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ECONOMICS OF NETWORKS AS APPLIED TO SOCIAL CHOICE MODEL OF LEVEL OF EDUCATION, SOCIAL DISTANCE, NEWS VERACITY AND PROSPECT THEORY AND A NOTE ON WARDROP EQUILIBRIUM AND BRAESS PARADOX

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Abstract
This paper investigates Barabási–Albert (BA) and Erdős-Rényi model and preferential attachment in different settings: social choice and social distance, news veracity and networks, Prospect theory with social choice, and Wardrop equilibrium with Braess paradox. News veracity is higher in Erdős-Rényi graph vs Barabási–Albert (BA), in the Erdős-Rényi graph spread of misinformation is higher than in the model with hubs (influential nodes). In the average social distance and education level model there is higher news veracity between most educated individuals (PhD's) more when two nodes have less than highest education. Probability of news being true (news veracity) is lowest for PhD's but gaslighting (probability of perception of being gaslighted) is highest. With a degree centrality included nodes (individuals) with lower level of education perceive higher new veracity. Price of Anarchy (PoA) is higher in Erdős-Rényi (ER) versus Barabási–Albert graph. With Prospect theory in news veracity network gain and loss lines are further way on the left from reference point, with a letter being more to left. Congestion in the market for news makes more and more agents further away from equilibrium state.
Keywords: Barabási–Albert, Erdős-Rényi, social distance, social choice, Wardrop equilibrium, Braess paradox
JEL code:E24,E32,J64

1.Introduction

(Neo) Classical economics is based on maximization of utility function of individual agent[1].But this so to say traditional economics has been based on methodological individualism, see McCormick, K. (1997).Akerlof(1997), wrote that this method should be extended in describing social decisions to include dependence of individuals utility on the utility or the actions of others. In that model education is presented as a social choice variable (demand for education). Dependence on the utilities of agents produce externalities that may slow down movement towards social equilibria, but also may create long-run low equilibrium traps. When people try to distance themselves in social space from their friends and relatives these are status seeking individuals, when individuals try to move themselves closer to friends and relatives, this is called conformist behavior, see Akerlof(1997). Some papers show that network can affect the equilibrium outcomes namely some of network measures such as lowest eigenvalue can if enough low provide unique equilibrium, while if large there are possibly many equilibria, see Bramoullé, Y., Kranton, R., D'Amours, M. (2014).When some person provides more public goods, their neighbors free-ride and provide less which forces their neighbors to provide more etc. So, the lowest eigenvalue, we find, captures the cumulative effects of agents' actions on others[2].In

[1] Stigler (1997), concisely defines the rational utility maximization hypothesis: His tastes are consistent, his cost calculations are correct, he makes those decisions that maximize utility
[2] The highest eigenvalue, which is positive, is important in games of pure complements. In economics and game theory, the decisions of two or more players are called strategic complements if they mutually

