УНИВЕРЗИТЕТ „ГОЦЕ ДЕЛЧЕВ” - ШТИП
ФАКУЛТЕТ ЗА ИНФОРМАТИКА

ISSN:1857-8691

ГОДИШЕН ЗБОРНИК 2016/2017

YEARBOOK 2016/2017

ГОДИНА 5 VOLUME V

GOCE DELCEV UNIVERSITY - STIP
FACULTY OF COMPUTER SCIENCE

УНИВЕРЗИТЕТ „ГОЦЕ ДЕЛЧЕВ“ – ШТИП
ФАКУЛТЕТ ЗА ИНФОРМАТИКА

ГОДИШЕН ЗБОРНИК 2016/2017

YEARBOOK 2016/2017

ГОДИНА 5 **МАЈ, 2017** **VOLUME V**

GOCE DELCEV UNIVERSITY – STIP
FACULTY OF COMPUTER SCIENCE

ГОДИШЕН ЗБОРНИК
ФАКУЛТЕТ ЗА ИНФОРМАТИКА
YEARBOOK
FACULTY OF COMPUTER SCIENCE

Редакција и администрација
Универзитет „Гоце Делчев“-Штип
Факултет за информатика
ул. „Крсте Мисирков“ 10-А
п. фах 201, 2000 Штип
Р. Македонија

Address of the editorial office
Goce Delcev University – Stip
Faculty of Computer Science
Krste Misirkov 10-A
PO box 201, 2000 Štip,
R. of Macedonia

С О Д Р Ж И Н А

STABILITY RESULTS FOR FIXED POINT ITERATION PROCEDURES

Rumen Tsanev Marinov[1], Diana Kirilova Nedelcheva[2]

[1] Department of Mathematics, Technical University of Varna, Bulgaria
marinov_r@yahoo.com
[2] Department of Mathematics, Technical University of Varna, Bulgaria
dkondova@abv.bg

Abstract. While solving inclusions numerically by an iterative procedure, usually we follow some theoretical model and deal with an approximate numerical sequence. If the numerical sequence converges to a point anticipated by the theoretical sequence, then we say that the iterative procedure is stable. This kind of study plays a vital role in computational analysis, game theory and computer programming. The purpose of this paper is to discuss stability of the Picard iterative procedure for pseudo-Lipschitz multivalued operators in metric spaces.
MSC (2010): 54H25; 65D15; 65D18; 47H10.
Key words: Set-valued mapping; fixed point theorem; Picard iterative procedure; pseudo-Lipschitz contraction; stability of iterative procedures.

1. Introduction

Let (X,d) be a metric space and $T: X \to X$. The solution of a fixed point equation $Tx = x$ for any $x \in X$, is usually approximated by a sequence $\{x_n\}$ in X generated by an iterative procedure $f(T,x_n)$ that converges to a fixed point of T. However, in actual computations, we obtain an approximate sequence $\{y_n\}$ instead of the actual sequence $\{x_n\}$. Indeed, the approximate sequence $\{y_n\}$ is calculated in the following manner. First, we choose an initial approximation $x_0 \in X$. Then we compute $x_1 = f(T,x_0)$. But, due to rounding off or discretization of the function, we get an approximate value y_1, say , which is close enough to x_1, i.e., $y_1 \approx x_1$. Consequently, when computing x_2, we actually compute $y_2 \approx x_2$. In this way, we obtain an approximate sequence $\{y_n\}$ instead of the actual sequence $\{x_n\}$. The iterative procedure $f(T,x_n)$ is considered to be numerically stable if and only if the approximate sequence $\{y_n\}$ still converges to the desired solution of the equation $Tx = x$. Urabe [1] initiated the study of this problem. The study of stability of iterative procedures plays a significant role in numerical mathematics due to chaotic behavior of functions and discretization of computations in computer programming. For a detailed discussion on the role of stability of iterative procedures, one may refer to Czerwik et al. [2,3], Harder and Hicks [4-6], Lim [7], Matkowski and Singh [8], Ortega and Rheinboldt [9], Osilike [10,11], Ostrowski [12], Rhoades [13,14], Rus et al. [15] and Singh et al. [16].

However, Ostrowski [12] was the first to obtain the following classical stability result on metric spaces.

