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СОДРЖИНА
CONTENT

WAVELET APPLICATION IN SOLVING ORDINARY DIFFERENTIAL EQUATIONS USING GALERKIN METHOD

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Abstract. The Galerkin method is one of the most used methods for finding numerical solutions of ordinary and partial differential equations. Its simplicity makes it suitable for many applications. In this paper we show that the wavelet-Galerkin method is an improvement over the standard Galerkin method for ordinary differential equations.

Keywords. condition number, sparse matrix, wavelet, scaling function, wavelet-Galerkin method.

1. Introduction

The concepts of wavelet theory were provided by Meyer, Mallat, Daubechies, and many others, [4], [8], [10]. Since the beginning, the number of applications where wavelets have been used has exploded. In areas such as time-series analysis, approximation theory and numerical solutions of differential equations, wavelets are recognized as powerful weapons not just tools, [1], [2], [3], [7], [11], [12], [13].

In general, it is not always possible to obtain exact solution of an arbitrary differential equation. This necessitates either to go for discretization of differential equations leading to numerical (approximate) solutions, or for qualitative study which is concerned with deduction of important properties of the solutions without actually solving them. In the early nineties, scientists were very optimistic because it seemed that many fine properties of wavelets can be directly applied and would automatically lead to efficient numerical method for solving differential equations. The reason for this optimism was the fact that many nonlinear partial differential equations (PDEs) have solution containing local phenomena and interactions between several scales. Such solutions can be well represented in wavelet basis because of its satisfactory properties such as compact support (locality in time domain) and vanishing moments (locality in frequency domain).

The Galerkin method is one of the best known methods for finding numerical solutions of ordinary and partial differential equations. Its simplicity makes it perfect for many applications. The wavelet-Galerkin method is an improvement over the standard Galerkin method by using a compactly supported orthogonal functional basis, [2], [11], [12], [13]. The translates of a wavelet for all dilations form an unconditional orthonormal basis of $L^2(\mathbb{R})$ and the translates of a scaling function for all dilations form an unconditional orthonormal basis for $V_j \subset L^2(\mathbb{R})$, which is a great improvement over the standard polynomial basis or a trigonometric basis which not necessarily have to be unconditional.

The aim of this article is to throw some light on this aspect of wavelet analysis for numerical and qualitative analysis of ordinary differential equations. Section 2 is of preliminary character; we describe the spaces of functions that we use throughout this paper, we also recall some basic wavelet tools such as multiresolution analysis (MRA) and define the condition number of a matrix. In Section 3 we describe the classical Galerkin method for numerical solving of Sturm-Liouville differential equation which comes down to solve a linear system of equations, or equivalently, a matrix equation $AX = Y$. For numerical purposes, there are two properties that we would like the matrix A to have. Firstly, we would like A to have a small condition number, to obtain stability of the solution under small perturbations in the data. Secondly, for performing with A quickly, we would like A to be sparse, which means that A should have a high proportion of entries that are 0. In this paper we show that the two desirable properties of matrix A can be achieved if we use the wavelets as basis vectors.

2. Preliminaries and Notations

2.1. Spaces of functions. $L^2(\mathbb{R})$ is a Hilbert space of square integrable functions on the real line with the inner product $\langle f, g \rangle = \int_{\mathbb{R}} f(t)\overline{g}(t)dt$, where $\overline{g}(t)$ is a complex conjugate of $g(t)$. The Fourier transform of a function $f \in L^2(\mathbb{R})$ is given with

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

The Hilbert space of square integrable functions on [0,1], with the inner product

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g}(t)dt,$$

is denoted by $L^2([0,1])$. $\mathbb{C}^2([0,1])$ is the space of functions on $[0,1]$ with continuous derivatives up to order 2.

2.2. The condition number of a matrix. The methods for numerically solving linear ordinary differential equation often come down to solving a linear system of equations, or equivalently, the matrix equation $AX = Y$. Theoretically, such a system is well understood: for a square matrix A, there exists a unique solution X for every Y if and only if A is an invertible matrix. However, in applications there are further issues that are of crucial importance. It is often observed that for two close values of Y, for example Y' and Y'', the appropriate obtained solutions X' and X'' are far apart. Such a linear system is called badly conditioned. In this situation, small errors in data Y can lead to large error in the solution X. A measure of the stability of the linear system $AX = Y$ under perturbation of the data Y is a condition number of a matrix A.