many economic settings, who interacts with whom matters. For instance, adolescents' consumption of tobacco and alcohol is influenced by their friends. Firms' investments decisions depend on the actions of other firms producing substitute and complementary goods[3]. Akerlof (1997) work was an extension of Becker earlier work for example: Becker (1964, 1968, 1971, 1973, 1974).Those works did not take into account social interactions.Networks can be viewed as facilitators of flow among the distributed entities. On the other hand, social and economic networks, these units (nodes) are individuals or firms, whereas links are information/interactions between nodes. At a broad level the study of networks can encompass all kinds of interactions. For instance: Information transmission, web links, information exchange, or trade, credit and financial flows, friendship trust, spread of epidemics, diffusion of ideas and innovation. Networks in this paper are constructed by using Barabási–Albert (BA) and Erdős-Rényi model. In Erdős-Rényi network, we assign N nodes, and then connect each pair with some probability p .This means that one node will have much higher degree than any other. In Barabási–Albert (BA) network, we assign N nodes, but to create them, we first start with a small set of connected nodes. Then we add nodes one at a time till we get N nodes. When we add a node, we connect it to a small number of existing nodes with probability proportional to the degree of the existing node. As a result, nodes with higher degree (the earlier ones) tend to get even higher degree. Let's think this way If you're born rich, you're practically handed money, but if you are born poor you have extra fees. This is known as the Matthew effect, from the book of Matthew: "For to everyone who has, more will be given, and to those who have nothing, even that will be taken away"[4]. Anyways, the result is that the network ends up with a powerlaw distribution. Power law distribution can be used to describe any exponential mathematical relationship, but one of the more commonly referred to power laws is $y = \frac{1}{x}$, which is represented as an asymptotic relationship between the x and y.[5]However, socio-economic studies show the importance of neighborhood effects, these effects are statistically significant for instance Borjas (1995) has found that the slow rate of convergence for different ethnic groups can be explained mainly by neighborhood fixed effects. One more recent study investigated how choices about social affiliation based on one attribute can exacerbate or attenuate segregation on another correlated attribute. Namely, this study has identified three population parameters: between-group inequality, within-group inequality, and relative group size—that determine how income inequality between race groups affects racial segregation, see Bruch EE.(2014).Also, there is a literature of spatial inequalities that is currently "fragmented" across ethnic segregation and built environment domains[6], see Patias, N., Rowe, F., Arribas-Bel, D. (2023).In this paper first we will investigate two types of network algorithms: Barabási–Albert, Erdős-Rényi,we will show rich get richer phenomenon (preferential attachment),then we will relate Akerlof (1997) model and Kranton et al.(2020).First, is a social distance social choice model,

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reinforce one another, and they are called strategic substitutes if they mutually offset one another, see Bulow, J. I., Geanakoplos, J. D., Klemperer, P. D. (1985)

[3] Social interaction theory will explain why social decisions-such as the demand for education, the practice of discrimination, the decision to marry,divorce, and bear children, and the decision whether or not to commit crimes-are not simple choices based primarily on individual considerations, see Akerlof (1997).
[4] For instance, a person who is already rich gets more and more and a person who is having less gets less. This is called the Rich getting Richer phenomena or Preferential Attachment.For instance a student with higher degree is rich and the student with low degree is poor, now if a new student comes to class he/she has to make friends, so he/she will select students with a higher degree and become friends with them which increases the degree of rich. This is called Rich getting Richer or Preferential Attachment.
[5] One attribute of power law distributions is their scale invariance. In mathematics, one can consider the scaling properties of a function or curve $f\ (x)$ under rescalings of the variable x. That is, one is interested in the shape of $f\ (\lambda x)$ for some scale factor λ, which can be taken to be a length or size rescaling. The requirement for $f\ (x)$ to be invariant under all rescalings is usually taken to be:$f(\lambda x) = \lambda^{\Delta} f(x)$, for some choice of exponent Δ, and for all dilations λ. This is equivalent to f being a homogeneous function of degree Δ. Dilations are a way to stretch or shrink shapes around a point called the center of dilation. The amount we stretch or shrink is called the scale factor.
[6] Ethnic segregated areas are often more disadvantaged in terms of unemployment, housing conditions and access to services.

second is news veracity and education in networks. Then we will include the case of Wardrop equilibrium and Braess paradox in the analysis. This paper is more about enabling understanding this literature and less about adding something new to this vast economic literature in this area.

2.Barabási–Albert model

In Erdös- Rényi Model (see Erdös- Rényi(1959)), the clustering coefficient is fixed to be $C = p$ [7]which means there is no clustering effect at all. However, in most real networks the clustering coefficient is typically much larger than it is in a comparable independent random network.But researchers have suggested that most real networks follow a power law distribution:

equation 1

$$P(k) \sim k^{\gamma}$$

γ (gamma) is the power-law exponent, typically in the range $2 < \gamma < 3$ for most real-world networks. Such models are called scale free, see Albert, R., Barabási, A.-L. (2002), and Barabási and Albert, (1999). The empirical result shows that many large networks are scale free, in other words, their degree distribution follows a power law for large k, which Erdös- Rényi Model cannot produce, see Li, Aoxi (2011).Growth and preferential attachment inspired the introduction of Barabási–Albert model[8]. Growth: Starting with a small number (m_0) of nodes, at every time step, we add a new node with $m(\leq m0)$ edges that link the new node to m different nodes already present in the system. Preferential attachment: When choosing the nodes to which the new node connects, we assume that the probability Π that a new node will be connected to node i depends on the degree[9] k_i of node i, such that following applies:

equation 2

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}$$

Barabási, A.-L. and Albert, R. (1999),model is simply defined as (for this part see Bianconi, Ginestra, (2018)) : At time $t = 1$ the network is formed by $n_0 \geq m$ nodes connected by m_0 links. At each time $t > 1$ two processes define the network evolution. In the mean-field approximation Barabási, A.-L. and Albert, R. (1999),the degree $k_i(t)$ that a node i arrived in the network at time t_i is taken to be a continuous deterministic variable depending on time t equal to the expected value of the degree of node i over different realizations of the stochastic network growth. If the degree k_i of a given node i is a continuous real variable, the rate at which k_i changes can be written as:

equation 3

$$\frac{\partial k_i}{\partial t} = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{N-1} k_j} = \frac{k_i}{2t}$$

In the initial condition $k_i(t_i) = m$, if $t \gg 1$,one can approximate the sum of the denominator;

equation 4

$$\sum_j k_j \simeq 2mt$$

[7] In Erdös- Rényi model $c = p = \frac{\langle k \rangle}{N}$; where N are nodes, p is probability , and the degree k_i of node i.

[8] The network models previously assumed that we start with a fixed number N of vertices that are then randomly connected or rewired, without modifying N. In contrast, most realworld networks describe open systems that grow by the continuous addition of new nodes. Starting from a small nucleus of nodes, the number of nodes increases throughout the lifetime of the network by the subsequent addition of new nodes. For example, the World Wide Web (WWW) grows exponentially in time by the addition of new web pages, and the research literature constantly grows by the publication of new papers. Preferential attachment means: nodes with high degrees are more likely to have more nodes connected to them. When a new edge is created, it is more likely to connect to a vertex that already has a large number of edges. This “rich get richer” effect is characteristic of the growth patterns of some real-world networks.

[9] In the study of graphs and networks, the degree of a node in a network is the number of connections it has to other nodes and the degree distribution is the probability distribution of these degrees over the whole network.The degree distribution is: $P_k = \frac{n_k}{n}$; $\exists n$ -nodes and n_k of them have degree k.

Now obtaining that the degree k_i of node i arrived in the network at time t_i increases with time as a power law,

equation 5

$$k_i = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}$$

This applies for $t \geq t_i$, which implies that older nodes have higher degree, from this expression, given the network at time t, the probability $P(k_i(t) > k)$ that a node has degree $k_i(t)$ greater than k is given by:

equation 6

$$P(k_i(t) > k) = P\left(m\left(\frac{t}{t_i}\right)^{\frac{1}{2}} > k\right) = P\left(t_i < t\left(\frac{m}{k}\right)^2\right)$$

the probability that a random node of the network is arrived at time $t_i < \tau$, in the mean-field approximation it is given by the following expression:

equation 7

$$P(t_i < \tau) = \frac{\tau}{t}$$

As long as $t \gg 1$ it follows that :

equation 8

$$P(k_i(t) > k) = \left(\frac{m}{k}\right)^2$$

So now the degree of distribution $P(k)$ is given as:

equation 9

$$P(k) = -\frac{dP(k_i > k)}{dk} = \frac{2m^2}{k^3}$$

The exact degree distribution now is given as:

equation 10

$$P(k) = \frac{2m(m+1)}{k(k+1)(k+2)}$$

the Bianconi–Barabási model see Bianconi, G. and Barabási, A.-L. (2001) (2001a) that assigns to each node a fitness describing its ability of nodes to acquire new links and includes growth and generalized preferential attachment, yields scale-free networks with tunable power-law exponent $\gamma \in (2, 3]$. Barabási–Albert model proposes preferential attachment as a basic mechanism to generate scale free networks. Or since $k_i(t_i) = m$ the solution to $\frac{\partial k_i}{\partial t} = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{N-1} k_j} = \frac{k_i}{2t}$ is given as:

equation 11

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta}; \beta = \frac{1}{2}$$

This solution was derived and previously. And by using the previous equation one can derive:

equation 12

$$P[k_i(t) < k] = P\left(t_i > \frac{m^{\frac{1}{\beta}} t}{k^{\frac{1}{\beta}}}\right)$$