Theorem 1.1. Let (X,d) be a complete metric space and $T: X \to X$ a Banach contraction with contraction constant q, i.e., $d(Tx,\ Ty) \le qd(x,\ y)$ for all $x,\ y \in X$, where $0 \le q < 1$. Let p be the fixed point of T. Let $x_0 \in X$ and $x_{n+1} = Tx_n,\ n = 0,\ 1,\ 2,\ldots$ Suppose that $\{y_n\}$ is a sequence in X and $\varepsilon_n = d(y_{n+1},\ Ty_n)$. Then

$$d(p,y_{n+1}) \le d(p,x_{n+1}) + q^{n+1}d(x_0,y_0) + \sum_{j=0}^{n} q^{n-j}\varepsilon_j.$$

Moreover, $\lim_{n\to\infty} y_n = p$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$.

This result has found a respectable place in numerical analysis and computer programming and further extended by Harder and Hicks [5,6], Jachymski [17], Osilike [10,11,18], Osilike and Udomene [19], Rhoades [13,14], Czerwik et al. [2] and Zhou [20].

The classical result on stability due to Ostrowski has been extended to multi-valued maps by Singh and Chadha [21] and further extended by Singh and Bhatnagar [22] and Singh et al. [23].

Furhter, stability of iterative procedures has a remarkable importance in fractal graphics while generating fractals. Its usefulness lies in the fact that in fractal graphics, fractal objects are generated by an infinite recursive process of successive approximations. An itertive procedure produces a sequence of results and tends towards one final object called a set attractor (fractal), which is independent of the initial choice. This stable character of set attractor is due to the stability of iterative procedure, else the system of underlying successive approximations would show chaotic behavior and never settle into a stationary state. However, fractals themselves have a variety of applications in digital imaging, mobile computing, architecture and construction, various branches of engineering and applied sciences. For recent potential applications of fractal geometry in related fields, one may refer to Batty and Longley [24], Buser et al. [25], Lee and Hsieh [26], Mistakeidis and Panagouli [27], Shaikh et al. [28] and Zmeskal et al. [29]. For connections of the round-off stability with the concept of limit shadowing for a fixed point problem involving multi-valued maps, one may refer to Petrusel and Rus [30].

The purpose of this article is to discuss the stability of Picard iterative procedure, i.e., $x_{n+1} \in f(T, x_n) = Tx_n$ for a map T satisfying pseudo-Lipschitz multi-valued contraction (cf. Definition 2.2).

2. Preliminaries

Let (X, d) be a metric space and

$$CB(X) = \{A : A \text{ is a nonempty closed bounded subset of } X\},$$

$$CL(X) = \{A : A \text{ is a nonempty closed subset of } X\}.$$

For $A, B \in CL(X)$ and $\varepsilon > 0$,

$$N(\varepsilon, A) = \{x \in X : d(x, a) < \varepsilon \quad \text{for some } a \in A\},$$

For sets A and B in X, the *excess* of A beyond B is defined by

$$e(A, B) = \sup_{x \in A} d(x, B),$$

where the convention is used that

$$e(\emptyset, B) = \begin{cases} 0 & when\ B \neq \emptyset \\ \infty & otherwise \end{cases}.$$

The Pompeiu–Hausdorff *distance* between A and B is the quantity

$$h(A, B) = \max\{e(A, B), e(B, A)\}.$$

Equivalently, these quantities can be expressed by

$$e(A, B) = \inf\{\varepsilon > 0 : A \subseteq N(\varepsilon, B)\},$$

$$h(A, B) = \inf\{\varepsilon > 0 : A \subseteq N(\varepsilon, B), B \subseteq N(\varepsilon, A)\}.$$

An orbit $O(x_0)$ of a multi-valued map T at a point x_0 is a sequence

$$\{x_n : x_n \in Tx_{n-1},\ n = 1, 2, \ldots\}.$$

For a single-valued map T, this orbit is $\{x_n : x_n = Tx_{n-1},\ n = 1, 2, \ldots\}$.

The study of fixed point theorems for multi-valued contractions was initiated by Markin [31] and Nadler [32]. The notion of multi-valued contractions have been generalized by many authors.

Definition 2.1. (Nadler [32])
A map $T : X \to CL(X)$ is called a Nadler multi-valued contraction if

$$h(Tx, Ty) \le q d(x, y)$$

for all $x, y \in X$, where $0 \le q < 1$.