Let A be a $n \times n$ matrix. The *operator norm*, or just *the norm* of A is defined by

$$\|A\| = \sup \frac{\|Az\|}{\|z\|}, \tag{2.1}$$

where the supremum is taken over all nonzero complex vectors z in $\mathbb{C}^n$.

Let A be an invertible $n \times n$ matrix. A *condition number* $C_{\#}(A)$ of A, is defined by

$$C_{\#}(A) = \|A\| \|A^{-1}\|,$$

where A^{-1} is the inverse matrix for *A*. It is clear that $C_{\#}(A) \geq 1$. It is known that if A is normal invertible matrix then

$$C_{\#}(A) = \frac{|\lambda|_{\max}}{|\lambda|_{\min}} \tag{2.2}$$

where $|\lambda|_{\max} = \max\{|\lambda| : \lambda \text{ is an eigenvalue of A}\}$ and $|\lambda|_{\min} = \min\{|\lambda| : \lambda \text{ is an eigenvalue of A}\}$. If A is unitary matrix, then $C_{\#}(A) = 1$. For irregular matrix A, $C_{\#}(A) = \infty$.

In applications, a small condition number (i.e. near 1) is desirable. In case when $C_{\#}(A)$ is high, the system $AX = Y$ can be replaced with the equivalent system $BAX = BY$, where B is a preconditioning matrix such that $C_{\#}(BA) < C_{\#}(A)$. In theory this is always possible, i.e. for an invertible matrix A, $B = A^{-1}$.

2.3. Wavelets and Multiresolution analysis (MRA). Let $\psi_{a,b}$, $a > 0, b \in \mathbb{R}$ be a family of functions defined as translations (or shifting) by factor *b* and dilatation (or scaling) by factor *a* of the function $\psi \in L^2(\mathbb{R})$

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right).$$

The function $\psi \in L^2(\mathbb{R})$ (called a *wavelet* or *mother wavelet*) is assumed to satisfy the admissibility condition

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty,$$

which implies that

$$\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(t)dt = 0 \,. \tag{2.3}$$

One can prove that, if $\int_{-\infty}^{\infty} \psi(t)dt = 0$ and $\int_{-\infty}^{\infty} (1+|t|^{\alpha})|\psi(t)|dt < \infty$ for some $\alpha > 0$, then $C_{\psi} < \infty$, [1].

In most situations, it is useful to restrict ψ to be well localized both in time and frequency domains. For time localization, $\psi(t)$ and its derivatives must decay very rapidly, while for frequency localization, $\hat{\psi}(\omega)$ must decay sufficiently fast as $|\omega| \to \infty$ and $\hat{\psi}(\omega)$ must become flat in the neighborhood of 0. The flatness is associated with the number of vanishing moments of $\psi(t)$ since

$$\int_{-\infty}^{\infty} t^k \psi(t)dt = 0 \Leftrightarrow \hat{\psi}^{(k)}(0) = 0 \tag{2.4}$$

for *k* = 0, 1, …, *n.* It means that larger number of vanishing moments more is the flatness ω is small.

The notion of multiresolution analysis (MRA) was introduced in 1988/89 by Mallat and Meyer as a natural approach to the wavelet orthonormal basis. One can easily obtain a wavelet basis associated to the particular multiresolution approximation as follows.

A *multiresolution analysis* (MRA) of space $L^2(\mathbb{R})$ consists of a sequence of closed subspaces $\{V_j\}_{j=-\infty}^{\infty}$ (called *approximation spaces*) with the following properties:

1. $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$;
2. $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$; $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
3. $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$;
4. $f(t) \in V_j \Leftrightarrow f(t-k) \in V_j$, $\forall k \in \mathbb{Z}$;
5. There exists a function ϕ (called *scaling function* or *father wavelet*) such that $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k), k \in \mathbb{Z}$ constitute orthonormal basis for corresponding subspace V_j.

