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ПРОИЗВОДИ НА ДИСТРИБУЦИИ ВО КОЛОМБООВА АЛГЕБРА

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Апстракт

Во овој труд се добиени некои производи на дистрибуции. Резултатите се добиени во Коломбоова алгебра од обопштени функции, која се покажала како релевантна алгебарска структура за решавање на нелинеарни проблеми поврзани со дистрибуциите на Schwartz.

Клучни зборови: *дистрибуција, Коломбоова алгебра, обопштени функции на Colombeau, производ на дистрибуции.*

Класификација на научни полиња: 46F10, 46F30.

PRODUCTS OF DISTRIBUTIONS IN A COLOMBEAU ALGEBRA

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Abstract

In this paper some products of distributions are derived. The results are obtained in Colombeau algebra of generalized functions, which is most relevant algebraic construction for dealing nonlinear problems of Schwartz distributions.

Keywords and Phrases: *distribution, Colombeau algebra, Colombeau generalized functions, multiplication of distribution.*

Mathematics Subject Classification 2010: 46F10, 46F30.

1. Introduction

At the early fifty's of the last century, the French mathematician Laurent Schwartz invented his theory of generalized functions, also known today as a Theory of distributions, which has a big impact in a various fields of sciense and engineering, because of the techniques it provided when dealing with different non-linear problems that classical theory of distributions was unable to solve. One of the most useful aspects of Schwartz's theory of distributions in applications is that discontinuous functions can be handled as easily as continuous or differentiable functions which provides a powerful tool in formulating and solving many problems of various fields of maths, physics, ect. [12].

According to the distribution theory, we can distinguish two complementary points of view:

The first one is that distribution can be considered as a continuous linear functional f acting on a smooth function φ with compact support, i.e. we have a linear map

$$\varphi \to \langle f, \varphi \rangle$$

where φ is called *test function*.

The second one is sequential approach: taking a sequence of a smooth functions (φ_n) converging to the Dirac δ -function, we obtain a family of regularization (f_n) by the convolution product

$$f_n(x) = (f * \varphi_n)(x) = \langle f(y), \varphi_n(x-y) \rangle$$

which converges weakly to the distribution f. We identify all the sequences that converge weakly to the same limit, we consider them as an equivalence class, so each distribution f correspondes to such equivalence calss. We call the elements of each equivalence class *representatives* of the aproppriete distribution f. This way we obtain sequential reprezentation of distributions. Some authors use the equivalence calsses of nets of regularization, i.e. the δ -net $(\varphi_\varepsilon)_{\varepsilon>0}$ defined with

$$\varphi_\varepsilon = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$$

But, two main problems when operating with distributions appear: not any two distributions can always be multiplied (the space of all distributions is not an algebra), and products of distributions not always satisfy the Leibniz rule for diferentiation. So, many attempts have been made to define products of distributions, or rather to enlarge the range of existing products [27]. Many attempts have been made to include the distributions into differential algebras (as an example one can see [25]).

The regularization method mentioned above is a result of a proposal for solving this problems: some modifications of functions are made in order to simlify them and make them more 'regular' (continuous, differentiable, finite, etc.). Actually, all the operations then are done with the regularized functions (the sequences of smooth functions; that's the meaning of the term '*operating in the sence of distributions*') and with the inverse process starting from the result, the function is returned from the regularization.

To overcome the problem with multiplication of distributions, the authors were searching for an associative and commutative algebra ($A(\Omega)$, +, $\circ$), where Ω is an open set in R^n, satisfying following properties:

1) The space $\mathcal{D}'(\Omega)$ of distributions over Ω is linearly embedded into $A(\Omega)$ and $f(x)\equiv 1$ is the unit in $A(\Omega)$;

2) There exist derivation operators $\partial i : A(\Omega) \to A(\Omega)$, i =1, 2, ..., n, that are linear and satisfy the Leibniz rule;

3) $\partial i|_{D'(\Omega)}$, i =1, 2, ..., n, is the usual partial derivative;

4) $\circ|_{C(\Omega)\times C(\Omega)}$ is the usual product of functions.