Let (X, d) be a metric space and $T : X \to CL(X)$. For a point $x_0 \in X$, let

$x_{n+1} \in T(x_n)$ denote some iteration procedure. Let the sequence $\{x_n\}$ be convergent to a fixed point p of T. Let $\{y_n\}$ be an arbitrary sequence in X and set

$$\varepsilon_n = h\left(y_{n+1}, T\left(y_n\right)\right), n = 0,\ 1,\ 2, \ldots.$$

If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = p$ then the iteration process is said to be T-stable or stable with respect to T (cf. [21]).

Ostrowski's stablity theorem [12] says that Picard iterative procedure for (single-valued) Banach contraction is stable. Following is the extension of this theorem to multivalued contractions given by Singh and Chadha [21].

Theorem 2.2. Let X be a complete metric space and $T : X \to CL(X)$ such that the condition given in Definition 2.1 holds for all $x, y \in X$. Let x_0 be an arbitrary point in X and $\{x_n\}_{n=1}^{\infty}$ an orbit for T at x_0 such that $\{x_n\}_{n=1}^{\infty}$ is convergent to a fixed point p of T. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in X, and set

$$\varepsilon_n = h\left(y_{n+1}, T(y_n)\right), n = 0,\ 1,\ 2, \ldots.$$

Then

$$d\left(p, y_{n+1}\right) \le d\left(p, x_{n+1}\right) + q^{n+1} d\left(x_0, y_0\right) + \sum_{j=0}^{n} q^{n-j} \varepsilon_j.$$

Further, if Tp is singleton then

$$\lim_{n\to\infty} y_n = p \text{ if and only if } \lim_{n\to\infty} \varepsilon_n = 0.$$

We shall need the following result.

Lemma 2.1. (Harder and Hicks [6])

If c is a real number such that $0 < |c| < 1$ and $\{b_k\}_{k=0}^{\infty}$ is a sequence of real numbers such that $\lim_{k\to\infty} b_k = 0$, then $\lim_{n\to\infty}\left(\sum_{k=0}^{n} c^{n-k} b_k\right) = 0.$

We denote by $B_a(x)$ the closed ball centered at x with radius a.

Theorem 2.3 (Contraction Mapping Principle for Set-Valued Mappings(Dontchev and Hager [33])).

Let X be a complete metric space with metric d, and consider a set-valued mapping $T : X \to CL(X)$ and a point $\bar{x} \in X$. Suppose that there exist scalars $a > 0$ and $q \in [0,1)$ such that

(a) $d\left(\bar{x}, T\left(\bar{x}\right)\right) < a(1-q)$;

(b) $e(T\left(u\right) \cap B_a\left(\bar{x}\right), T\left(v\right)) \le q d\left(u, v\right)$ for all $u, v \in B_a\left(\bar{x}\right)$.

Then T has a fixed point in $B_a\left(\bar{x}\right)$; that is, there exists $x \in B_a\left(\bar{x}\right)$ such that $x \in T\left(x\right)$. If T is single-valued, then x is the unique fixed point of in $B_a(x)$.

Definition 2.2.

A map $T : X \to CL(X)$ is called a pseudo-Lipschitz multi-valued contraction in $\bar{x} \in X$ if there exist scalars $a > 0$ and $q \in [0,1)$ such that

(a) $d\left(\bar{x}, T\left(\bar{x}\right)\right) < a(1-q)$;

(b) $e(T\left(u\right) \cap B_a\left(\bar{x}\right), T\left(v\right)) \le q d\left(u, v\right)$ for all $u, v \in B_a\left(\bar{x}\right)$ for all $u, v \in X$.

3. Main result

Theorem 3.1. Let X be a complete metric space and $T: X \to CL(X)$ a pseudo-Lipschitz multi-valued contraction in $x_0 \in X$ with parameters $a > 0$ and $q \in [0,1)$ (cf. Definition 2.2). Let $\{x_n\}_{n=1}^{\infty}$ be an orbit for T at x_0 such that $\{x_n\}_{n=1}^{\infty}$ is convergent to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be a sequence in $B_a(x_0)$ and set $\varepsilon_n = h(y_{n+1}, T(y_n)), n = 0, 1, 2, \ldots$. Then

$$d(p, y_{n+1}) \le d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + \sum_{j=0}^{n} q^{n-j} \varepsilon_j.$$

Further, if Tp is singleton then

$$\lim_{n\to\infty} y_n = p \text{ if and only if } \lim_{n\to\infty} \varepsilon_n = 0.$$