Let $\phi \in L^2(\mathbb{R})$ be compactly supported scaling function of MRA. Then

$$\int_{-\infty}^{\infty} \phi(t)dt \neq 0 \,, \tag{2.5}$$

and ϕ satisfies the following dilatation equation

$$\phi(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} a_k \phi(2t-k) \tag{2.6}$$

where a_k are real coefficients and $a_k \neq 0$ for only finitely many $k \in \mathbb{Z}$ (the number of nonzero coefficients a_k in the series (2.6) is denoted by *L*). Since $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$, $j,k \in \mathbb{Z}$ are orthonormal in $L^2(\mathbb{R})$, we have

$$\int_{-\infty}^{\infty} \phi(t-n)\phi(t-k)dt = \delta_{k,n} \tag{2.7}$$

where $\delta_{k,n}$ is the Kronecker delta function such that $\delta_{k,n} = 0$ for $n \neq k$ and $\delta_{k,n} = 1$ for $n = k$.

If $\phi \in L^2(\mathbb{R})$ be compactly supported scaling function of MRA, one can construct the wavelet ψ such that $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k), j,k \in \mathbb{Z}$ constitute an orthonormal basis for $L^2(\mathbb{R})$. It can be shown [4], that if $\hat{\phi}$ and $\hat{\psi}$ are the Fourier transforms of the scaling function and its corresponding wavelet, then, the following relation holds

$$\hat{\psi}(\omega) = \left(\left(\hat{\phi}(\omega/2)\right)^2 - \left(\hat{\phi}(\omega)\right)^2 \right)^{1/2} e^{i\omega/2}, \tag{2.8}$$

or equivalently,

$$\psi(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} (-1)^k \overline{a_{-1-k}}\phi(2t-k). \tag{2.9}$$

The simplest example of MRA is the Haar multiresolution analysis. In this case

$$\phi(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}. \tag{2.10}$$

Consequently to (2.8), we obtain that $\psi(t) = \phi(2t) - \phi(2t-1)$, that is

$$\psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise} \end{cases}. \tag{2.11}$$

The Haar wavelet $\psi(t)$ is developed by Alfred Haar in 1910, long before anyone began speaking of wavelets.

3. Wavelet-Galerkin method for Sturm-Liouville equation

3.1. Sturm-Liouville equation. We consider the class of ordinary differential equations (known as Sturm-Liouville equations) of the form

$$Lu(t) = -\frac{d}{dt}\left(a(t)\frac{du}{dt} \right) + b(t)u(t) = f(t),\ 0 \le t \le 1, \tag{3.1}$$

with Dirichlet boundary conditions

$$u(0) = u(1) = 0. \tag{3.2}$$

Let $a(t)$, $b(t)$ and $f(t)$ be a real-valued functions, such that $f(t)$ and $b(t)$ are continuous functions and $a(t)$ has a continuous derivative on $[0,1]$. Note that *L* may be differential operator with variable coefficient because $a(t)$ and $b(t)$ are not necessarily constants. We assume that the operator *L* is *uniformly elliptic*, which means that there exist constants $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that

$$0 < C_1 \le a(t) \le C_2 \text{ and } 0 \le b(t) \le C_3 \text{ for all } t \in [0,1]. \tag{3.3}$$

By the theory of ordinary differential equations, it is known that there is a unique function *u* satisfying equation (3.1) and the boundary conditions (3.2).

3.2. Galerkin method for ordinary differential equations. For the Galerkin method [9], [12], we suppose that $\{v_j\}$ is a complete orthonormal system (orthonormal basis) for $L^2([0,1])$, and that every v_j is $\mathrm{C}^2([0,1])$ function that satisfies

$$v_j(0) = v_j(1) = 0 .$$

We select some finite set Λ of indices j and consider the subspace

$$S = span\{v_j, j \in \Lambda\},$$

i.e. the set of all finite linear combination of the elements $\{v_j\}$, $j \in \Lambda$.

