УНИВЕРЗИТЕТ „ГОЦЕ ДЕЛЧЕВ" - ШТИП
ФАКУЛТЕТ ЗА ИНФОРМАТИКА

ISSN:1857-8691

ГОДИШЕН ЗБОРНИК 2016/2017

YEARBOOK 2016/2017

ГОДИНА 5 **VOLUME V**

GOCE DELCEV UNIVERSITY - STIP
FACULTY OF COMPUTER SCIENCE

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YEARBOOK
FACULTY OF COMPUTER SCIENCE

За издавачот:
Проф д-р Цвета Мартиновска Банде

Редакција и администрација	Address of the editorial office
Универзитет „Гоце Делчев“-Штип	Goce Delcev University – Stip
Факултет за информатика	Faculty of Computer Science
ул. „Крсте Мисирков“ 10-А	Krste Misirkov 10-A
п. фах 201, 2000 Штип	PO box 201, 2000 Štip,
Р. Македонија	R. of Macedonia

С О Д Р Ж И Н А

INVERSE FUNCTION THEOREM WITH STRONG METRIC REGULARITY

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Abstract. In this paper we prove an extension of the contraction mapping principle for single-valued mappings dealing with more general assumptions containing modulus instead of pseudo-contractive functions. In [4] A. L. Dontchev and R.T. Rockaffelar supposethe strong metric regularity of set-valued mapping F and the Lipschitz continuity of the function g with given nonnegative constants and prove the strong metric regularity of $g+F$, while we assume the properties of F and g with modulus functions and prove a generalization of their result.
MSC (2010): 47J07; 49J53; 49K40; 47H10.
Key words: Metric regularity; inverse function theorem ; set-valued mapping; generalized equations; fixed point theorem .

1. Introduction

The notion of regular multifunction has emerged out of the pioneering work of Ljusternik and Sobolev [7] through the efforts of several workers. In its initial form it was conceived as a mean for solving equations or inequalities under more or less classical surjectivity assumptions on some approximations ([2], [3], [6], [10], [11], [13], [14]). In [15] Penot prove the equivalence of the definitions for Aubin continuous of the inverse mapping, metric regularity and openness with linear rate of a set-valued mapping for the case when the Lipschitz modulus is not just a constant but it is a function.
In this paper we prove an extension of the contraction mapping principle for single-valued mappings where we use more general assumptions containing modulus instead of pseudo-contractive functions. Using this result we prove an inverse function theorem. We consider the parametric generalized equation

$$\text{find} \quad x\in X \quad \text{such that} \quad 0\in f(p,x)+F(x),$$

where f is a function, F is a set valued mapping acting from a metric space to another metric space and prove an implicit mapping theorem. Some iterative methods for solving parametric generalized equation can be found in [8],[9], and [12]. The using of a modified variational iteration method for solving nonlinear coupled equations is shown in [5].

In this paper we deal with metric spaces (P,π) for p and a complete metric space (X,ρ) for x. The space (Y,σ) is linear space equipped with *shift-invariant metric* σ for the range of f and g, that is:

$$\sigma(y+z,y'+z)=\sigma(y,y') \quad \text{for} \quad \text{all} \quad y,y',z\in Y.$$

In such spaces the standard definitions, e.g. of the ball in X with center x and radius r is defined by:

$$B_r(\bar{x})=\{x\in X \mid \rho(x,\bar{x})\le r\}$$

and the distance from a point x to a set C in X is denoted by:

$$d(x,C)=\inf_{x'\in C}\rho(x,x').$$

The subset C of a complete metric space is *closed* when $d(x;C)=0 \Rightarrow x\in C$. Also a set C is *locally closed* at a point $x\in C$ if there is a neighborhood U of x such that the intersection $C\cap U$ is closed.
The *graph* of F is the set $\operatorname{gph} F=\{(x,y)\in X\times Y \mid y\in F(x)\}$ and the *inverse* of F is the mapping $F^{-1}:Y\,\Gamma\,\ X$ defined by $F^{-1}(y)=\{x \mid y\in F(x)\}$.
The *solution mapping* associated with the generalized equation $0\in f(p,x)+F(x)$ is the potentially set-valued mapping $S:P\,\Gamma\,\ X$ defined by

$$S:p\mapsto\{x \mid f(p,x)+F(x)\ni 0\} \quad \text{for} \quad p\in P. \tag{1}$$

Graphical localization of S at $\bar{p}$ for $\bar{x}$, where $\bar{x} \in S(\bar{p})$ is a set-valued mapping with its graph having the form $(Q \times U) \cap \operatorname{gph} S$ for some neighborhoods Q of $\bar{p}$ and U of $\bar{x}$. The localization is *single-valued* when this mapping reduces to a function for Q into U. If it is not only single-valued but Lipschitz continuous on Q, we speak of *Lipschitz localization.*

Definition 1.1. An *modulus* is said to be an increasing function $w:[0,\infty) \to [0,\infty)$, which is continuous in 0 and $w(0)=0$.

