# УНИВЕРЗИТЕТ „ГОЦЕ ДЕЛЧЕВ" - ШТИП
# ФАКУЛТЕТ ЗА ИНФОРМАТИКА

ISSN:1857-8691

# ГОДИШЕН ЗБОРНИК
# 2014
# YEARBOOK
# 2014

ГОДИНА 3 VOLUME III

## GOCE DELCEV UNIVERSITY - STIP
## FACULTY OF COMPUTER SCIENCE

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**CONTENT**

# NUMERICAL ANALYSIS OF BEHAVIOR FOR LORENZ SYSTEM WITH MATHEMATICA

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**Abstract:** In this paper I study the Lorenz system using Runge – Kuta method. The Runge – Kuta method is numerical method and it does not give precise analysis, but in absence of the explicit solutions for the Lorenz system, it can be a good method for analysis. We will give intervals for t, where the system converges to a fixed point or there is a chaos but that is not so precise assessment. The analysis will be made by changing the parameters *σ, r, b* so that: we will change one parameter and two others remain fixed, we will change two parameters and the third remains fixed or we will change all three parameters in the parametric space $(\sigma, r, b)$. Numerical analysis of the system was made with mathematical package Mathematica.

**Mathematics Subject Classification 2010:** 65Y15, 37M99, 37D45, 65D25, 65P20.
**Key words:** Fixed points, Runge – Kuta method, parameters, parametric space, convergence, chaos.

## 1. Introduction

Lorenz system is a nonlinear autonomous dynamic system. It does not have solution, but its behavior is studied in the mathematical literature [1], [2]. Using the mathematical package Mathematica for studying the solution behavior of the Lorenz system and other dynamical systems is found in the mathematical literature [3], [4]. In [5] is used mathematical method Runge-Kuta for Lorenz system.

The Lorenz system has a form

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(r - z) - y$$
$$\dot{z} = xy - bz$$

It depends on three parameters *σ, r, b* and initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0)$, so its solution $(x(t), y(t), z(t))$ depends only on three parameters *σ, r, b*. We made an analysis of system behavior in the parametric space $(\sigma, r, b)$.

We obtain the fixed points of the Lorenz system from $\dot{x} = 0$, $\dot{y} = 0$, $\dot{z} = 0$. If x=0 then y=z=0 and the first fixed point is *O* (0,0,0). If $x \neq 0$ then the fixed points are $O_{1/2}$ ($\pm\sqrt{b(r-1)}$ ;$\pm\sqrt{b(r-1)}$ ; r-1).

The fixed point *O* (0,0,0) exists independently of the values of the parameters, but the fixed points $O_{1/2}$ ($\pm\sqrt{b(r-1)}$ ;$\pm\sqrt{b(r-1)}$ ; r-1) not always exist in real space. They exist only when the following conditions for the parameters are satisfied: $b > 0, r > 1$ or $b < 0, r < 1$. For r=1, we have fixed point *O* (0,0,0). We will only keep the system parameters $\sigma$ >0, r>0, b>0, when the system has a physical meaning.

Numerical analysis of the system was made with mathematical package Mathematica, which numerically approximates the solutions of the system by the method of Runge – Kuta.

We have considered the following cases:

1. One of the parameters is changing, other two parameters are fixed. This is the case, when the parameters are changing along a line in the parametric space $(\sigma, r, b)$;
2. Two of the parameters are changing, the third is fixed. This is the case, when the parameters are moving in a plane in the parametric space $(\sigma, r, b)$;
3. All the three parameters are changing. This is the case, when all three parameters are moving in the parametric space $(\sigma, r, b)$.

On 2-D graphs, the red color correspondents with $x = x(t)$, the green color correspondents with $y = y(t)$ and the blue color correspondents with $z = z(t)$.

Regardless of the above cases, for t=0 the curves $x = x(t), y = y(t), z = z(t)$ always start from initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0)$ in 2-D coordinate systems Oxt, Oyt, Ozt respectively and the curve $(x(t), y(t), z(t))$ always start from initial value $(x_0, y_0, z_0)$ in 3-D coordinate system Oxyz.

Regardless of the above cases, under the condition $\sigma > 0, b > 0, 0 < r \le 1$ for $t \to \infty$, the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the zero in 2-D coordinate systems Oxt, Oyt, Ozt respectively and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O$ in 3-D coordinate system Oxyz.