Schwartz in his famous impossibility result [28] has shown that there isn't any algebra satisfying all these conditions. The optimal solution of this problem was offered by French mathematician Jean -Francois Colombeau who menaged to construct such an algebra in a way that the condition $C(\Omega)$ in the 4)th property above is replaced with $C^\infty(\Omega)$, i.e

4') $\circ|_{C^\infty(\Omega)\times C^\infty(\Omega)}$ is the usual product of functions

Colombeauin his books [3,1,2] described the problem of multiplication of distributions and introduced a differential algebra $\mathcal{G}$, called Colombeau algebra, as a way for overcoming that problem. He introduced several variants of Colombeau algebras, but all of them with the property that C^∞ functions are faithful differential subalgebra of $\mathcal{G}$, something that Colombeau noticed as essential in overcoming Schwartz impossibility result. This new theory of generalized functions of Colombeau is more general then the theory of distributions. From the view point of diferentiation, these new generalized functions have the same properties as distributions. But, from the view point of product of distributions and non-linear operations, these new generalized functions have properties completly different from the properties of distributions: every finite product of generalized functions is generalized function, and moreover, the algebra of these generalized functions is closed under many non - linear operations, too. Also, every product of distributions is generalized function (in this new sense) and may not be a distribution.

In a few words, the differential Colombeau algebra $\mathcal{G}$ as a powerful tool for treating linear and nonlinear problems including singularities has almost the optimal properties while the problem of multiplication of Schwartz distributions is concerned: it is an associative differential algebra of generalized functions, contains the algebra of smooth functions as a subalgebra, the distribution space $\mathcal{D}'$ is linearly embedded in it as a subspace and the multiplication is compatible with the operations of differentiation and products with C^∞-differentiable functions. The notion 'association' in $\mathcal{G}$ is a faithful generalization of the equality of distributions and enables obtaining results in terms of distributions again.

Colombeau's theory was primarly been used for dealing with nonlinear partial differential equations with singularities and was developed during the years and it has now a big appliance in a different fields (physics, geometry, etc. see [20, 13, 14, 21, 30, 18, 17, 26, 27, 16]).

About embedding of the space of distributions into the space of Colombeau generalized functions one can read papers [15, 25, 11, 29]. We can see some products of distributions in a Colombeau algebra in [10, 5, 6, 8, 9, 4, 7] and other papers written by this author.

Following this approach, we evaluate in this paper some product of distributions, as embedded in Colombeau algebra, in terms of associated distributions. Similar results are obtained in [23, 24, 19].

2 Colombeau algebra

In this section we will introduce basic notations and definitions from Colombeau theory.

N0 is the set of non-negative integers, i.e. N0 = N *{0}*.

For *q* N0 we denote

$$A_q(\mathbf{R}) = \left\{ \varphi(x) \in D(\mathbf{R}) \middle| \int_{\mathbf{R}} \varphi(x)\,dx = 1 \, and \int_{\mathbf{R}} x^j \varphi(x)\,dx = 0, j = 1,\dots,q \right\}$$

where $\mathcal{D}(\mathbf{R})$ is the space of all C^∞ functions $\varphi : \mathbf{R} \to \mathbf{C}$ with compact support. The elements of the set $A_q(\mathbf{R})$ are called *test functions*.

It is obvious that $A_1 \supset A_2 \supset A_3 \dots$ Also, $A_k \neq \emptyset$ for all $k \in \mathbf{N}$ (this is proved in [3]).

For $\varphi \in A_q(\mathbf{R})$ and $\varepsilon > 0$ it is denoted $\varphi_\varepsilon = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ and $\check{\varphi}(x) = \varphi(-x)$.

Wanting to obtain an algebra containing the space of distributions, which elements could be multiplied and diferentiated as well as C^∞ functions, Colombeau started with $\mathcal{E}(\mathbf{R})$, the algebra of functions $f(\varphi, x) : A_0(\mathbf{R}) \times \mathbf{R} \to \mathbf{C}$ that are infinitely differentiable with respect to the second variable, *x*. The embedding of distributions into such an algebra must be done in a way that the embedding of C^∞ functions will be identity. Let *f* and *g* be C^∞ functions. Taking the sequence $(f * \varphi_\varepsilon)_{\varepsilon>0}$, which converges to *f* in $\mathcal{D}'$, as a representative of *f*, we obtain an embedding of $\mathcal{D}'$ into $\mathcal{E}(\mathbf{R})$. So, if we consider *f* and *g* as a distributions, we look at the sequences $(f * \varphi_\varepsilon)_{\varepsilon>0}$ and $(g * \varphi_\varepsilon)_{\varepsilon>0}$. The product of *f* and *g* as a distributions not always coincide with their classical product considered as a distribution, i.e.

$$(f * \varphi_\varepsilon)(g * \varphi_\varepsilon) \neq (fg) * \varphi_\varepsilon$$

The idea therefore is to find an ideal $\mathcal{I}[\mathbf{R}]$ such that this diffrence will vanish in the resulting quotient. In order to determine $\mathcal{I}[\mathbf{R}]$ it is obviosly enough to find an ideal containing the differences $((f * \varphi_\varepsilon) - f)_{\varepsilon>0}$.