We look for an approximation u_s of the exact solution u of the equation (3.1) in the form

$$u_s = \sum_{k\in\Lambda} x_k v_k \in S ,$$

(3.4)

where the coefficients $x_k, k \in \Lambda$ are unknown. Our criterion for determining the coefficients x_k is that u_s should behave like the true solution u on the subspace S, i.e.

$$\langle Lu_s, v_j \rangle = \langle f, v_j \rangle , \ \forall j \in \Lambda .$$

(3.5)

If we substitute equation (3.4) in equation (3.5) we obtain

$$\sum_{k\in\Lambda} \langle Lv_k, v_j \rangle x_k = \langle f, v_j \rangle , \ \forall j \in \Lambda .$$

(3.6)

Let X denote the vector $(x_k)_{k\in\Lambda}$ and let Y be the vector $(y_k)_{k\in\Lambda}$ where $y_k = \langle f, v_k \rangle$. Let $A = [a_{j,k}]_{j,k\in\Lambda}$ where $a_{j,k} = \langle Lv_k, v_j \rangle$. Thus, (3.6) is a linear system of equations

$$\sum_{k\in\Lambda} a_{j,k} x_k = y_j \quad j \in \Lambda$$

(3.7)

or,

$$AX = Y .$$

(3.8)

For each subset Λ we obtain an approximation $u_s \in S$ to the true solution u, by solving the linear system (3.8) for X and then we determine u_s by equation (3.4).

We expect that as we increase our set Λ in some systematic way, our approximations u_s should converge to the true solution u. Now, our main concern is the nature of the linear system, resulting from the choice of wavelet basis as opposed to some other basis, for example, Fourier basis. For numerical purposes, there are two properties that we would like the matrix A in the linear system (3.8) to have. First, we would like A to have a small condition number, to obtain stability of the solution under small perturbations in the data. Second, for performing with A quickly, we would like A to be sparse, which means that A should have a high proportion of entries that are 0. In the rest of the paper we will show that the two desire properties of matrix A can be achieved if we use the wavelets as basis vectors.

3.3. Wavelet-Galerkin method for ordinary differential equations. As we emphasized, the family of wavelets $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k), j,k \in \mathbb{Z}$ constitute an orthonormal basis for $L^2(\mathbb{R})$. We assume the possibility of modifying the wavelet system for $L^2(\mathbb{R})$, so as to obtain a complete orthonormal system $\{\psi_{j,k}\}_{(j,k)\in\Gamma}$ for $L^2([0,1])$. The set Γ is a certain subset of $\mathbb{Z}\times\mathbb{Z}$ that we do not specify. The functions $\psi_{j,k}$ are not exactly the same functions as in a wavelet basis for $L^2(\mathbb{R})$, but they

are similar. In particular, $\psi_{j,k}$ has a scale of about 2^{-j}, and is concentrated near the point $2^{-j}k$, and $\psi_{j,k}$ is 0 outside an interval centered at $2^{-j}k$ of length proportional to 2^{-j}. Wavelets concentrated well into the interior of $[0,1]$ are nearly the same as usual wavelets, but those concentrated near the boundary points are substantially modified. After the modifications, $\forall(j,k)\in\Gamma$, $\psi_{j,k}$ should be $\mathbb{C}^2$ function and satisfy the boundary conditions

$$\psi_{j,k}(0)=\psi_{j,k}(1)=0.$$

Now, we rewrite the equations (3.4) and (3.6) using the fact that the wavelets are indexed by two integers, in the form

$$u_s=\sum_{(j,k)\in\Gamma}x_{j,k}\psi_{j,k},$$

and

$$\sum_{(j,k)\in\Gamma}\left\langle L\psi_{j,k},\psi_{l,m}\right\rangle x_{j,k}=\left\langle f,\psi_{l,m}\right\rangle,\ \forall(l,m)\in\Gamma. \tag{3.9}$$

We can still regard this, as a matrix equation $AX=Y$, where the vectors $X=(x_{j,k})_{(j,k)\in\Gamma}$ and $Y=(y_{j,k})_{(j,k)\in\Gamma}$, $y_{j,k}=\left\langle f,\psi_{j,k}\right\rangle$ are indexed by pairs $(j,k)\in\Gamma$, and $A=[a_{l,m;j,k}]_{(l,m);(j,k)\in\Gamma}$, $a_{l,m;j,k}=\left\langle L\psi_{j,k},\psi_{l,m}\right\rangle$. The pairs (l,m) and (j,k) represent row and column of A respectively.

Next, we will prove that if the matrix A does not have a low condition number or is not sparse, then the system $AX=Y$ can be replaced with the equivalent system $MZ=V$, for which the new matrix M has the desire properties, i.e. M is sparse matrix and has smaller condition number than *A*. Indeed, we define matrix $M=[m_{l,m;j,k}]_{(l,m),(j,k)\in\Gamma}$ by

$$M=D^{-1}AD^{-1}, \tag{3.10}$$

where $D=[d_{l,m;j,k}]_{(l,m),(j,k)\in\Gamma}$,

$$d_{l,m;j,k}=\begin{cases}2^j, & (l,m)=(j,k)\\ 0, & (l,m)\neq(j,k)\end{cases}$$

is diagonal matrix.

