УНИВЕРЗИТЕТ „ГОЦЕ ДЕЛЧЕВ" - ШТИП
ФАКУЛТЕТ ЗА ИНФОРМАТИКА

ISSN 1857- 8691

ГОДИШЕН ЗБОРНИК 2012

YEARBOOK 2012

ГОДИНА 1 VOLUME I

GOCE DELCEV UNIVERSITY - STIP
FACULTY OF COMPUTER SCIENCE

УНИВЕРЗИТЕТ „ГОЦЕ ДЕЛЧЕВ“ – ШТИП
ФАКУЛТЕТ ЗА ИНФОРМАТИКА

ГОДИШЕН ЗБОРНИК
2012
YEARBOOK
2012

ГОДИНА 1 **МАРТ, 2013** **VOLUME I**

GOCE DELCEV UNIVERSITY – STIP
FACULTY OF COMPUTER SCIENCE

ГОДИШЕН ЗБОРНИК
ФАКУЛТЕТ ЗА ИНФОРМАТИКА
YEARBOOK
FACULTY OF COMPUTER SCIENCE

Редакција и администрација
Универзитет „Гоце Делчев“-Штип
Факултет за информатика
ул. „Крсте Мисирков“ 10-А
п. фах 201, 2000 Штип
Р. Македонија

Address of the editorial office
Goce Delcev University – Stip
Faculty of Computer Science
Krste Misirkov 10-A
PO box 201, 2000 Štip,
R. of Macedonia

СОДРЖИНА
CONTENT

THE BLACK-SCHOLES MODEL AND VALUATION OF THE EUROPEAN CALL OPTION

Limonka Lazarova[1], Marija Miteva[2] and Natasa Stojkovik[3]

[1] limonka.lazarova@ugd.edu.mk [2] marija.miteva@ugd.edu.mk
[3] natasa.maksimova@ugd.edu.mk

Faculty of computer science, University “Goce Delcev” – Stip

Abstract: In this paper will be considered the simple continuous time model of Black-Scholes. The Black-Scholes formula for valuation of the European Call Option will be shown. It will be given a review of the background of this model and also the basic concepts of stochastic or Ito calculus that are necessary to explore the model.
Keywords and Phrases: Geometric Brownian motion, Ito integral, Ito formula, martingale, Black-Scholes formula.
Mathematics Subject Classification 2010: 91G80.

1. Introduction

The Black-Scholes Model was first discovered in 1973 by Fischer Black and Myron Scholes who developed a formula for valuation of European contingent claims based on geometric Brownian motion model for the stock price process. Robert Merton developed another method to derive the formula with more applicability and generalized the formula in many directions. It was for the development of the Black-Scholes Model that Scholes and Merton received the Nobel Prize of Economics in 1997 (Black had passed away two years earlier). The idea of the Black-Scholes Model was first published in "The Pricing of Options and Corporate Liabilities", of the Journal of Political Economy by Fischer Black and Myron Scholes, [4] and then elaborated in "Theory of Rational Option Pricing" by Robert Merton in 1973. Within six months of the publication of the Black-Scholes Model article, Texas Instruments had incorporated the Black-Scholes Model into their calculator, announcing the new feature with a half-page ad in The Wall Street Journal. The three young Black-Scholes Model researchers, which were still in their twenties, set about trying to find an answer to derivatives pricing using mathematics, exactly the way a physicist or an engineer approaches a problem. They had shown that mathematics could be applied using a little known technique known as stochastic differential equations and that discovery led to the development of the Black-Scholes Model that we know today.

Stochastic calculus or Ito calculus is one of the basic and main tools in finance, especially in the construction of the finance models in the theory of options. By using of the theory of probability and stochastic processes and by introducing of coincidence in the coefficients of differential equations is derived more realistic mathematical models. In [1], Bernt Oksendal elaborates some examples of stochastic models.

In order to describe the Black-Scholes model, the following definitions given in [6], [7] and [10] are necessary.