Expanding the last term in a Taylor series and having in mind propertis of $\varphi(x)$ as an element of $A_q(\mathbf{R})$ we can see that it will vanish faster then any power of ε, uniformly on compact sets, in all derivatives. The set of these differences will not be an ideal in $\mathrm{E}(\mathbf{R})$ but in a set of a sequences which derivatives are bounded uniformly on compact sets by negative power of *ε*. These sequences are called 'moderate' sequences and the set containing them is denoted with $\mathrm{E}_M[\mathbf{R}]$.

Finaly, the generalized functions of Colombeau are elements of the quotient algebra

$$\mathrm{G} \equiv \mathrm{G}(\mathbf{R}) = \frac{\mathrm{E}_M[\mathbf{R}]}{\mathrm{I}[\mathbf{R}]}$$

where, as explained before, $\mathrm{E}_M[\mathbf{R}]$ is the subalgebra of 'moderate' functions such that for each compact subset *K* of R and any *p* N0 there is a *q* N such that, for each $\varphi \in A_q(\mathbf{R})$ there are *c*>0*, η*>0 and it holds:

$$\sup_{x\in K}\left|\partial^p f(\varphi_\varepsilon, x)\right| \le c\varepsilon^{-q}$$

for 0 <*ε*<*η* and $\mathrm{I}[\mathbf{R}]$ is an ideal of $\mathrm{E}_M[\mathbf{R}]$ consisting of all functions $f(\varphi, x)$ such that for each compact subset *K* of R and any *p* N0 there is a *q* N such that for every *r*≥*q* and each $\varphi \in A_r(\mathbf{R})$ there are *c*>0*, η*>0 and it holds:

$$\sup_{x\in K}\left|\partial^p f(\varphi_\varepsilon, x)\right| \le c\varepsilon^{r-q}$$

for 0 <*ε*<*η*. Elements of $\mathrm{I}[\mathbf{R}]$ are also known as 'null' functions or 'negligible' functions.

The Colombeau algebra $\mathrm{G}(\mathbf{R})$ contains the distributions on R canonically embedded as a C-vector subspace by the map:

$$i : \mathrm{D}'(\mathbf{R}) \to \mathrm{G}(\mathbf{R}) : u \to \tilde{u} \doteq \left\{ \tilde{u}(\varphi, x) = \left(u * \check{\varphi}\right)(x) : \varphi \in A_q(\mathbf{R}) \right\}$$

where denotes the convolution product of two distributions and is given by:

$$(f * g)(x) = \int_{\mathbf{R}} f(y) g(x - y) dy$$

According to the above, we can also write: $\tilde{u}(\varphi, x) = \langle u(y), \varphi(y - x)\rangle$ where $\langle u, \varphi\rangle$ denotes the integral $\int_{\mathbf{R}} u(x)\varphi(x)\,dx$.

An element $f \in \mathrm{G}$ (a generalized function of Colombeau) is actually an equivalence class $[f] = [f_\varepsilon + \mathrm{I}]$ of an element $f_\varepsilon \in \mathrm{E}_M$ which is called *representative* of *f*. Multiplication and differentiation of generalized functions are performed on arbitrary representatives of the respective feneralized functions.

With the next two deffinitions, we introduce the notion of 'association'.

Definition 2.1 *Generalized functions f, g* $\mathrm{G}(\mathbf{R})$ *are said to be associated, denoted f ≈g, if for some representatives* $f(\varphi_\varepsilon, x)$ *and* $g(\varphi_\varepsilon, x)$ *and arbitrary* $\psi(x) \in \mathrm{D}(\mathbf{R})$ *there is a q* N0 *such that for any* $\varphi(x) \in A_q(\mathbf{R})$

$$\lim_{\varepsilon\to 0_+}\int_{\mathbf{R}}\left|f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x)\right|\psi(x)\,dx = 0 .$$

Definition 2.2 *A generalizedfunction f* $\mathrm{G}(\mathbf{R})$ *is said to admit some u D*'(R) *as 'associated* *distribution', denoted f ≈u, if for some representative* $f(\varphi_\varepsilon, x)$ *of f and any* $\psi(x) \in \mathcal{D}(\mathbf{R})$ *there is a q* N0 *such that for any* $\varphi(x) \in A_q(\mathbf{R})$

$$\lim_{\varepsilon \to 0_+} \int_{\mathbf{R}} f(\varphi_\varepsilon, x)\psi(x)\,dx = \langle u, \psi \rangle .$$

These definitions are independent of the representatives chosen and the distribution associated, if it exists, is unique. The association is a faithful generalization of the equality of distributions.