Now, by assuming that we add the nodes at nodes at equal time intervals, the ti values have a constant probability density:

equation 13

$$P(t_i) = \frac{1}{m_0 + t}$$

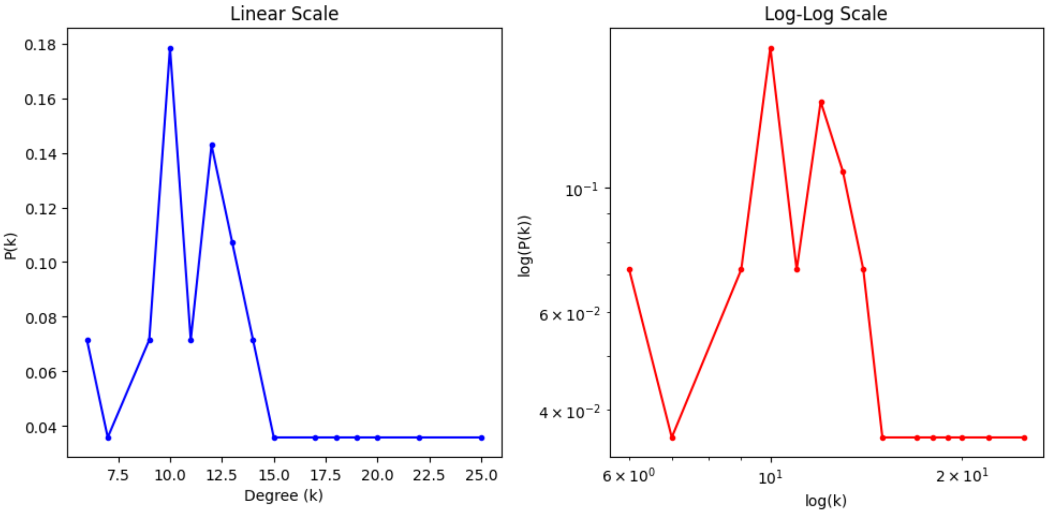
So now we can obtain :

equation 14

$$P\left(t_i > \frac{m^{\frac{1}{\beta}}t}{k^{\frac{1}{\beta}}}\right) = 1 - \frac{m^{\frac{1}{\beta}}t}{k^{\frac{1}{\beta}}(t+m_0)} \Rightarrow P(k) = \frac{\partial P[k_i(t) < k]}{\partial k} = \frac{2m^{\frac{1}{\beta}}t}{m_0 + t}\frac{1}{k^{\frac{1}{\beta}+1}} \sim 2m^{\frac{1}{\beta}}k^{-\gamma}$$

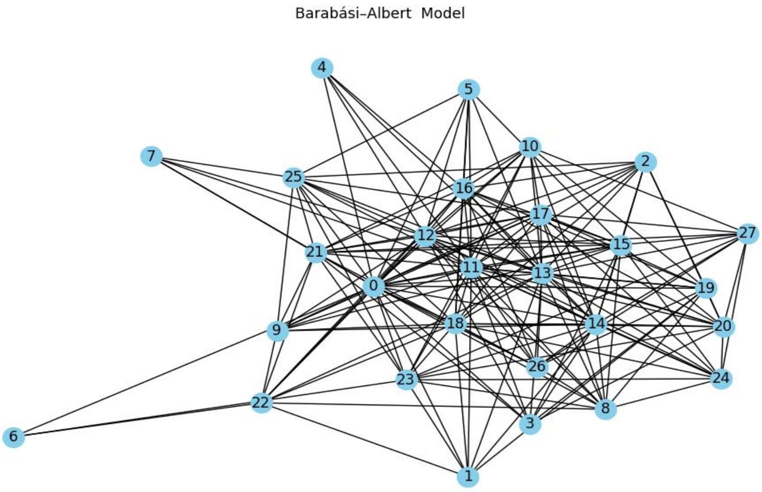
Where $\gamma = \frac{1}{\beta} + 1 = 3$. Next we show linear and log-log scale in Barabási–Albert model.