## 2. Numerical analysis of behavior for Lorenz system when changing only one parameter

When a parameter of the system changes and the other two parameters are fixed then it moves in a straight line in the parameter space $(\sigma, r, b)$.

We have the following results:

Under the restrictions $0 < \sigma \le 1, b > 0, r > 1$, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ are approaching to the fixed point $O_{1/2}$ in 3-D coordinate system Oxyz;

a) When $b > 0, r > 1$ are fixed, $\sigma > 1$ is changing, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ oscillate around the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D system and the curve $(x(t), y(t), z(t))$ oscillate around the point $O_{1/2}$ in 3-D coordinate system Oxyz, then their behavior can become chaotic;

b) When the parameters $\sigma > 1$, $b > 0$ are fixed, the parameter $r > 1$ is changing for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system for smal values of r. For big values of r, the behavior is chaotic. For $0 < b \le 1$, the values for r where the behavior is chaotic are smaller for bigger $\sigma$. For $b > 1$, the values for r where the behavior is chaotic are bigger for bigger $\sigma$;

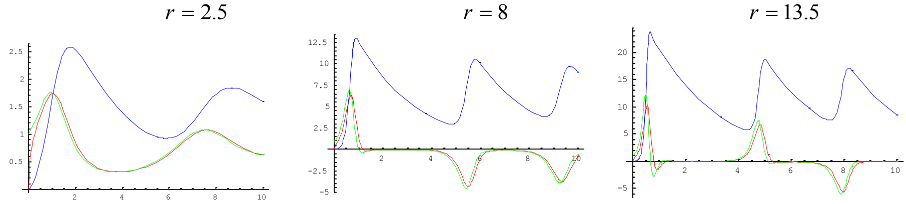
c) When $\sigma \ge 1, r > 1$ are fixed and

i) by changing the parameter $0 < b \le 1$, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ entered into chaos. For small values of the parameter r, subintervals for b appear in which the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system. With increasing $\sigma$ these r are getting smaller and the subintervals for b in which we have convergence are also getting smaller.

ii) by changing the parameter b>1, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system. For r>8 in some small subintervals of b starting from 1, chaos occurs. Outside those subintervals of b the curves converge.

*Example 1:* The parameter r is changing of 2.5 to 30 by step 5.5. Let $\sigma$ =10, b=0.4 and the initial values $x_0 = 0, y_0 = 1, z_0 = 0$. This is a case when there is a chaos for r>5. For r=2.5, there is approaching to $0.77,\ 0.77,\ 1.5$ respectively in 2-D system and $O_{1/2}$ ( $0.77,\ 0.77,\ 1.5$ ) in 3-D coordinate system (figure 1).