Since $\det(D)=2^{jn}$, its inverse matrix D^{-1} is

$$D^{-1}=\frac{1}{\det(D)}adjD=\frac{1}{2^{jn}}\begin{bmatrix}2^{j(n-1)} & 0 & 0 & \dots & 0\\ 0 & 2^{j(n-1)} & 0 & \dots & 0\\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 2^{j(n-1)}\end{bmatrix}_{n\times n}=$$

$$=\frac{1}{2^{jn}}2^{j(n-1)}\begin{bmatrix}1 & 0 & 0 & \dots & 0\\ 0 & 1 & 0 & \dots & 0\\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1\end{bmatrix}_{n\times n}=\begin{bmatrix}2^{-j} & 0 & 0 & \dots & 0\\ 0 & 2^{-j} & 0 & \dots & 0\\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 2^{-j}\end{bmatrix}_{n\times n}.$$

Then, for the elements of matrix *M* we have

$$m_{l,m;j,k} = 2^{-l-j} a_{l,m;j,k} = 2^{-l-j} \left\langle L\psi_{j,k}, \psi_{l,m} \right\rangle \tag{3.11}$$

Since $AX = Y \Leftrightarrow D^{-1}AD^{-1}DX = D^{-1}Y$, setting $Z = DX$ and $V = D^{-1}Y$, we obtain the equivalent system $MZ = V$.

The matrix *M* is sparse, because of the good localization (compact support) of the wavelets. Namely, $\psi_{j,k}$ is 0 outside an interval of length $c2^{-j}$ around the point $2^{-j}k$, for some constant c (depending on the choice of wavelet system). Because the operator L involves only differentiation and multiplication by another function, it does not change this localization property. So, $L\psi_{j,k}$ is 0 outside this interval as well. Similarly, $\psi_{l,m}$ is 0 outside an interval of length $c2^{-l}$ around the point $2^{-l}m$. As j and l get large, fewer and fewer of these intervals intersect, so more and more of the matrix elements

$$m_{l,m;j,k} = 2^{-l-j} a_{l,m;j,k} = 2^{-l-j} \left\langle L\psi_{j,k}, \psi_{l,m} \right\rangle = 2^{-l-j} \int_0^1 L\psi_{j,k}(t)\, \overline{\psi_{l,m}(t)}\, dt$$

are 0. So *M* is sparse, which makes computation with it easier.

It is proved in [11] that if $\{\psi_{j,k}\}_{(j,k)\in\Gamma}$ is wavelet system, then there exist constants $C_4, C_5 > 0$ such that for all functions g of the form $g = \sum_{(j,k)\in\Gamma} c_{j,k}\psi_{j,k}$ (the sum is finite), it holds

$$C_4 \sum_{(j,k)\in\Gamma} 2^{2j} \left|c_{j,k}\right|^2 \le \int_0^1 \left|g'(t)\right|^2 dt \le C_5 \sum_{(j,k)\in\Gamma} 2^{2j} \left|c_{j,k}\right|^2 . \tag{3.12}$$

The next proposition shows that the condition number of matrix *M* is bounded, independently of the set of indices, so the new equivalent system *MZ=V* is well conditioned.

Proposition 3.1. [11, Theorem 1.2] Let *L* be a uniformly elliptic Sturm-Liouville operator. Let $\{\psi_{j,k}\}_{(j,k)\in\Gamma}$ be a complete orthonormal system for $L^2([0,1])$ such that $\psi_{j,k}$ is in $C^2([0,1])$, satisfies $\psi_{j,k}(0) = \psi_{j,k}(1) = 0$ and (3.12) holds. Let Λ be a finite subset of Γ.

Then the condition number of *M* defined by (3.10) satisfies the following inequality

$$C_{\#}(M) \le \frac{(C_2 + C_3)C_5}{C_1 C_4}$$

for any finite set Λ, where the constants C_1, C_2, C_3 are determined by (3.3), and C_4, C_5 by (3.12).