Figure 1 liner scale k vs P(k) and log-log scale log(P(k))vs log(k) in Barabási–Albert model with n_initial = 14 # Initial number of nodes; n_final = 28 # Final number of nodes m = 10 # Parameter m (m <= m_0)



Source: Author's own calculation

Figure 2 Barabási–Albert model n_initial = 14 # Initial number of nodes; n_final = 28 # Final number of nodes m = 10 # Parameter m (m <= m_0)



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2.1 Preferential attachment

The Barabási–Albert assumes that the probability $P(k)$ that a node attaches to node i is proportional to the degree k of node i,see Barabási; Albert (1999)[10]. This assumption involves two hypotheses: first,

[10] Here we remember following equation: $\Pi(k_i) = \frac{k_i}{\sum_j k_j}$

that $P(k)$ depends on k, in contrast to random graphs in which $P(k) = p$, and second, that the functional form of $P(k)$ is linear in k. The functional form of $P(k)$ can be determined for networks for which we know the time at which each node joined the network ,(see Jeong, Néda, and Barabási,(2003) and Pastor-Satorras et al., (2001))[11]. Now, let's consider the state of the network at a given time, and record the number of ''old'' nodes present in the network and their degrees. Next measure the increase in the degree of the ''old'' nodes over a time interval ΔT, much shorter than the age of the network. Now, degree distribution will be:

equation 15

$$\Pi(k) = \frac{\Delta k_i}{\Delta k}$$

Δk represents the number of edges added at additional time ΔT . Cumulative distribution function is :

equation 16

$$k(k) = \sum_{k_i=0}^{k} \Pi(k_i)$$

Now $\Pi(k) \sim k^a$ follows the power law distribution. In some cases, such as the Internet $a \simeq 1$, for other networks dependence is sublinear $a \simeq 0.8 \pm 0.1$.Non-linear preferential attachment was investigated by Krapivsky, Redner, and Leyvraz (2000).Previous authors calculate the average number of $n_k(t)$ of nodes with $k-1$ incoming edges at time t by the rate equation approach. Now, the differential equation for time evolution is given as:

equation 17

$$\frac{dn_t}{dt} = \frac{1}{\mathcal{M}_a}[(k-1)^a n_{k-1} - k^a n_k] + \delta_k$$

where $M_a(t) = \sum k^a n_k(t)$ the ath moment of $N\ k(t)$.In the previous equation: the first term accounts for new nodes that connect to nodes with $k-1$ edges, thus increasing their degree to k. The second term describes new nodes connecting to nodes with k edges, turning them into nodes with $k-1$ edges and hence decreasing the number of nodes with k edges. The third term accounts for the continuous introduction of new nodes with a single outgoing edge. In the sublinear case $a < 1$, $M_a(t) = \mu t$, with a prefactor $1 \leq \mu = \mu(a) \leq 2$, now the degree distribution is :

equation 18

$$P(k) = \frac{\mu}{k^a} \prod_{j=1}^{k} \left(1 + \frac{\mu}{j^a}\right)^{-1}$$

In conclusion, the analytical calculations Krapivsky, Redner, and Leyvraz (2000)demonstrate that the scale-free nature of the network is destroyed for nonlinear preferential attachment. The only case in which the topology of the network is scale free is that in which the preferential attachment is asymptotically linear:

equation 19

$$\prod(k_i) \sim a_\infty k_i\,; k_i \to \infty; P(k) \sim k^{-\gamma}; \gamma = 1 + \frac{\mu}{a_\infty}$$

About the growth In the Barabási-Albert model the number of nodes and edges increases linearly in time, and consequently the average degree of the network is constant. In general about the non-linear preferential attachment we have:

equation 20

$$p_i = \frac{k_i^a}{\sum_j k_j^a}$$

If $a > 1$ the model is super linear, and a small number of nodes connect to almost all other nodes in the network.The probability n_{kl} of finding a link that connects a node of degree k to an ancestor node of degree l when $m = 1$ is given by:

[11] Such dynamical data are available for the co-authorship network of researchers, the citation network of articles, and the Internet at the domain level.

equation 21

$$n_{kl} = \frac{4(l-1)}{k(k+1)(k+l)(k+l+1)(k+l+2)} + \frac{12(l-1)}{k(k+l-1)(k+l)(k+l+1)(k+l+2)}$$

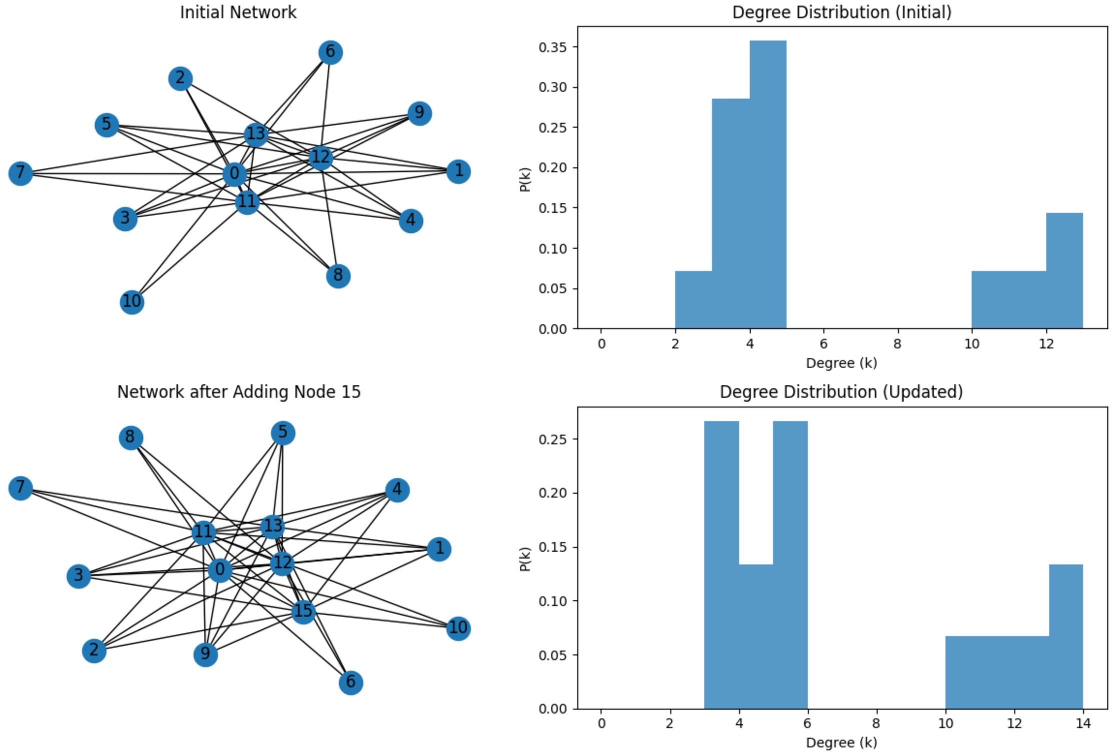
Clustering coefficient[12] is given as:

equation 22

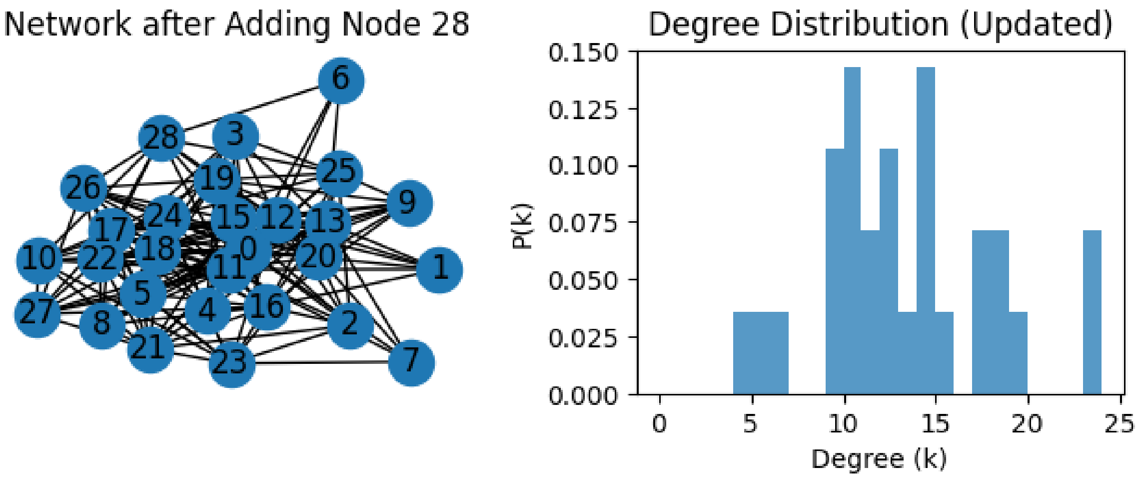
$$C_{rand} = \frac{\langle k \rangle}{n}$$