Definition 1.2. A function $f: X \to Y$ is said to be *Lipschitz continuous with modulus function* relative to a set D, if $D \subset \operatorname{int}\operatorname{dom} f$ and there exists a modulus function k, such that

$$\sigma(f(x'), f(x'')) \le k\left(\rho(x', x'')\right) \text{ for all } x', x'' \in D, \tag{2}$$

It is said to be a *Lipschitz continuous around* $\bar{x}$ when this holds for some neighborhood D of $\bar{x}$.

Definition 1.3. A function f:P$\times X \to Y$ is said to be *Lipschitz continuous* with respect to x uniformly in p around $(\bar{p},\bar{x}) \in \operatorname{int}\operatorname{dom} f$ with modulus function, when there are neighborhoods Q of $\bar{p}$ and U of $\bar{x}$ along with a modulus function k, such that

$$\sigma(f(p,x'), f(p,x'')) \le k\rho(x',x'') \text{ for all } x',x'' \in U \text{ and } p \in Q.$$

Definition 1.4. A set valued mapping $F: X \,\Gamma\, Y$ is said to be *metrically regular* at $\bar{x}$ for $\bar{y}$ when $\bar{x} \in F(\bar{y})$ and there is a constant $k \ge 0$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(x, F^{-1}(y)) \le kd(y, F(x)) \qquad \textit{for all } (x,y) \in U \times V.$$

The infimum of k of all such combinations of k, U and V is called the *regularity modulus* for F at $\bar{x}$ for $\bar{y}$.

Definition 1.5. A mapping $F: X \,\Gamma\, Y$ with $(\bar{p},\bar{x}) \in \operatorname{gph} F$ is called *strongly regular* at $\bar{x}$ for $\bar{y}$ if its inverse F^{-1} has a Lipschitz localization at $\bar{y}$ for $\bar{x}$.

Definition 1.6. Consider a function $f: P \times X \to Y$ and a point $(\bar{p},\bar{x}) \in \operatorname{int}\operatorname{dom} f$. A function $h: X \to Y$ with $\bar{x} \in \operatorname{int}\operatorname{dom} h$ is said to be a *strict estimator* of f with respect to x uniformly in p at $(\bar{p},\bar{x})$ with modulus μ if $h(\bar{x}) = f(\bar{p},\bar{x})$ and there exists scalars a and τ such that

$$\sigma(r(p,x'), r(p,x)) \le \mu(\rho(x',x)) \text{ for all } x, x' \in B_a(\bar{x}) \text{ and } p \in B_\tau(\bar{p}),$$

where $r(p,x) = f(p,x) - h(x)$.

Definition 1.7. A nondecreasing function $\varphi: J \to J$ is said to be a *Bianchini-Grandolfi gauge function* [1] on J, where J is an interval on R_+ containing 0 if

$$s(t) = \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \qquad \text{for all} \quad t \in J.$$

Here we denote by φ^n to be the n-th iteration of the function $\varphi: J \to J$, where J is an interval on R_+ containing 0, $\varphi^0(t) = t$.

Note that Ptak [16] called a function $\varphi: J \to J$ satisfying the above condition a *rate of convergence* on J and noticed that φ satisfies the following functional equation

$$s(t) = t + s(\varphi(t)).$$

2. Main result

We will prove an extension of the contraction mapping principle for set-valued mappings stated in [4] by A. L. Dontchev and R.T. Rockaffelar. The difference is that we use more general assumptions containing modulus instead of pseudo-contractive multifunctions.

Theorem 2.1. (Contraction mapping principle) Let (X,ρ) be a complete metric space and $\bar{x} \in X$.

Consider a single-valued mapping $\Phi : X \to X$. Assume that the increasing function $\varphi : J \to J$, where J is an interval on bR_+ containing 0, is a Bianchini-Grandolfi gauge function. Suppose that there exist $a > 0$ and $r \in J$ and a constant $r \in J$, such that the following assumptions hold:

(a) $\rho(\bar{x}, \Phi(\bar{x})) \le r, \quad where \quad s(r) \le a;$

(b) $\rho(\Phi(u), \Phi(v)) \le \varphi(\rho(u,v)) \quad for \quad all \quad u, v \in B_a(\bar{x}).$

Then Φ has a fixed point in $B_{s(r)}(\bar{x})$; that is, there exists $x \in B_{s(r)}(\bar{x})$ such that $x \in \Phi(x)$.

