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CONTENT

CORRESPONDENCE BETWEEN ONE-PARAMETER GROUP OF LINEAR TRANSFORMATIONS AND LINEAR DIFFERENTIAL EQUATIONS THAT DESCRIBE DYNAMICAL SYSTEMS

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Abstract: Mathematical formalization of the notion of determined process leads to the notion of one-parameter group of linear transformations. In this paper we define one-parameter group of diffeomorphisms and see their relationship with vector fields, which connect the one-parameter group of diffeomorphisms with differential equations.

Key Words: deterministic process, phase velocity vector, vector field, diffeomorphism, phase space, phase flow, phase curve

1. Introduction and basic concepts

Starting with the definition of the derivative and differentiation of function, we know that the derivative characterizes some change. Therefore, many different processes in the nature, due to some change, can be described by differential equations.

The theory of ordinary differential equations is one of the basic instrument for the application of mathematics. Mathematical modeling of various scientific and engineering processes is based on systems of differential equations as a boundary problem. The most explored class of systems of differential equations, which are constructed by the experimentally established lawfulness in the process, is the class of autonomous systems. Apart from analytical solution of differential equation systems, the qualitative theory also uses geometric interpretation of the solution, defining phase space, so that any solution is a trajectory (curve) in that phase space, but the most important is kinematics interpretation which describes the solution as a motion of a point along a curve. The main assignment of the theory of differential equations is defining and exploring the motion of systems along phase velocity vector field. Or, in other words, exploring the problem for the type of phase curves, whether they remain in a bounded area or unlimited grow (periodic, stable, unstable solutions) [2, 4, 6, 9].

Basic concepts and definitions given bellow are taken from [10, 5, 3].

Definition 1.1: *A process is said to be deterministic if its entire future course and its entire past are uniquely determined by its state at the present instant of time. The set of all possible states of a process is called its phase space.*

Definition 1.2: *A process is said to be differentiable if its phase space has the structure of a differentiable manifold and if its change of state with time is described by differentiable functions.*

The motion of systems is usually described by the motion of the points along a curve in the phase space. The velocity of a motion of the phase point along that curve is determined by the point itself. That way, every point of the phase space determines a vector, called *phase velocity vector*. All phase velocity vectors form a *phase velocity vector field* in the phase space. This vector field determines the differential equation of the process.

Definition 1.3: *Transformation on a set is one-to-one mapping of the set onto itself.*

Definition 1.4: *A collection of transformations of a set is called a transformation group if it contains the inverse f^{-1} of each of its transformations f and the product fg of any two of its transformations f and g, with $(fg)(x) = f(g(x))$.*

If we consider this transformation group as a set A with defined two operations: $A \times A \to A$ and $A \to A$ ($(f, g) \to fg$ and $f \to f^{-1}$), then we are faced with the algebraic notion of abstract group. Here, the operation composition (product) of mappings is the basic operation, f^{-1} is inverse, and identity mapping is the unit in this group.

Now, let G be a group and M is a set.

Definition 1.5: *We say that it is given an action of the group G on a set M if for each element $g \in G$ there is a corresponding transformation $T_g : M \to M$ such that the corresponding transformation to the product of each two elements of G is a transformation which is product of the transformations corresponding to those two elements, while the corresponding transformation to each inverse element is appropriate inverse transformation, i.e.*

$$T_{fg} = T_f T_g \qquad T_{f^{-1}} = (T_f)^{-1}$$

We should mention that every transformation group on a set is action on that set. Actually, $T_g \equiv g$.

Transformation T_g is also called *action of the element* $g \in G$ on the set M . The action of the group G on the set M determines a mapping $T : G \times M \to M$ defined with: $(g, m) \to T_g m$. The element $T_g m = gm$ is obtained by the action of g on m . If we fix an element $m \in M$ and act on it by all the elements of the group G we will obtain the set $\{gm \mid g \in G\} \subseteq M$. We call this set the *orbit* of the point m .

Definition 1.6: *A one–parameter group of transformations of a set is an action on that set by the group of all real numbers.*

One–parameter group of transformations of a set M is usually denoted with $\{g^t\}$. Here, $g^t : M \to M$ is a transformation corresponding to the point $t \in R$. Actually, a one–parameter group of transformations on a set M is a collection of transformations g^t parametrized by the real parameter t such that for any real numbers s and t the following two relations hold:

1) $g^{t+s} = g^t g^s$
2) $g^{-t} = \left(g^t\right)^{-1}$.