Myltiplying two distributions in $\mathcal{G}$ as a result it is in general obtained a generalized function which may not always be associated to the third distribution.

By *Colombeau product of distributions* is meant the product of their embedding in $\mathcal{G}$ whenever the result admits an associated distribution.

If the regularized model product of two distributions exists, then their Colombeau product also exists and it is same with the first one.

The relation *f ≈u* is asymmetric, the distribution *u* stands on the r.h.s.; the relation *f ≈u* is an equivalent relation in $\mathcal{G}$ so it is symmetric in $\mathcal{G}$ and it can also be written as *f −u* ≈0.

As a final conclusion of this introduction we have a fact that we operate with the elements of $\mathcal{G}$ exactly the same as with the C^∞-functions, because we actually operate with their representatives which are C^∞-functions.

3 Results on some products of distributions

In this part, we will obtain the product of distributions x_+^p and ln x_+ as embedded in Colombeau algebra.

For the embedding of the distribution x_+^p using the embedding rule, we have:

$$\widetilde{x_+{}^p} = \int_0^\infty y^p \varphi_\varepsilon(y-x)dy = \frac{1}{\varepsilon}\int_0^\infty y^p \varphi\left(\frac{y-x}{\varepsilon}\right)dy$$

where $\varphi \in A_0(\mathbf{R})$. Without lost of generality we can take $\operatorname{supp}\varphi \subseteq [-l, l]$. Using substitution $v = \dfrac{y-x}{\varepsilon}$ we obtain the representative of x_+^p in a Colombeau algebra:

$$\widetilde{x_+{}^p} = \int_0^\infty y^p \varphi_\varepsilon(y-x)dy = \frac{1}{\varepsilon}\int_0^\infty y^p \varphi\left(\frac{y-x}{\varepsilon}\right)dy$$

The same way

$$\widetilde{\ln x_+} = \int_0^\infty \ln y\, \varphi_\varepsilon(y-x)\,dy = \frac{1}{\varepsilon}\int_0^\infty \ln y \varphi\left(\frac{y-x}{\varepsilon}\right)dy$$

and with the substitution $u = \dfrac{y-x}{\varepsilon}$ we obtain:

$$\widetilde{\ln x_+} = \int_{-\frac{x}{\varepsilon}}^{l} \ln(x+\varepsilon u)\varphi(u)\,du$$

Now, for arbitrary $\psi(x)$ $\mathcal{D}(\mathbf{R})$ we have:

$$\left\langle \widetilde{x_+^p}\left(\varphi_\varepsilon, x\right) \cdot \widetilde{\ln x_+}, \psi(x) \right\rangle =$$

$$= \int_{-\infty}^{\infty} \psi(x)\, dx \int_{-\frac{x}{\varepsilon}}^{l} (x+\varepsilon v)^p \varphi(v)\, dv \int_{-\frac{x}{\varepsilon}}^{l} \ln(x+\varepsilon u)\varphi(u)\, du$$

But $\psi(x)$ has also compact support, so we have:

$$\left\langle \widetilde{x_+^p}\left(\varphi_\varepsilon, x\right) \cdot \widetilde{\ln x_+}, \psi(x) \right\rangle = \int_{-l\varepsilon}^{l\varepsilon} \psi(x)\, dx \int_{-\frac{x}{\varepsilon}}^{l} (x+\varepsilon v)^p \varphi(v)\, dv \int_{-\frac{x}{\varepsilon}}^{l} \ln(x+\varepsilon u)\varphi(u)\, du$$