The stochastic process will be considered in complete filtrate space $(\Omega, \Im, P, (\Im_t, t \in T))$.

Definition 1.1 *The real valued stochastic process* M *is called martingale if:*

i) $(\forall t \geq 0), \ \exists E(M_t)$,

ii) $s < t \ \Rightarrow \ E(M_t \mid \Im_s) = M_s$.

X *is semi martingale if* $X = X_0 + M + A$, *where* X_0 *is* $\Im_0$ *- measurable random variable,* M *is local martingale with value 0 in 0 and* A *is adapted process with continuous trajectories on the right with value 0 in 0 and paths are with finite variation.*

Definition 1.2 *The real valued stochastic process* $B = (B_t \mid t \in [0, +\infty))$ *is called Brownian motion if:*

i) B *is adapted with a filtration* $(\Im_t, t \in [0, +\infty))$;

ii) B *has independent increments i.e.*
$\forall s,t \ (0 \leq s < t)$: $P(B_t - B_s \in A \mid \Im_s) = P(B_t - B_s \in A)$ *for every Borel set* B;

iii) B *has stationary increments i.e.*
$\forall s,t \ (0 \leq s < t)$: $B_t - B_s$ *has normal distribution* $N(\mu(t-s), \sigma^2(t-s))$

iv) $P(B_0 = x) = 1, \quad (x \in R)$.

The most important Brownian motion for modeling of the financial models is geometric Brownian motion.

Definition 1.3 *If* $\{B(t), t \geq 0\}$ *is Brownian motion, then stochastic process* $\{Y(t), t \geq 0\}$, *defined by* $Y(t) = e^{B(t)}$ *is geometric Brownian motion.*

The fact that traditional differentiation and integration could not be applied in stochastic calculus implies finding new methods and procedures of differentiation and integration. These new tools were developed for the first time by Ito (1944), who proved Ito lemma and Ito formula. This formula is very important in stochastic calculus, especially in stochastic models in the finance.

The following definitions are given in [5].

Definition 1.4 *Let* $f(t,\omega):[0,\infty)\times\Omega \to R$, *such hat* $f(t,\omega)$ *e* $B \times \Im$ *- measurable (*B *is Borel* σ *- algebra on* $[0,\infty)$*),* $f(t,\omega)$ *is* $\Im_t$ *- adapted and*

$$E\left[\int_0^t f(s,\omega)^2 ds\right] < \infty .$$

Ito integral to the function f *is defined by:*

$$\int_0^t f(s,\omega)dB_s(\omega) = \lim_{n\to\infty}\int_0^t H_n(s,\omega)dB_s(\omega) , \qquad \text{(limit in } L^2 \text{)}$$

(1)

Where (H_n) *is sequence elementary function, for which:*

$$E\left[\int_0^t \left(f(s,\omega) - H_n(s,\omega)\right)^2 ds\right] \to 0, \quad when \ n \to \infty .$$

The Ito integral is very important in financial mathematics. For example, if stochastic process Z_t is price of the stock, then stochastic integral $\int_0^t H_s dZ_s$ is showing the gain or the loss at the disposal with H_s shares of the stock at time s .

Definition 1.5 *Let* $f : R \to R$ *is* C^2 *- function and* B_t *is Brownian motion. The one dimensional Ito formula is given by:*

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)dB_s \qquad (2)$$

2. The Black Scholes continuous time model

The most famous model, which is used in finance, long time ago is the model in which the stock price can be described with Brownian motion. At the beginning this model, in which the stock price is described with many Brownian motions, was not accepted because of two reasons. One of the reasons is that the Brownian motion can receive a negative value, but the option price cannot be negative. The other reason we can see in the following example: if an investor invests 10000$ on sale of shares of the stock, each of 100$ and if the stock price is increased to 200$, then the investor will have profit 10000$. Also, if that 10000$ are invested on sale of shares, each of the stock 1000$, and then the price is increased on 2000$, so in this case the investor will have the same profit of 10000$. It follows that there are proportional increments, but the Brownian motion has stationary and independent increments. Because of that, the random variables $\frac{\Delta S_t}{S_t}$ can be described with the process of Brownian motion, where S_t is stock price process.