$r = 2.5$ $r = 8$ $r = 13.5$



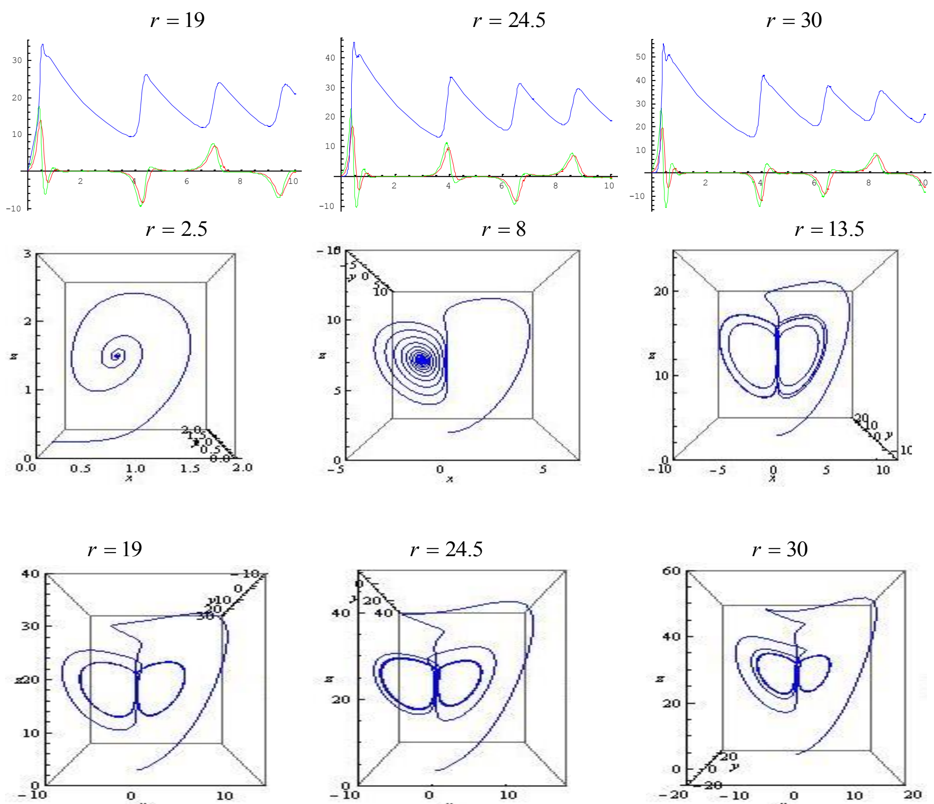



*Figure 1:* The graphs in 2-D and 3-D coordinate system for parametar r of 2.5 to 30 by step 5.5

## 3. Numerical analysis of behavior for Lorenz system when two of the parameters are changed, one parameter is fixed

When two of the parameters are changing, the third remains fixed then they are moving in a plane in the parametric space $(\sigma, r, b)$.

1. The parameters $\sigma$ , r are changing and the parameter b remains fixed.

i) When $0 < \sigma \le 1, r > 1$ are changing and the parameter b>0 is fixed, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system.

ii) When $\sigma > 1, r > 1$ are changing and the parameter b>0 is fixed, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system for small $\sigma$ and big r. For big $\sigma$ and small r, chaos occurs.

2. The parameters $\sigma$ , b are changing and the parameter r remains fixed.
   For r>1 fixed and:

i) the parameters $0 < \sigma \le 1, b > 1$ are changing, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system.

ii) the parameters $0 < \sigma \le 1, 0 < b \le 1$ are changing, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system for small r. For the parameter $r \ge 7$ chaos occurs for some intervals for $\sigma$ and b.

iii) the parameters $\sigma > 1, 0 < b \le 1, r \ge 10$ for $t \to \infty$ the curves in 2-D and 3-D coordinate system are entering into chaos. For r<10, in some intervals for $\sigma$ and b the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system.

iv) the parameters $\sigma > 1, b > 1$ for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system for small b. For big b chaos occurs.

3. The parameters r, b are changing and the parameter $\sigma$ remains fixed.

- For $r > 1,\ 0 < b \le 1$ and

i) the parameter $0 < \sigma < 1$ is fixed, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system;

ii) the parameter $0 < \sigma < 1$ is fixed, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system for some subintervals for b and r. For some intervals of r and b , when 9<r<22, chaos occurs.

iii) the parameter $\sigma > 1$ is fixed, for $t \to \infty$, are appearing certain intervals for b and $0 < r < 6$ in the plane br in which chaos is occurring. Out of them the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system. For 9<r<22 the there is no a convergence for any b. For r>21 despite the chaos. For some intervals of b the curves converge. When r increases the number of those intervals of b where the curves converge is increasing;

- For the parameters r>1, b>1 and

i) the parameter $0 < \sigma \le 1$ is fixed, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system.

ii) the parameter $\sigma > 1$ is fixed, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system for small r. For 14<r<18, $\sigma > 1$ some subintervals of b, chaos occurs.

*Example 2:* The parameter b is changing of 0.1 to 1 by step 0.3 and r is changing of 2.5 to 24.5 by step 11. For $\sigma$ =0.5 and the initial values $x_0 = 30, y_0 = 20, z_0 = 10$. There is convergence to the fixed point. (figure 2, figure 3).