Next, we will show the plot of the Barabási-Albert model, the initial network and its degree distribution, and then iteratively adding nodes while visualizing the network and the degree distribution after each step (three in this example for space requirements).

Figure 3 plot of the Barabási-Albert model, the initial network and its degree distribution, and then iteratively adding nodes while visualizing the network and the degree distribution after each step (three in this example for space requirements)



[12] There is no analytical prediction for the Barabási-Albert model. In graph theory, a clustering coefficient is a measure of the degree to which nodes in a graph tend to cluster together. Evidence suggests that in most real-world networks, and in particular social networks, nodes tend to create tightly knit groups characterized by a relatively high density of ties; this likelihood tends to be greater than the average probability of a tie randomly established between two nodes, see Holland and Leinhardt, (1971), and Watts and Strogatz, (1998).



Source: Author's own calculations

2.2 Preferential attachment some proofs

We are studying sequence of graphs $(G_t)_{t\in n} = (G_0, G_1, \dots, G_t..)$ where G_0 is some random arbitrary graph at time $t = 0$ with vertices n_0 and edges e_0 .If we are given G_t then G_{t+1} is constructed by adding one additional vertex v to the graphs and connecting it to m vertices from G_t,where the probability of connection is proportional to the degree of vertices in G_t .This is also called, what is called preferential attachment. The probability to connect to vertex w is :

equation 23

$$\frac{\deg w}{\sum_{u\in v(G_t)} \deg u}$$

Here all the connections are independent meaning that it is possible to have multiple edges. Now, we have that :

equation 24

$$\begin{matrix} |V(G_t)| = n_0 + t \\ |E(G_t)| = m_t + e_0 \end{matrix} \Rightarrow \sum_{u\in V(G_t)} \deg u = 2(m_t + e_0) = c_t t$$

$c_t \rightarrow 2m$ as $t \rightarrow \infty$.Now, the model will be denoted as:

equation 25

$$\mathcal{G}(m) = (G_0, G_1, \dots,)$$

Now, let $N_{k,t}$ be the random variable that is equal to the number of vertices of degree k in G_t.Here we will show that the degree distribution $N_{k,t}$ follows the power law. Now, it will be convenient to consider the scaled random variable $\frac{N_{k,t}}{t}$ which gives us the proportion of the vertices of degree k.Here we will introduce lemma:

Lemma 1 Let x_t, y_t, η_t, r_t be the real numbers satisfying:

equation 26

$$x_{t+1} - x_t = \eta_{t+1}(y_t - x_t) + r_{t+1}, t \in \mathbb{N}$$

And also :

equation 27

$$\lim_{(t\to\infty)} y_t = x$$
$$\eta_t > 0; \eta_t < 1 \; for\; suff.large\; t$$
$$\sum_{t=1}^{\infty} \eta_t = \infty$$
$$\lim_{(t\to\infty)} \frac{r_t}{\eta_t} = 0$$

This means that :

equation 28

$$\lim_{t\to\infty} x_t = x$$

Now we will show the proof.