The different stocks have different volatility σ. It is expected that the rate of return μ is greater than risk free rate r, because every investor expects higher profit, with which would be recover the takeover risk.

For modeling stock price it will be used stochastic differential equation $\frac{dS_t}{S_t} = \sigma dB_t + \mu dt$, or equivalent integral form:

$$S_t = S_0 + \int_0^t S_s \sigma dB_s + \int_0^t S_s \mu ds \quad , \qquad (3)$$

where B_t is Brownian motion.

This stochastic differential equation can be solved explicitly, and its solution is given in the following theorem:

Theorem 2.1 *The solution of the stochastic differential equation* $dS_t = \sigma S_t dB_t + \mu S_t dt$ *is given with process of geometric Brownian motion.*

Proof: From [12], is following that there is at most one solution of $dS_t = \sigma S_t dB_t + \mu S_t dt$.

We assume that the stock price in initial moment 0 is $S_0 = 1$. We will apply the one-dimensional Ito formula (2) to the function $f(x) = e^x$, so we get

$$S_t = e^{X_t} = e^{X_0} + \int_0^t e^{X_s} dX_s + \frac{1}{2}\int_0^t e^{X_s} d\langle X \rangle_s = 1 + \int_0^t S_s \sigma dB_s + \int_0^t S_s \left(\mu - \frac{1}{2}\sigma^2 \right) ds + \frac{1}{2}\int_0^t$$

$$= 1 + \int_0^t S_s \sigma dB_s + \int_0^t S_s \mu ds.$$

[

Ve assume that the interest rate r e 0, without loss of generality. Let s vestor buys Δ_0 shares of stock at time t_0. The investing in shares of the stock at time t_1 has changed to Δ_1 shares of the stock, Δ_2 shares of the stock at time t_2 etc. In this case, the investor's wealth at time t is given by:

$$X_{t_0} + \Delta_0 \left(S_{t_1} - S_{t_0} \right) + \Delta_1 \left(S_{t_2} - S_{t_1} \right) + \ldots + \Delta_i \left(S_{t_{i+1}} - S_{t_i} \right) \quad .$$

(4)

That means that at the time t_0 the investor has an initial wealth X_{t_0}. If investor buys Δ_0 shares of stock and each of the share with price S_{t_0}, it will cost $\Delta_0 S_{t_0}$ and if at the moment t_1, buys Δ_0 shares of the stock by price S_{t_1}, then the investor's wealth will be $X_{t_0} + \Delta_0 \left(S_{t_1} - S_{t_0} \right)$. This procedure continues in the other time points $t_1, t_2, \ldots$ so we get the relation (4). The relation (4) can be written in the form:

$$X_{t_0} + \int_0^t \Delta(s) dS_s \quad ,$$

(5)

where $t \geq t_{i+1}$ and $\Delta(s) = \Delta_i$ for $t_i \leq s < t_{i+1}$. The wealth of the investor is given by Ito integral (1) in terms of the stock price. The integrand must be adapted process with respect to the filtration on the probability space $(\Omega, \Im, P)$, because the number of shares of the stock which investor posses at time s cannot be based on future information.

Next, we will consider the case when interest rate r is not equal to 0. Let $P_t = e^{-rt} S_t$ is current stock price, and let $P_0 = S_0$. If we have Δ_i shares of the stock in the time period $[t_i, t_{i+1})$, the wealth of the investor will be $\Delta_i \left(P_{t_{i+1}} - P_{t_i} \right)$. So the process of investor's wealth will be:

$$X_{t_0} + \int_{t_0}^{t} \Delta(s) dP_s .$$

If we apply the Ito product formula given in [8], [11], we derive the stochastic differential equation

$$dP_t = \sigma P_t dB_t + (\mu - r) P_t dt \qquad (6)$$

and its solution $P_t = P_0 \exp\left(\sigma B_t + \left(\mu - r - \frac{\sigma^2}{2}\right)t\right)$ is similar with the solution to the equation (3).