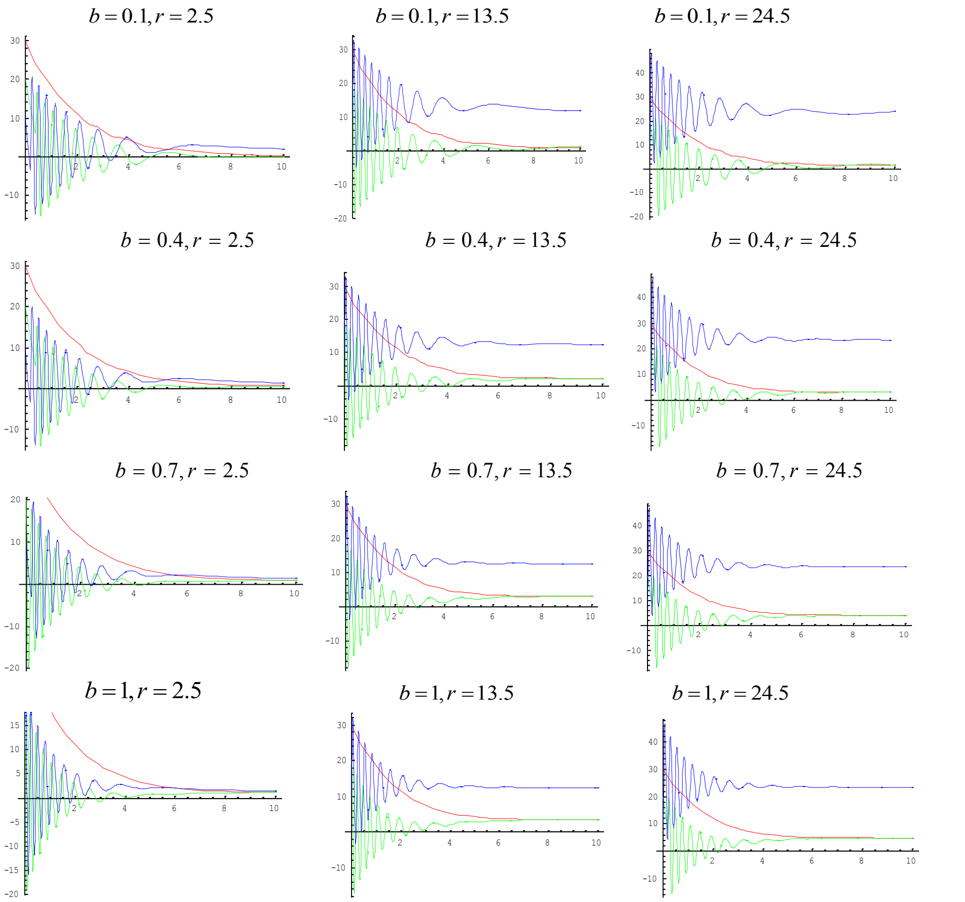
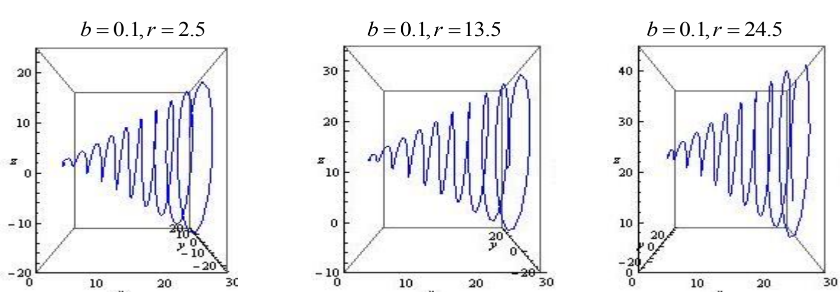



*Figure 2:* The graphs in 2-D coordinate system for parametar r of 2.5 to 24.5 by step 11 and b of 0.1 to 1 by step 0.3