Using substitution $w = -\frac{x}{\varepsilon}$, Taylor theorem for ψ and changing the order of integration we obtain the integral:

$$\left\langle \widetilde{x_+^p}\left(\varphi_\varepsilon, x\right) \cdot \widetilde{\ln x_+}, \psi(x) \right\rangle = \varepsilon^{p+1}\psi(0) \int_{-l}^{l} \varphi(v)\, dv \int_{v}^{l} \varphi(u)\, du \int_{v}^{u} (v-w)^p \ln(\varepsilon u - \varepsilon w) dw + O(\varepsilon)$$

With the new substitution $t = \frac{w-v}{u-v}$ we obtain

$$\left\langle \widetilde{x_+^p}\left(\varphi_\varepsilon, x\right) \cdot \widetilde{\ln x_+}, \psi(x) \right\rangle =$$

$$= -\varepsilon^{p+1}\psi(0)\int_{-l}^{l}\varphi(v)dv\int_{v}^{l}(v-u)^{p+1}\varphi(u)du\left[\ln(\varepsilon u-\varepsilon v)\int_0^1(1-t)^p dt+\int_0^1(1-t)^p\ln t dt\right]+O(\varepsilon)$$

We know that $\int_0^1 (1-t)^p dt = \frac{1}{p+1}$ and $\int_0^1 (1-t)^p \ln t dt = -\frac{\sigma_{p+1}}{p+1}$ where $\sigma_p = \sum_{k=1}^{p}\frac{1}{k}$ for p N and $\sigma_0 = 0$ so for the integral above we have:

$$\left\langle \widetilde{x_+^p}\left(\varphi_\varepsilon, x\right) \cdot \widetilde{\ln x_+}, \psi(x) \right\rangle = -\frac{\varepsilon^{p+1}}{p+1}\psi(0) \int_{-l}^{l} \varphi(v)\, dv \int_{v}^{l} (v-u)^{p+1}\varphi(u) \ln(\varepsilon u - \varepsilon v)\, du$$

$$-\frac{\varepsilon^{p+1}\sigma_{p+1}}{p+1}\psi(0)\int_{-l}^{l}\varphi(v)dv\int_{v}^{l}(v-u)^{p+1}\varphi(u)du+O(\varepsilon)$$

By the properties of test functions, $\int u^k \varphi(u) du = 0$ for $k \geq 1$ and $\int \varphi(u) du = 1$, so the second integral above is zero and we have:

$$\left\langle \widetilde{x_+^p}\left(\varphi_\varepsilon, x\right) \cdot \widetilde{\ln x_+}\left(\varphi_\varepsilon, x\right), \psi(x) \right\rangle = -\frac{\varepsilon^{p+1}}{p+1}\psi(0) \int_{-l}^{l} \varphi(v)\, dv \int_{v}^{l} (v-u)^{p+1}\varphi(u) \ln(\varepsilon u - \varepsilon v)\, du + O(\varepsilon)$$

Next, using integration by part for the integral $\int_{v}^{l}(v-u)^{p+1}\varphi(u)\ln(\varepsilon u-\varepsilon v)du$

we have:

$$\left\langle \widetilde{x_+^p}(\varphi_\varepsilon,x)\cdot\widetilde{\ln x_+}(\varphi_\varepsilon,x),\psi(x)\right\rangle = -\frac{\varepsilon^{p+1}}{p+1}\psi(0)\int_{-l}^{l} v^{p+1}\varphi(v)\ln(\varepsilon u-\varepsilon v)\,dv$$

$$+\frac{\varepsilon^{p+1}}{p+1}\psi(0)\int_{-l}^{l}\varphi(v)\,dv\int_{v}^{l}\frac{v^{p+1}}{u-v}du+O(\varepsilon)$$

Applying integration by parts for the first integral above we estimate its value is 0 whereas

$$\left\langle \widetilde{x_+^p}(\varphi_\varepsilon,x)\cdot\widetilde{\ln x_+}(\varphi_\varepsilon,x),\psi(x)\right\rangle = \frac{\varepsilon^{p+1}}{p+1}\psi(0)\int_{-l}^{l} v^{p+1}\varphi(v)\,dv\int_{v}^{l}\frac{du}{u-v}+O(\varepsilon)$$

$$=\frac{\varepsilon^{p+1}}{p+1}\psi(0)\int_{-l}^{l} v^{p+1}\varphi(v)\ln|u-v|\,dv+O(\varepsilon)$$

Applying integration by parts once again,we obtain that

$$\widetilde{x_+^p}(\varphi_\varepsilon,x)\cdot\widetilde{\ln x_+}(\varphi_\varepsilon,x)\approx 0$$

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