The following theorem shows the completeness of the continuous time model, [9].

Theorem 2.2 *If P_t is geometric Brownian motion and if the price of option V is $\Im_t$ - measurable and square-integrable, then there is a constant c and adapted process K_s, such that $V = c + \int_0^t K_s dP_s$. Moreover there is a probability measure $\overline{P}$ in terms of which P_t is martingale.*

Proof: Let P_t is geometric Brownian motion and let $P_t = P_0 \exp\left(\sigma B_t + \left(\mu - r - \frac{\sigma^2}{2}\right)t\right)$, i.e. in differential form $dP_t = \sigma P_t dB_t + (\mu - r) P_t dt$. We define a new probability measure $\overline{P}$ with:

$$\frac{d\overline{P}}{dP} = M_t = \exp\left(aB_t - \frac{a^2 t}{2}\right).$$

From Girsanov theorem [9], $\widetilde{B}_t = B_t - at$ is Brownian motion with respect to the probability measure $\overline{P}$. It follows that:

$$dP_t = \sigma P_t d\widetilde{B}_t + \sigma a P_t dt + (\mu - r) P_t dt .$$

If we choose that $a = -\frac{(\mu - r)}{\sigma}$, we obtain:

$$dP_t = \sigma P_t d\widetilde{B}_t \quad . \tag{7}$$

$\widetilde{B}_t$ is Brownian motion with respect to the probability measure $\overline{P}$, so it follows that P_t is martingale, because it is presented as Ito integral, which is martingale. The equation (7), can be written as

$$d\widetilde{B}_t = \sigma^{-1} P_t^{-1} dP_t \quad . \tag{8}$$

If the price of the option V is $\Im_t$ - measurable, from the martingale representation theorem in [2] it follows that exists adapted process H_s such that $\int_0^t H_s^2 ds < \infty$ and $V = c + \int_0^t H_s d\widetilde{B}_s$. By using to the equation (8) follows that:

$$V = c + \int_0^t H_s \sigma^{-1} P_t^{-1} dP_s \ . \qquad \square$$

3. Valuation of the European Call Option with Black Scholes formula

We will derive a formula for valuation of the arbitrary option. Let $T \geq 0$ is fixed real number. If the price of arbitrary option V is $\Im_T$ - measurable, then from theorem 2.2, it follows that

$$V = c + \int_0^t K_s dP_s . \tag{9}$$

and with respect to the probability measure $\overline{P}$, the process P_t is martingale.

Theorem 3.1 *The price of the option V is $\overline{E}V$.*

Proof: This is no arbitrage principle (risk-free profit). Suppose that the price of the option V, in initial time is W_0. If investor starts with 0$, then he can sell the option V for W_0 dollars and to use this money to buy and trade with shares of stock. If he uses c dollars of this income and if invests in accordance with the strategy of owning K_s shares of stock at time s, then at the time of maturity of the option V, i.e. at time T will be:

$e^{rT}(W_0 - c) + V$ $.

At the time of maturity T, the buyer of the option V uses the option and the seller of the option used V dollars to fulfill his obligation. In this way the profits of the seller of the option would be $e^{-rT}(W_0 - c) + V$ if $W_0 > c$, without risk. From here it follows that $W_0 \leq c$. If $W_0 < c$, then the opposite happens, i.e. the investor buys option instead of to sell the option and to own $-K_s$ shares of the option at the time s. Since we cannot have a risk-free profit, it is following that $W_0 \geq c$ or $W_0 = c$.