Proof: Let n be a nonnegative integer. Using the triangle inequality, we have

$$d(p, y_{n+1}) \le d(p, x_{n+1}) + d(x_{n+1}, y_{n+1}). \tag{1}$$

Since

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\le d(y_{n+1}, Tx_n \cap B_a(x)) \\ &\le e(Tx_n \cap B_a(x), Ty_n) + d(Ty_n, y_{n+1}), \\ &\le qd(x_n, y_n) + \varepsilon_n \end{aligned} \tag{2}$$

we derive analogously

$$d(x_n, y_n) \le qd(x_{n-1}, y_{n-1}) + \varepsilon_{n-1}. \tag{3}$$

Therefore, using (1) and (2) in (1), we obtain

$$\begin{aligned} d(p, y_{n+1}) &\le d(p, x_{n+1}) + qd(x_n, y_n) + \varepsilon_n \\ &\le d(p, x_{n+1}) + q^2 d(x_{n-1}, y_{n-1}) + (\varepsilon_n + q\,\varepsilon_{n-1}). \end{aligned}$$

Repeat this process $(n-1)$ times to obtain

$$d(p, y_{n+1}) \le d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + \sum_{j=0}^{n} q^{n-j} \varepsilon_j. \tag{4}$$

Further, if Tp is singleton then

$$\begin{aligned} \varepsilon_n &= d(y_{n+1}, Ty_n) \\ &\le d(y_{n+1}, p) + d(p, Tp) + h(Tp, Ty_n) \\ &\le d(y_{n+1}, p) + d(p, Tp) + qd(p, y_n). \end{aligned}$$

This yields $\varepsilon_n \to 0$ as $n \to \infty$, since $Tp = \{p\}$ by hypothesis.

Conversly, suppose that $\varepsilon_n \to 0$ as $n \to \infty$. Note that $q \in [0,1)$.

If $q = 0$ then (4) yields $\lim_{n\to\infty} y_n = p$. So, assume that $q \in (0,1)$. Then $q^{n+1} d(x_0, y_0) \to 0$ as $n \to \infty$. Since $\lim_{n\to\infty} \varepsilon_n = 0$, by Lemma 2.1 $\sum_{k=0}^{n} q^{n-k} \varepsilon_k \to 0$ as $n \to \infty$.

Hence from (4), $\lim_{n\to\infty} y_n = p$. □

4. Conclusion

In this study we discuss the stability of Picard iterative procedure, i.e., $x_{n+1} \in Tx_n$ for a map T satisfying pseudo-Lipschitz multi-valued contraction. Stability of iterative procedures has a remarkable importance in fractal graphics while generating fractals. Its usefulness lies in the fact that in fractal graphics, fractal objects are generated by an infinite recursive process of successive approximations. Our results extend and complement many theorems in the literature

References

[1] Urabe, M: *Convergence of numerical iteration in solution of equations*. J Sci Hirishima Univ Sér A. 19, 479–489 (1956)

[2] Czerwik, S, Dlutek, K, Singh, SL: *Round-off stability of iteration procedures for operators in b-metric spaces*. J Natur Phys Sci. 11, 87–94 (1997)

[3] Czerwik, S, Dlutek, K, Singh, SL: *Round-off stability of iteration procedures for set-valued* operators in b-metric spaces. J Natur Phys Sci. 15(1-2):1–8 (2001)

[4] Harder, AM: *Fixed point theory and stability results for fixed point iterative procedures*. Ph.D. Thesis, Univ of Missouri-Rolla. (1987)

[5] Harder, AM, Hicks, TL: *A stable iteration procedure for non-expansive mappings*. Math Japon. 33, 687–692 (1988)

[6] Harder, AM, Hicks, TL: *Stability results for fixed point iteration procedures*. Math Japon. 33, 693–706 (1988)

[7] Lim, TC: *On fixed point stability for set-valued contractive mappings with applications to generalized differential equations*. J Math Anal Appl. 110, 436–441 (1985). doi:10.1016/0022-247X(85)90306-3

[8] Matkowski, J, Singh, SL: *Round-off stability of functional iterations on product spaces*. Indian J Math. 39(3):275–286(1997)

[9] Ortega, JM, Rheinboldt, WC: *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, New York (1970)

[10] Osilike, MO: *Stability results for fixed point iteration procedures*. J Nigerian Math Soc. 14, 17–29 (1995)

[11] Osilike, MO: *Stability results for the Ishikawa fixed point iteration procedure*. Indian J Pure Appl Math. 26, 937–945(1995)