The parameter t is usually called *time* and the transformation g^t is called *transformation in time t .*

The one–parameter group of transformations of a set is mathematical equivalent of the physical concept two-sided deterministic process. Let M

be a phase space of the process. Each point of that space is a definite state of the process. Assume that at the moment $t = 0$ the process was in the state x. Then at another moment t the process will be at another state. Let us denote this new state of the process $g^t x$. This way for every $t \in R$ we define a mapping $g^t : M \to M$ from the phase space of the process into itself. The mapping g^t takes the state of the process at the moment 0 to the state at the moment t. We call this *transformation of the process in time* t*.*

The mapping g^t defined as above is really a transformation. This follows from the fact that, according to the definition of determinacy of the process, each state uniquely determines both the past and the future of the process. $g^{t+s} = g^t g^s$ property is also satisfied: suppose that at the initial moment the process was in the state x. The process could pass in a new state at the moment $t + s$ either directly ($x \to g^{t+s} x$), either first getting the state $g^t x$ in the time t and then see where this state $g^t x$ moves in time s.

Some concrete examples for application of one-parameter group of transformations while solving differential equations were given in [1].

The one-parameter group of transformations on a set M is also called *phase flow* with phase space M. The orbits of a phase flow are called its *phase curves* or *trajectories*. The points lying on a phase curves are called *fix point* of the flow.

Definition 1.7: *A smooth mapping, which inverse mapping is also smooth is called diffeomorphism (all coordinate functions and their inverse functions are smooth).*

Definition 1.8: *One-parameter group of diffeomorphisms is one-parameter group of transformations whose elements are diffeomorphisms satisfying the additional condition that* $g^t x$ *depends smoothly on both of the arguments* t *and* x*.*

See [8] for an application of diffeomorphisms in a dynamical systems.

Definition 1.9: *One-parameter group of linear transformations is a one-parameter group of diffeomorphisms whose elements are linear transformations.*

The phase velocity vector of the flow $\{g^t\}$ at the point $x \in M$ is the velocity with which the point $g^t x$ leaves x, *i.e.*

$$v(x) = \frac{d}{dt}\Big|_{t=0} (g^t x) \qquad (1)$$

The phase velocity vectors of a flow at all points of M form a smooth vector field (because $g^t x$ depends smoothly on t and x). It is the *phase velocity field*.

Definition 1.10: *The points where the phase velocity vector vanishes are called equilibrium points or singular points of the phase velocity field.*

Remark 1.1: *The fixed points of the flow are actually the singular points of the phase velocity field, i.e. the points where the phase velocity vector vanishes and vice versa.*

Definition 1.11: *Let* $A: R^n \to R^n$ *be a linear operator. A linear differential equation is an equation with the phase space* R^n, *defined by the vector field* $v(x) = Ax$, *i.e.*

$$x' = Ax \qquad (2)$$

If we fix a coordinate system x_i, $i = 1,...,n$ in R^n then the equation $x' = Ax$ can be written as a system of n equations

$$x_i' = \sum_{i=1}^{n} a_{ij} x_j\ , \qquad i = 1,...,n \qquad (3)$$

where $\lfloor a_{ij} \rfloor$ is the matrix of the operator A in the considered coordinate system.

So, the differential equation $v(x) = Ax$ is actually a system of n first order linear ordinary differential equations with constant coefficients.

Definition 1.12: *Let* $A: R^n \to R^n$ *be a linear operator. The operator* e^A *is defined on the following two equivalent ways:*

1) $e^A = E + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ... + \frac{A^n}{n!} + ...$

2) $e^A = \lim_{n \to \infty} \left(E + \frac{A}{n} \right)^n$, *where* E *denotes identity operator.*

2. Correspondence between one-parameter group of linear transformations and differential equations

Let us fix a point x_0 and consider its motion under the action of the phase flow g^t. In other words, consider the mapping $\varphi : R \to M$ defined as follows:

$$\varphi(t) = g^t x_0 \tag{4}$$

Theorem 2.1: *The mapping φ is solution of the differential equation $x' = v(x)$ with initial condition $\varphi(0) = x_0$.*

Proof: Let

$$\varphi'(t) = \frac{d}{dt}\varphi(t) \tag{5}$$

be the first derivative of φ at the point t. We can also write it as:

$$\varphi'(t) = \frac{d}{d\tau}\Big|_{\tau=t}\varphi(\tau) \tag{6}$$

According to the definition of φ we obtain:

$$\frac{d}{d\tau}\Big|_{\tau=t}\varphi(\tau) = \frac{d}{d\tau}\Big|_{\tau=t} g^{\tau} x_0 \tag{7}$$

Introducing a new variable $u = \tau - t$ we obtain the expression:

$$\frac{d}{du}\Big|_{u=0}(g^{u+t}x_0) = \frac{d}{du}\Big|_{u=0}(g^u g^t x_0) = \frac{d}{du}\Big|_{u=0} g^u (g^t x_0) \tag{8}$$

According to (1), the last expression is equal to

$$v\left(g^t x_0\right) = v\left(\varphi\left(t\right)\right) \tag{9}$$

So, $\varphi(t)$ is the solution of the equation $x' = v(x)$.