Remark. In most practical situations, it is usefully to restrict ψ to have a larger number of vanishing moments (see eq. (2.4)). So, if *f* (*t*) is a polynomial, then $y_{j,k} = \left\langle f, \psi_{j,k} \right\rangle = 0$, i.e. *B* is a null-matrix. So, according to (2.5), it is much more suitable to work with the scaling function ϕ and not with the actual wavelet ψ.

In the next example, we will use the Haar scaling function and the corresponding wavelet. It will be shown that the numerical results obtained by using the Haar scaling function are better than the ones obtained by the Haar wavelet.

Example. We consider the differential equation

$$u''(t) - u(t) = t - 1,\ 0 \le t \le 1$$

with Dirihlet boundary conditions

$$u(0) = u(1) = 0\,.$$

Its exact solution is

$$u(t) = -\frac{1}{1-e^2}e^t + \frac{e^2}{1-e^2}e^{-t} - t + 1.$$

Now, we will obtain the approximate solution u_s^1 using a Haar wavelet (2.11). Let $\Gamma = \{(3,0);(3,1);(3,2);(3,3);(3,4);(3,5);(3,6);(3,7)\}$. The differential operator *L* is $Lu(t) = u''(t) - u(t)$, so we obtain

$$L\psi_{j,k} = -\psi_{j,k},\ \forall (j,k) \in \Gamma,\ \text{and}\ \ a_{l,m;j,k} = \left\langle L\psi_{j,k}, \psi_{l,m} \right\rangle = \begin{cases} -1, & (j,k) = (l,m) \\ 0, & (j,k) \neq (l,m) \end{cases}.$$

Since $y_{j,k} = \left\langle f, \psi_{j,k} \right\rangle$, we have

$$y_{3,k} = \left\langle f, \psi_{3,k} \right\rangle = -\frac{1}{64\sqrt{2}},\ \forall k = \overline{0,7}\,.$$

Solving the linear system $AX = Y$ where $X = (x_{j,k})_{(j,k)\in\Gamma}$, $Y = (y_{j,k})_{(j,k)\in\Gamma}$, and $A = [a_{l,m;j,k}]_{(l,m);(j,k)\in\Gamma}$, $a_{l,m;j,k} = \left\langle L\psi_{j,k}, \psi_{l,m} \right\rangle$ we get

$$x_{3,k} = \frac{1}{64\sqrt{2}},\ \forall k = \overline{0,7}$$

So the approximate solution is

$$u_s^1 = \sum_{(j,k)\in\Gamma} x_{j,k}\psi_{j,k} = 2^{3/2}\left(x_{3,0}\psi(2^3t) + x_{3,1}\psi(2^3t-1) + x_{3,2}\psi(2^3t-2) + x_{3,3}\psi(2^3t-3) + \right.$$

$$\left. + x_{3,4}\psi(2^3t-4) + x_{3,5}\psi(2^3t-5) + x_{3,6}\psi(2^3t-6) + x_{3,7}\psi(2^3t-7)\right).$$

In a similar way we obtain the approximate solution u_s^2 using the Haar scaling function (2.10).

Table 1. Comparison of the results using Haar wavelet and Haar scaling function

t	Exact solution u	Numerical solution u_s^1	Absolute error of u_s^1	Numerical solution u_s^2	Absolute error of u_s^2
0.0	0	0	0	0	0
0.1	0.0265183	-0.03125	0.057768	0.09375	0.672317
0.2	0.0442945	-0.03125	0.075545	0.08125	0.036956
0.3	0.0545074	0.03125	0.023257	0.06875	0.014243
0.4	0.0582599	0.03125	0.027009	0.05625	0.00201
0.5	0.0565906	-0.03125	0.087841	0.05625	0.00034
0.6	0.0504834	-0.03125	0.072128	0.04375	0.00673
0.7	0,0408782	-0.03125	0.002570	0.03125	0.009628
0.8	0,0286795	0.03125	0.016484	0.01875	0.009929
0.9	0.0147663	0.03125	0.03125	0.00625	0.008516
1.0	0	0	0	0	0

Remark. Let us note that here we have used the simplest scaling function which is not even a smooth function. The results can be improved using scaling functions with better properties. For comparison, in [9],

the same equation is solved using the cubic spline scaling function and the obtained results are much better.

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