Proof. We will proceed by induction. Denoting $x^1 = \Phi(\bar{x})$ and consider the sequence

$$x^{k+1} \in \Phi(x^k) \quad for \;\; k = 1, 2 \ldots.$$

We will prove that this sequence satisfies the following inequalities:

$$\rho(x^k, x^{k+1}) \le \varphi^k(r);$$
$$\rho(x^{k+1}, \bar{x}) \le s(r).$$

We denote $x^0 = \bar{x}$. Then

$$\rho(x^1, \bar{x}) \le r = \varphi^0(r).$$

Now we suppose that there exists

$$x^{k+1} = \Phi(x^k) \quad за \;\; k = 0, 1 \ldots, n-1.$$

We will prove that the above mentioned inequalities hold for $k = n$. Using (b) we have

$$\rho(x^n, x^{n+1}) = \rho(x^n, \Phi(x^n)) = \rho(\Phi(x^{n-1}), \Phi(x^n))$$
$$\le \varphi(\rho(x^{n-1}, x^n)) \le \varphi(\varphi^{n-1}(r)) = \varphi^n(r).$$

Hence

$$\rho(x^{n+1}, x^n) \le \varphi^n(r).$$

By the triangle inequality we have

$$\rho(x^{n+1}, \bar{x}) \le \sum_{k=0}^{n} \rho(x^{k+1}, x^k) \le \sum_{k=0}^{n} \varphi^k(r) < s(r).$$

Hence $x^{n+1} \in B_{s(r)}(\bar{x})$ and this completes the induction.

Since for all $k > m > 1$ we have

$$\rho(x^k, x^m) \le \sum_{i=m}^{k-1} \rho(x^{i+1}, x^i) \le \sum_{i=m}^{k-1} \varphi^i(r)$$

and $\varphi(t)$ is Bianchini-Grandolfi gauge function, the sequence $\{x^n\}$ is Cauchy sequence. But X is complete metric space, then $\{x^n\}$ converges to x.

Let $n \to \infty$ in the following inequality

$$\rho(x^{n+1}, \bar{x}) \le s(r).$$

Then we have

$$\rho(x, \bar{x}) \le s(r)$$

and consequently $x \in B_{s(r)}(\bar{x})$. Since Φ is continuous and $x^n = \Phi(x^{n-1})$ as $n \to \infty$ we have $x = \Phi(x)$, i.e. x is fixed point for Φ and $x \in B_{s(r)}(\bar{x})$.

Now we will prove that x is unique. Let $\tilde{x} \in X$ and $\tilde{x} = \Phi(\tilde{x})$. Then

$$\rho(x, \tilde{x}) = \rho(\Phi(x), \Phi(\tilde{x})) \le \varphi(\rho(x, \tilde{x})).$$

By the definition of Bianchini-Grandolfi gauge function we have $\varphi(0) = 0$ and $\varphi(t) < t$ for $\forall t > 0$. By the equality $s(t) = t + s(\varphi(t))$ we have $s(\varphi(t)) < s(t)$. Consequently $\rho(x, \tilde{x}) = 0 \Rightarrow x \equiv \tilde{x}$ and this completes the proof of Theorem 2.1.,

The next theorem is a generalization of Theorem 5F.1, p.292 [4].

Theorem 2.2. (Inverse function theorem with strong metric regularity) Let (X,ρ) be a complete metric space, (Y,σ) be a linear space with shift-invariant metric σ and $F: X\,\Gamma\,\; Y$ is a set-valued mapping, $(\bar{x},\bar{y})\in \operatorname{gph} F$, U is neighborhood of $\bar{x}$, V is neighborhood of $\bar{y}$. The function $g: X\to Y$ is Lipschitz continuous with modulus function μ in U. Let $y\mapsto F^{-1}(y)\cap U$ is Lipschitz function in V with modulus function k and the function $\varphi(t)=k(\mu(t))$, $t\in J$, where J is interval in R_+, containing 0, is Bianchini-Grandolfi gauge function, such that

$$s(t)=\sum_{n=0}^{\infty}\varphi^n(t)<\infty, \qquad \text{for all} \quad t\in J.$$

Let $k^{-1}-\mu$ is increasing function.

Then there exist neighborhoods U' of $\bar{x}$ and V' of $\bar{y}$ such that the mapping

$y\to(g+F)^{-1}(y)\cap U'$ is Lipschtz function in $g(\bar{x})+V'$ with modulus function $(k^{-1}-\mu)^{-1}$.