[12] Ostrowski, AM: *The round-off stability of iterations*. Z Angew Math Mech. 47, 77–81 (1967). doi:10.1002/zamm.19670470202

[13] Rhoades, BE: *Fixed point theorems and stability results for fixed point iteration procedures*. Indian J Pure Appl Math. 21, 1–9 (1990)

[14] Rhoades, BE: *Fixed point theorems and stability results for fixed point iteration procedures-II*. Indian J Pure Appl Math. 24, 697–703 (1993)

[15] Rus, IA, Petrusel, A, Petrusel, G: *Fixed Point Theory 1950-2000: Romanian Contributions*. House of the Book of Science, Cluj-Napoca (2002)

[16] Singh, SL, Mishra, SN, Chadha, V: *Round-off stability of iterations on product spaces*. C R Math Rep Acad Sci Canada.16(3):105–109 (1994)

[17] Jachymski, JR: *An extension of A. Ostrowski's theorem on the round-off stability of iterations*. Aequations Math. 53,242–253 (1997). doi:10.1007/BF02215974

[18] Osilike, MO: *Iterative solution of nonlinear equations of f-strongly accretive type*. J Math Anal Appl. 200, 259–271 (1996). doi:10.1006/jmaa.1996.0203

[19] Osilike, MO, Udomene, A: *Short proofs of stability results for fixed point iteration procedures for a class of contractivetype mappings*. Indian J Pure Appl Math. 30(12):1229–1234 (1999)

[20] Zhou, HY: *Stable iteration procedures for strong pseudocontractions and nonlinear equations involving accretive operators without Lipschitz assumption*. J Math Anal appl. 230, 1–10 (1999). doi:10.1006/jmaa.1998.6108

[21] Singh, SL, Chadha, V: *Round-off stability of iterations for multivalued operators*. C R Math Rep Acad Sci Canada. 17(5):187–192 (1995)

[22] Singh, SL, Bhatnagar, C: *Stability of iterative procedures for multivalued operators*. Proc of the Conference on Recent Trends in Mathematics and Its Applications. pp. 25–34.Ind. Soc. Math. Math. Sci., Gorakhpur (2003)

[23] Singh, SL, Bhatnagar, C, Hashim, AM: *Round-off stability of Picard iterative procedure for multivalued operators*. Nonlinear Anal Forum. 10(1):13–19 (2005)

[24] Batty, M, Longley, P: *Fractal cities: a geometry of form and function*. Academic Press, London (1994)

[25] Buser, P, Tosan, E, Weinand, Y: *Fractal Geometry and its applications in the field of construction*. Available at: http://150.217.71.164/ambientlab2010/docs/105c06_project_plan.pdf

[26] Lee, WL, Hsieh, KS: *A robust algorithm for the fractal dimension of images and its applications to the classification of natural images and ultrasonic liver images*. Signal Process. 90(6):1894–1904 (2010). doi:10.1016/j.sigpro.2009.12.010

[27] Mistakeidis, ES, Panagouli, OK: *Strength evaluation of retrofit shear wall elements with interfaces of fractal geometry*.Eng Struct. 24(5):649–659 (2002). doi:10.1016/S0141-0296(01)00131-6

[28] Shaikh, YH, Khan, AR, Pathan, JM, Patil, A, Behere, SH: *Fractal pattern growth simulation in electrodeposition and studyof the shifting of center of mass*. Chaos, Solitons & Fractals. 42(5):2796–2803 (2009). doi:10.1016/j.chaos.2009.03.192

[29] Zmeskal, O, Vala, M, Weiter, M, Stefkova, P: *Fractal-cantorian geometry of space-time*. Chaos, Solitons & Fractals. 42(3):1878–1892 (2009). doi:10.1016/j.chaos.2009.03.106

[30] Petrusel, A, Rus, IA: *The theory of a metric fixed point theorem for multivalued operators*. Proc Ninth International Conference on Fixed Point Theory and its Applications, Changhua, Taiwan, July 16-22 (2009). pp. 161–175.Yokohama Publ (2011)

[31] Markin, JT: *A fixed point theorem for set-valued mappings*. Bull Am Math Soc. 74, 639–640 (1968). doi:10.1090/S0002-9904-1968-11971-8

[32] Nadler, SB Jr: *Multivalued contraction mappings*. Pacific J Math. 30, 475–488 (1969)

[33] Dontchev, AL., W. W. Hager, *An inverse mapping theorems for set-valued maps*, Proc. Amer. Math. Soc. 121(1994) 481-489.