a_3+p+q-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\right\}.\end{aligned}$$

Upon using (2.3) and considering the definition (1.5), we are finally led to the right-hand side of equation (3.6) and thereby (3.6) is proved. The proofs of formulae (3.2) to (3.5) run parallel to that of formula (3.6), so the details are skipped.

4. Generating functions

By virtue of the operational representations for the quadruple series $X_i^{(4)}$ ($i = 6,7,8,9,10$) in section 3 and also expansion formula (3.1), we find the following generating functions :

$$\begin{aligned}&\exp\left(y\left[D_{\alpha_1}D_{\alpha_2}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\alpha_2\right]+u\left[D_{\alpha_1}D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\alpha_2\right]\right)\\&\cdot\exp\left(t\left(1-x\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\left[D_{\alpha_1}\beta^{-1}D_{\beta}^{-1}\alpha_1\right]-z\left[D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_2\right]\left[D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\alpha_2\right]\right)\frac{\alpha_2}{\beta}\right)\\&\cdot\left\{\alpha_1^{a_1-1}\alpha_2^{-1}\beta^{-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\}\\&\qquad=\alpha_1^{a_1-1}\alpha_2^{-1}\beta^{-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\sum_{k=0}^{\infty}\frac{\alpha_2^k\,t^k}{\beta^k\,k!}X_6^{(4)}(a_1,a_1,k,a_1,a_1,k,k,k;c_1,c_1,c_2,c_2;x,y,z,u);\end{aligned}\tag{4.1}$$

$$\begin{aligned}&\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+t\left(1-y\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)+v\left(1-z\left[D_{\alpha_1}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\right]\right)+u\left[D_{\alpha_1}^2\beta_4^{-1}D_{\beta_4}^{-1}\alpha_1^2\right]\right)\\&\cdot\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\right\}\\&\qquad=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\sum_{k,r=0}^{\infty}\frac{t^k\,v^r}{k!\,r!}X_7^{(4)}(a_1,a_1,a_1,a_1,a_1,-k,-r,a_1;c_1,c_2,c_3,c_4;x,y,z,u);\end{aligned}\tag{4.2}$$

$$\begin{aligned}&\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+t\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)+v\left(1-z\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)+u\left[D_{\alpha_1}^2\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1^2\right]\right)\\&\cdot\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&\qquad=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\sum_{k,r=0}^{\infty}\frac{t^k\,v^r}{k!\,r!}X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,-k,-r,a_1;c_1,c_1,c_2,c_3;x,y,z,u);\end{aligned}\tag{4.3}$$

$$\begin{aligned}&\exp\left(x\left[D_{\alpha_1}^2\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1^2\right]+t\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)+v\left(1-z\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)+u\left[D_{\alpha_1}^2\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1^2\right]\right)\\&\cdot\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&\qquad=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\sum_{k,r=0}^{\infty}\frac{t^k\,v^r}{k!\,r!}X_9^{(4)}(a_1,a_1,a_1,a_1,a_1,-k,-r,a_1;c_2,c_1,c_1,c_3;x,y,z,u);\end{aligned}\tag{4.4}$$

$$\begin{aligned}&\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+z\left[D_{\alpha_1}D_{\alpha_3}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\alpha_3\right]+t\left(1-y\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]-u\left[D_{\alpha_3}\beta_4^{-1}D_{\beta_4}^{-1}\alpha_3\right]\right)\right)\\&\cdot\left\{\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\right\}\\&\qquad=\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\sum_{k=0}^{\infty}\frac{t^k}{k!}X_{10}^{(4)}(a_1,a_1,a_1,-k,a_1,-k,a_3,a_3;c_1,c_2,c_3,c_4;x,y,z,u);\end{aligned}\tag{4.5}$$

<u>*Proof of the generating functions*</u>

To prove relation (4.1), in (3.2), let us put $a_2=-b=k$, by multiplying both sides by $\frac{t^k}{k!}$ and then sum, we finally get

$$\begin{aligned}&\exp\left(y\left[D_{\alpha_1}D_{\alpha_2}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\alpha_2\right]\right)+u\left[D_{\alpha_1}D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\alpha_2\right]\\&\cdot\sum_{k=0}^{\infty}\frac{\left(t\left(1-x\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\left[D_{\alpha_1}\beta^{-1}D_{\beta}^{-1}\alpha_1\right]-z\left[D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_2\right]\left[D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\alpha_2\right]\right)\frac{\alpha_2}{\beta}\right)^k}{k!}\left\{\alpha_1^{a_1-1}\alpha_2^{-1}\beta^{-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\}\\&\qquad=\alpha_1^{a_1-1}\alpha_2^{-1}\beta^{-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\sum_{k=0}^{\infty}\frac{\alpha_2^k\,t^k}{\beta^k\,k!}X_6^{(4)}(a_1,a_1,k,a_1,a_1,k,k,k;c_1,c_1,c_2,c_2;x,y,z,u).\end{aligned}$$

Now, we use the definition (3.1) of the exponential expansion, which completes the proof of relation (4.1).
In the same manner, one can prove relations (4.2) to (4.5).

Remark: in (4.2) to (4.4) let $a_2=-k$ and $a_3=-r$ and $a_2=-k$ in (4.5), respectively.

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