About the initial condition,

$$\varphi(0) = g^0 x_0 = x_0 \tag{10}$$

thus we conclude that the initial condition is satisfied, too. □

Remark 2.1: *The converse is also true, i.e. the solution of differential equation* $x' = v(x)$ *with initial condition* $\varphi(0) = x_0$ *has the form* $\varphi(t) = g^t x_0$. *The proof follows directly from the existence and uniqueness theorem for the solution of first order differential equation which satisfies given initial condition.*

Thus, for each one-parameter diffeomorphism group there is associated differential equation, determined by the phase velocity vector field, which solution is a motion of the phase points under the action of the phase flow. If the phase flow describes any process with arbitrary initial conditions, then the differential equation defined by its phase velocity vector field determines the local low of evolution of the process. Knowing this local low of evolution, the theory of differential equations is supposed to reconstruct the past and predict the future. Establishing any low in the nature in a form of a differential equation reduces any problem about the evolution of the process (physical, chemical, ecological, biological process etc.) to a geometric problem for the behavior of the phase curves of given vector field in the corresponding phase space.

The phase flow of the differential equation $x' = v(x)$ is the one-parameter diffeomorphism group such that v is its phase velocity vector field.

Finding the phase flow of a differential equation, it suffices to find the solution of that equation. $g^t x_0$ is the value of the solution φ at the moment t with initial condition $\varphi(0) = x_0$.

Let $\{g^t \mid t \in R\}$ be a one-parameter group of linear transformations. Consider the motion $\varphi : R \to R^n$ of a point $x_0 \in R^n$. Let $A : R^n \to R^n$ be a linear operator defined by the relation

$$Ax = \frac{d}{dt}\Big|_{t=0} (g^t x), \quad \forall x \in R^n \tag{11}$$

Then, $\varphi(t)$ will be the solution of a differential equation $x' = Ax$ with the initial condition $\varphi(0) = x_0$. So, for describing a one-parameter group of linear transformations it is enough to explore the solutions of the linear equation $x' = Ax$ (the correspondence between one parameter group of linear transformations an differential equations is one-to-one and onto: each operator $A : R^n \to R^n$ defines a one-parameter group $\{g^t\}$).

Remark 2.2*: Let* $A : R^n \to R^n$ *be a linear operator. The family of all linear operators* $e^{tA} : R^n \to R^n$, $t \in R$ *(*A *is fixed) is one-parameter group of linear transformations, i.e.*

$$e^{(t+s)A} = e^{tA}e^{sA} \tag{12}$$

$$\frac{d}{dt}\left(e^{tA}\right) = Ae^{tA} \tag{13}$$

This can easily be proved just using the definition of operator e^A with an exponential series.

Theorem 2.2: *The solution $x = \varphi(t)$ of the equation $x' = Ax$ with initial condition $\varphi(0) = x_0$ has the form*

$$\varphi(t) = e^{tA}x_0 \tag{14}$$

Proof:

$$\varphi'(t) = \frac{d\varphi}{dt} = Ae^{tA}x_0 = A\varphi(t) \tag{15}$$

$$\varphi(0) = e^0 x_0 = x_0 \tag{16}$$

Thus the theorem is proved. □

Theorem 2.3: *Let $\{g^t : R^n \to R^n\}$ be a one-parameter group of linear transformations. There exist a linear operator $A : R^n \to R^n$ such that*

$$g^t = e^{tA} \tag{17}$$

Proof: Set

$$A = \frac{dg^t}{dt}\Big|_{t=0} = \lim_{t\to 0}\frac{g^t - E}{t} \tag{18}$$

According to Definition 1.11 and Theorem 2.1, the motion $\varphi(t) = g^t x_0$ is a solution of the equation $x' = Ax$ with initial condition $\varphi(0) = x_0$. Then, according to the Theorem 2.2 we have

$$\varphi(t) = e^{tA}x_0 \tag{19}$$

and we now obtain

$$g^t x_0 = e^{tA}x_0 \tag{20}$$

i.e.

$$g^t = e^{tA}$$ □

So, using this approach, we stated a correspondence between linear differential equations $x' = Ax$ and their flows $\{g^t\}$.

3. Conclusion

The notion of one-parameter group of transformations is actually a geometrization of the solution of system of differential equations. This could be particularly useful in the qualitative theory of differential equations which explores the behavior of systems in the phase space, instead of finding the explicit solution to them. If we consider a non-linear vector field with small non-linearity, we can linearized it if we expand it in a Taylor series in a neighborhood of the equilibrium point and omit the non-linear terms. That way we omit infinitely small terms of higher order, thus we can consider that the behavior of linearized and non-linear system in a neighborhood of the equilibrium point are closely related [7].

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