With respect to the probability measure $\overline{P}$, the process P_t is martingale. The mathematical expectation of the expression (9) is:

$$\overline{E}V = \overline{E}\left[c + \int_0^t K_s dP_s\right] = c = W_0 = V .$$

□

For valuation of the arbitrary option, the formula (9) is not appropriate. Suppose that V is standard European call option, where

$V = e^{-rt}(S_t - K)^+ = (e^{-rt}S_t - e^{-rt}K)^+ = (P_t - e^{-rt}K)^+$.

With respect to the probability measure $\overline{P}$, the stock price is given by $dP_t = \sigma P_t d\widetilde{B}_t$, where $\widetilde{B}_t$ is Brownian motion with respect to the probability measure $\overline{P}$. It follows that for the stock price we have $P_t = P_0 \exp\left(\sigma\widetilde{B}_t - \left(\frac{\sigma^2}{2}\right)t\right)$. It follows that:

$$\overline{E}V = \overline{E}\left[\left(P_T - e^{-rT}K\right)^+\right] = \overline{E}\left[\left(P_0 \exp\left(\sigma\widetilde{B}_T - \left(\frac{\sigma^2}{2}\right)T\right) - e^{-rT}K\right)^+\right]. \tag{10}$$

Because $\widetilde{B}_T$ is process of the Brownian motion, i.e. its increments are random variables with normal distribution, it follows that the density function is given by $p_{\widetilde{B}_T} = \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{(2T)}}$.

If we do some calculations we can obtain the Black-Scholes formula:

$$W_0 = x\Phi\left(g(x,T)\right) - Ke^{-rT}\Phi\left(h(x,T)\right),$$

where $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2}}dy, \quad x = P_0 = S_0$.

$$g(x,T) = \frac{\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad h(x,T) = g(x,T) - \sigma\sqrt{T}.$$

The Black-Scholes formula depends of the volatility σ, but it does not depend of the rate of return μ. The stock price is given by $dP_t = \sigma P_t d\widetilde{B}_t$, where $\widetilde{B}_t$ is Brownian motion with respect to the probability measure $\overline{P}$. It follows that for the stock price the equation $P_t = P_0 \exp\left(\sigma\widetilde{B}_t - \left(\frac{\sigma^2}{2}\right)t\right)$ hold. The rate of return μ is not present in this expression, because when the Girsanov theorem is applied, we have probability $\overline{P}$, which is risk neutral.

From the equation (10) we obtain:

$$\overline{E}V = \overline{E}\left[\left(P_T - e^{-rT}K\right)^+\right] =$$

$$= \overline{E}\left[\left(P_0 \exp\left(\sigma\widetilde{B}_T - \left(\frac{\sigma^2}{2}\right)T\right) - e^{-rT}K\right)^+\right] = \overline{E}\left[\left(x\exp\left(\sigma\widetilde{B}_T - \left(\frac{\sigma^2}{2}\right)T\right) - e^{-rT}K\right)^+\right]$$

where $\widetilde{B}_t$ is Brownian motion with respect to the probability measure $\overline{P}$. Instead of $S_0 = P_0$ we will use x. Because $\widetilde{B}_T$ is normal random variable with mathematical expectation 0 and variance T, then we can write as $\sqrt{T}Z$ where Z is random variable with standard normal distribution i.e. with mathematical expectation 0 and variance 1.

Because the expression $\quad x\exp\left(\sigma\widetilde{B}_T - \left(\frac{\sigma^2}{2}\right)T\right) > e^{-rT}K$

hold, if and only if: $\quad \log x + \sigma\sqrt{T}Z - \left(\frac{\sigma^2}{2}\right)T > -rT + \log K$

or: $\quad Z > \left(\frac{\sigma^2}{2}\right)T - rT + \log K - \log x,$

then if we write $z_0 = \left(\frac{\sigma^2}{2}\right)T - rT + \log K - \log x$, and if we consider that for $\Phi \quad \Phi(-z) = 1 - \Phi(z)$, hold, we will have:

$$\overline{E}V = \overline{E}\left[\left(P_T - e^{-rT}K\right)^+\right] =$$

$$= \overline{E}\left[\left(P_0\exp\left(\sigma\sqrt{T}z - \left(\frac{\sigma^2}{2}\right)T\right) - e^{-rT}K\right)^+\right] = \overline{E}\left[\left(x\exp\left(\sigma\sqrt{T}z - \left(\frac{\sigma^2}{2}\right)T\right) - e^{-rT}K\right)^+\right] =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty}\left(x\exp\left(\sigma\sqrt{T}z - \left(\frac{\sigma^2}{2}\right)T\right) - e^{-rT}K\right)^+ \exp\left(-\frac{z^2}{2}\right)dz =$$

$$= x\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty}\exp\left(-\frac{1}{2}\left(z^2-2\sigma\sqrt{T}z+\sigma^2T\right)\right)dz-e^{-rT}K\int_{z_0}^{\infty}\exp\left(-\frac{z^2}{2}\right)dz$$

$$= x\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty}\exp\left(-\frac{1}{2}\left(z-\sigma\sqrt{T}\right)^2\right)dz-e^{-rT}K\left(1-\Phi(z_0)\right)=$$

$$= x\frac{1}{\sqrt{2\pi}}\int_{z_0-\sigma\sqrt{T}}^{\infty}\exp\left(-\frac{1}{2}y^2\right)dy-e^{-rT}K\Phi(-z_0)=$$

$$= x\left(1-\Phi\left(z_0-\sigma\sqrt{T}\right)\right)-e^{-rT}K\Phi(-z_0)=x\Phi\left(\sigma\sqrt{T}-z_0\right)-Ke^{-rT}\Phi(-z_0).$$

The last formula is formula for valuation of the European call option with strike price K and time to maturity T, if $\sigma\sqrt{T}-z_0=g(x,T)$ and $-z_0=h(x,T)$.

4. Conclusion

In this paper we have reviewed the Black-Scholes model and we have applied Black-Scholes formula for valuation of European Call Option. It is shown that the Brownian motion can be used for modeling stock price from some financial data. We can conclude that despite their popularity and wide spread use, the model is built on some non-real life assumptions about the market, [3]. It assumes stocks move in a manner referred to as a random walk, risk free rate, no transaction costs and it assumes European-style options which can only be exercised on the expiration date.

References

[1] B. Oksendal, Stochastic differential equation. An introduction with applications, 5th. ed. Springer-Verlag Heidelberg New York,(2000) p. 1-4.

[2] D. Nularet , Stochastic Processes, p.68-113.

[3] D. Teneng, Limitations of the Black-Scholes Model, International Research Journal of Finance and Economics

ISSN 1450-2887 Issue 68 (2011), © EuroJournals Publishing, Inc. 2011

[4] F.Black, M. Scholes, The Pricing of Options and Corporate Liabilities, The Journal of Political Economy, Volume 81, Issue 3, (May-June, 1973), p.637-654.

[5] E.Allen, Modelling with Ito Stochastic Differential Equations, Springer (2007),

[6] I. Karatzas and S. Shreve. Brownian Motion and Stochastic Calculus. Springer. 1998, p.1-22

[7] J. Janssen, R. Manca, E. Volpe di Prignano, Mathematical Finance, Deterministic and Stochastic Models, p.517-540, 641-672

[8] R. F. Bass, The Basics of Financial Mathematics, Department of Mathematics, University of Connecticut, (2003), p.1-43

[9] R.J. Williams. Introduction to the Mathematics of Finance. American Mathematical Society. 2006

[10] S. M. Ross, Stochastic Processes, University of California, Berkeley, John Wiley \& Sons, Inc (1996), p.295-405

[11] S.Shreve, Stochastic Calculus and Finance, Carnegie Mellon University, (1997), p.153-175.

[12] T. G. Kurtz, Lectures on Stochastic Analysis, Departments of Mathematics and Statistics University of Wisconsin - Madison (2009), p.42-53