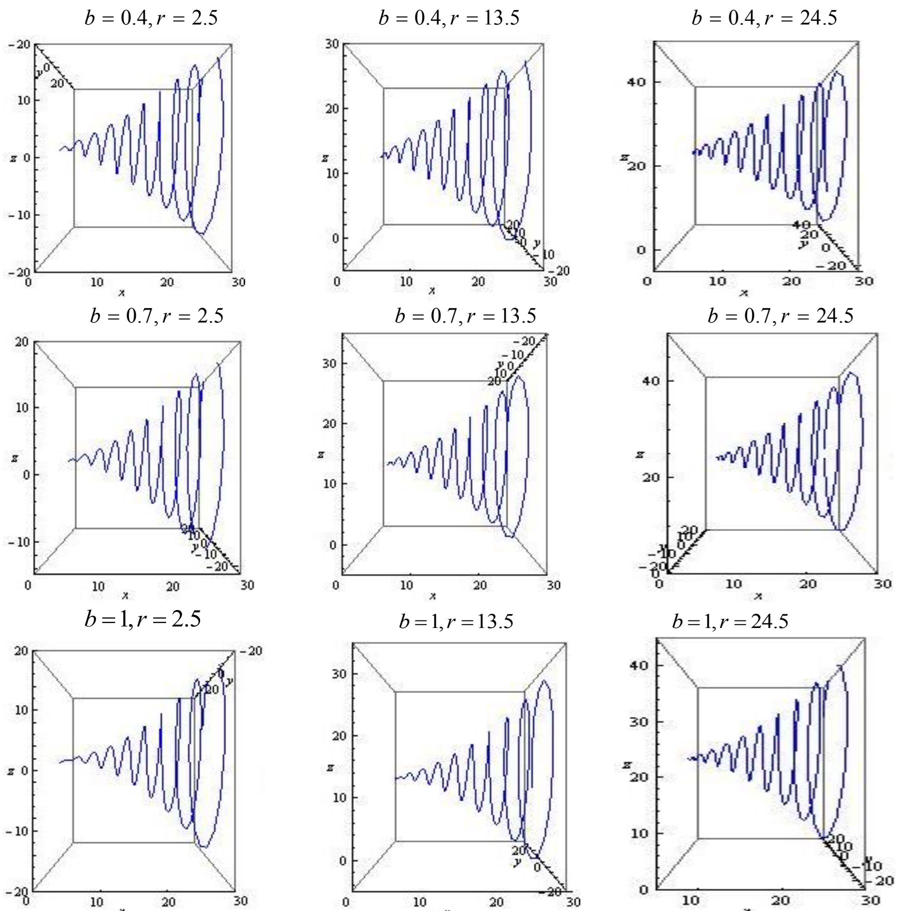



*Figure 3:* The graphs in 3-D coordinate system for parametar r of 2.5 to 24.5 by step 11 and b of 0.1 to 1 by step 0.3

## 4. Numerical analysis of behavior for Lorenz system when all the three parameters are changed

When all the three parameters are changing then the three parameters are moving in the parametric space $(\sigma, r, b)$.

- For the parameters $0 < \sigma \le 1$, r>1, b>0, for $t \to \infty$ the curves $x = x(t), y = y(t), z = z(t)$ are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system. For some small subintervals of $\sigma$, $16 < r < 27$ and $0 < b \le 1$ the system enters into chaos;
- For $\sigma$ about 1, r>21, $0 < b < 1$ and $4 < \sigma < 13$, 1<r≤19, $0 < b < 1$ and some small subintervals of $\sigma > 1$, r>8 and by increasing of $b > 1$ for $t \to \infty$ the system enters into chaos. There are some small parameter subintervals when $\sigma > 1$, r>1, $b > 0$ and the system enters into chaos. In other intervals for $\sigma > 1$, r>1, $b > 0$, for $t \to \infty$ the curves are approaching to the points $\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1$ respectively in 2-D coordinate system and the curve $(x(t), y(t), z(t))$ is approaching to the fixed point $O_{1/2}$ in 3-D coordinate system.

*Example 3:* The parameter $\sigma$ is changing of 2 to 11 by step 3, the parameter b is changing of 0.1 to 1 by step 0.3 and the parameter r is changing of 2.5 to 30 by step 5.5, with the initial values

$x_0 = 30, y_0 = 20, z_0 = 10$. Here are the graphical representations of most interesting parametric subintervals in 2-D and 3-D coordinate systems (figure 4, figure 5).

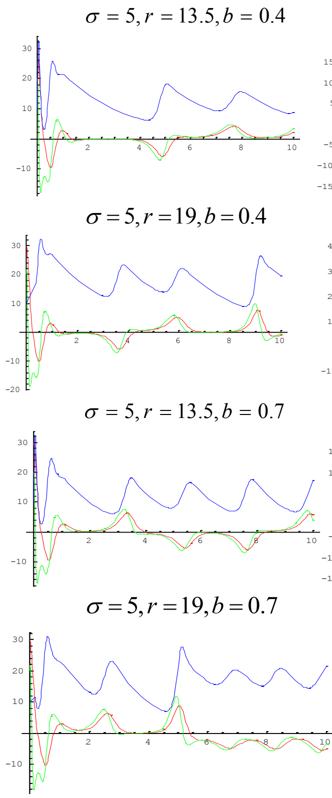
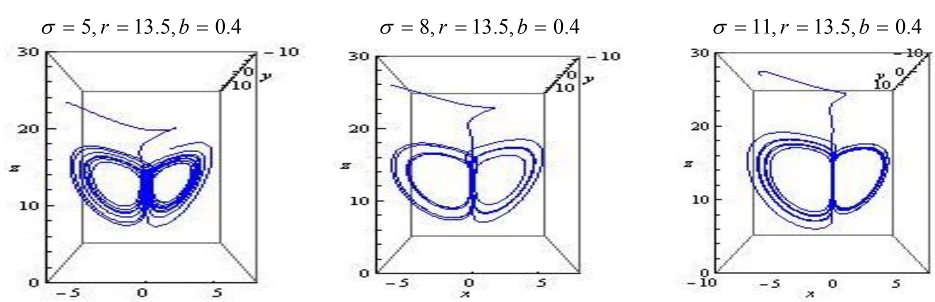