Proof. Let $r(y)=F^{-1}(y)\cap U$, $y\in V$. By the statements in the theorem we have

$$\rho(r(y),r(y'))\le k(\sigma(y,y')), \quad \textit{for all } y,y'\in V, \tag{1}$$

$$\sigma(g(x),g(x'))\le \mu(\rho(x,x')), \quad \textit{for all } x,x'\in U. \tag{2}$$

Choose $a>0$ and $b>0$ such that $B_a(\bar{x})\subset U$, $B_{b+\mu(a)}(\bar{y})\subset V$ and

$$s(k(b))\le a. \tag{3}$$

For all $y\in B_b(g(\bar{x})+\bar{y})$ and $x\in B_a(\bar{x})$, using that σ shift invariant metric we have

$$\begin{aligned}\sigma(-g(x)+y,\bar{y}) &= \sigma(y,g(x)+\bar{y})\\ &\le \sigma(y,g(\bar{x})+\bar{y})+\sigma(g(x),g(\bar{x}))\\ &\le b+\mu(\rho(x,\bar{x}))\le b+\mu(a).\end{aligned}$$

Then $-g(x)+y\in V\subset \operatorname{dom} r$. Fix $y\in B_b(g(\bar{x})+\bar{y})$ and consider the mapping

$$\Phi_y: x\mapsto r(-g(x)+y) \quad \text{for} \quad x\in B_a(\bar{x}).$$

Using (1), we have

$$\begin{aligned}&\rho(\bar{x},\Phi_y(\bar{x}))\\ &\quad=\rho(r(\bar{y}),r(-g(\bar{x})+y))\\ &\quad\le k(\sigma(y,\bar{y}+g(\bar{x})))\le k(b).\end{aligned}$$

By (3) we have $s(k(b))\le a$ and this showes that assumption (a) of Theorem 2.1 is satisfied. Then for all $u,v\in B_a(\bar{x})$ by (1) and (2) we come to

$$\begin{aligned}&\rho(\Phi_y(u),\Phi_y(v))\\ &\quad=\rho(r(-g(u)+y),r(-g(v)+y))\\ &\quad=(\sigma(-g(u),-g(v)))\\ &\quad=(\sigma(-g(u)+g(u)+g(v),-g(v)+g(u)+g(v)))\\ &\quad\le k(\sigma(g(u),g(v)))\le k(\mu(\rho(u,v))).\end{aligned}$$

Since $k(\mu)$ is increasing Bianchini-Grandolfi gauge function, we have assumption (b) of Theorem 2.1. Now we use Theotem 2.1 for the mapping Φ_y and we conclude that there exists unique fixed point $x=\Phi_y(x)$ and $x\in B_{s(k(b))}(\bar{x})\subset B_a(\bar{x})$. Then $x(y)=r(y-g(x(y)))$ and

$$\begin{aligned}&\rho(x(y),x(y'))\\ &\quad=\rho(r(-g(x(y))+y),r(-g(x(y'))+y'))\end{aligned}$$

$$\leq k(\sigma(y,y')+\sigma(g(x(y)),g(x(y'))))$$
$$\leq k(\sigma(y,y')+\mu(\rho(x(y),x(y')))).$$

Consequently

$$\rho(x(y),x(y'))\leq k(\sigma(y,y')+\mu(\rho(x(y),x(y')))).$$

Since k is continuous and increasing function we have

$$k^{-1}(\rho(x(y),x(y')))\leq \sigma(y,y')+\mu(\rho(x(y),x(y'))),$$
$$(k^{-1}-\mu)(\rho(x(y),x(y')))\leq \sigma(y,y').$$

By the continuity of $k^{-1}-\mu$ we come to

$$\rho(x(y),x(y'))\leq (k^{-1}-\mu)^{-1}(\sigma(y,y')).$$

But x and y arbitrarily chosen in the corresponding neighborhoods of $\overline{x}$ and $\overline{y}$. Then the above mentioned relation means that $y\to(g+F)^{-1}\cap V'$ is Lipschitz function in $g(\overline{x})+V'$ with modulus $(k^{-1}-\mu)^{-1}$. ,

In [4] A. L. Dontchev and R.T. Rockaffelar suppose the strong metric regularity of F and the Lipschitz continuity of g with given nonnegative constants and prove the strong metric regularity of $g+F$, while we assume the properties of F and g with modulus functions. So this theorem is a generalization of their result.

3. Conclusion

In this study, under more general assumptions containing modulus instead of pseudo-contractive functions, we examine an extension of the contraction mapping principle for single-valued mappings and inverse function theorem for strong metric regularity mapping. Our results extend and complement many theorems in the literature.

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