*Figure 4:* The graphs in 2-D coordinate system for parametar $\sigma$ of 2 to 11 by step 3, r of 2.5 to 30 by step 5.5 and b of 0.1 to 1 by step 0.3



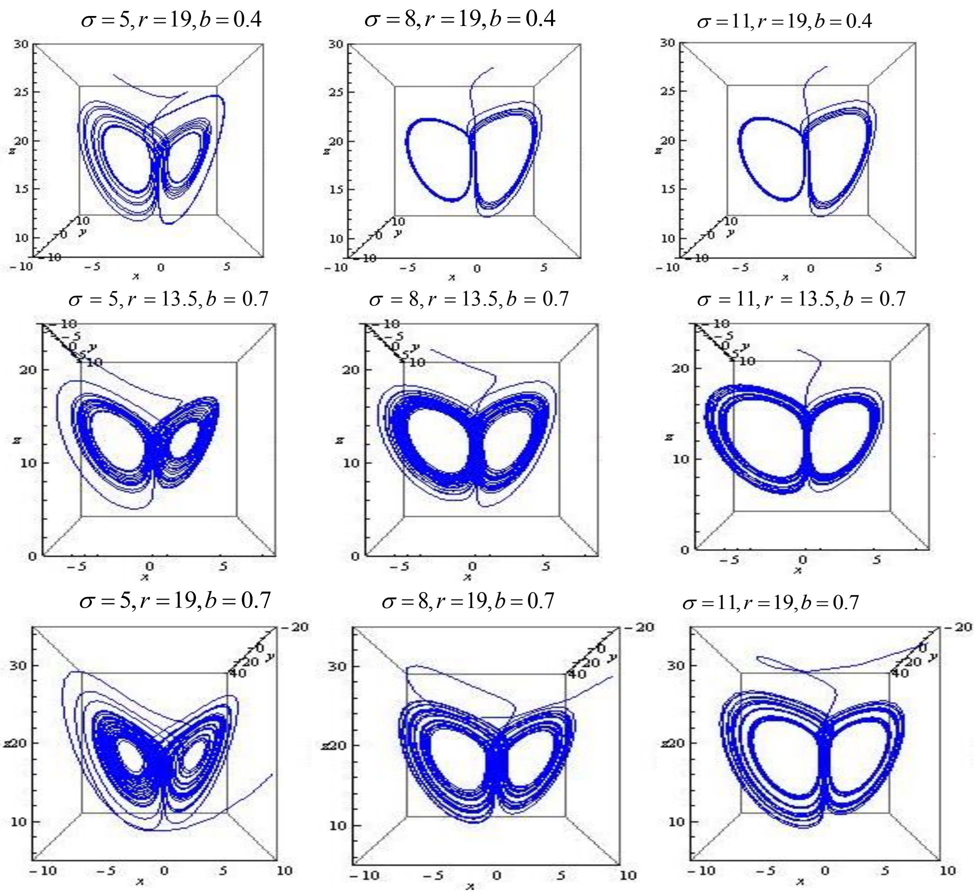



*Figure 5:* The graphs in 3-D coordinate system for parametar $\sigma$ of 5 to 8 by step 3, r =13.5 and b of 0.4 to 0.7 by step 0.3

## 5. Conclusion

The Lorenz system does not have explicit solutions because we can't analyze its behavior. The Runge – Kuta method is numerical method and it does not give precise analysis, but in the absence of explicit solutions it can be a good method for analysis. Numerical analysis of the Lorenz system with Runge – Kuta method still gives us a picture of its behavior: convergence to fixed point or chaos. The results obtained in this paper are expected and known in some papers that deal with numerically solving of Lorenz system such as the papers [6], [7], [8]. For a complite analysis of dynamical system in the whole spectrum of parameters is very difficult. These numerical studies by using of a Maximal Lyapunov exponents and Auto software for parametric space of Lorenz system and finding the chaotic regions in planes ($\sigma, b$) for fixed r, ($r, \sigma$) for fixed b, (r,b) for fixed $\sigma$ and in the three-parametric space $(\sigma, r, b)$ can look in the paper [6], [7], [8].

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