Proof: Because of $\lim_{(t\to\infty)} \frac{r_t}{\eta_t} = 0$ we have:

equation 29

$$\eta_{t+1}\left((y_t - x_t) + \frac{r_{t+1}}{\eta_{t+1}}\right) \to \eta_{t+1}(y_t - x_t), t \to \infty$$
$$\Downarrow$$
$$x_{t+1} - x_t = \eta_{t+1}(y_t - x_t) \Rightarrow x_{t+1} = x_t\big(1 - \eta_{(t+1)}\big) + \eta_{t+1} y_t$$

We assume large N so that $|y_t - x| < \frac{\epsilon}{2}$ for some arbitrary $\epsilon > 0$ .Now if $x_t > x - \epsilon$ we will consider that:

equation 30

$$x_{t+1} = x_t(1 - \eta_{t+1}) + \eta_{t+1} y_t > x - \epsilon + \eta_{t+1}(\epsilon + y_t - x) > x - \epsilon$$

Also, if $x_t < x + \epsilon$ it implies that $x_{t+1} < x + \epsilon$ now if $x_t < x - \epsilon$, then we have:

equation 31

$$\sum_{t=N}^{\infty}(x_{t+1} - x_t) > \sum_{t=N}^{\infty} \eta_t + \frac{1}{2}\epsilon = \infty$$

On the left-hand side partial sum is: $x_{N+k} - x_N, \Rightarrow x_t \to \infty$ which contradicts $x_t < x - \epsilon$ and for a sufficiently large t,

equation 32

$$|x_t - x| \le \epsilon$$

Now, follows the proof that $\big(E(N_{k,t})\big)_{k\in\mathbb{N}}$ converges to power law distribution.

Theorem 1 In the previously mentioned random graph process $\mathcal{G}(m)$ following applies:

equation 33

$$E(N_{k,t}) \to \frac{2m(m+1)}{k(k+1)(k+1)}; t \to \infty ; k \ge m \ge 1$$

This theorem gives power law degree distribution with the exponent $\alpha = 3$.

Proof: consider sequence of graphs $\mathcal{G}(m) = (G_0, G_1, \dots,)$.This sequence supposed to be an event in the algebra $\mathcal{g}_t = \alpha(G_0, \dots, G_t)$. This algebra is also generated by the decomposition $\mathcal{D}_t = \mathcal{D}_{(N_{k,l})k\in N, l=1,\dots t}$so the conditional expectation with respect to $\mathcal{g}_t$ is given as:

equation 34

$$E(N_{k,t+1}|\mathcal{g}_t) = \sum_{d=0}^{m} N_{k-d,t}\binom{m}{d}\left(1 - \frac{k-d}{c_t t}\right)^{m-d}\left(\frac{k-d}{c_t t}\right)^{d} + \delta_{m,k}$$

Given the graph G_t with$N_{k-d,t}$ vertices of degree $k-d$,the expectednumberof vertices o degree k in G_{t+1} is :

equation 35

$$E(N_{k,t}) = N_{k-d,t}\binom{m}{d}\left(1-\frac{k-d}{c_t t}\right)^{m-d}\left(\frac{k-d}{c_t t}\right)^{d}$$

We need here to sum all d , and $\delta_{m,k}$ is Kronecker delta[13].Kronecker delta is equal to 1 if $m = k$ and 0 if $m \neq k$ .Next, we will consider the expression: $E\left(\frac{N_{t,k+1}}{t+1}|\mathcal{g}_t\right) - \frac{N_{k,t}}{t}$ that is given by:

equation 36

$$\frac{1}{t+1}\left(\sum_{d=0}^{m}\frac{N_{k-d,t}}{t}t\binom{m}{d}\left(\frac{c_t t-k-d}{c_t t}\right)^{m-d}\left(\frac{k-d}{c_t t}\right)^{d} + \delta_{m,k}(t+1) - \frac{N_{k,t}}{t}(t+1)\right)$$

By using the binomial formula one can find:

equation 37

$$\frac{m}{(t+1)c_t}\left(\frac{N_{k-1,t}}{t}(k-1) + \frac{c_t}{m}\delta_{m,k} - \frac{N_{k,t}}{t}\left(k+\frac{c_t}{m}\right) + \sum_{j=1}^{m}t^{-j}\sum_{d=0}^{j}C_{k,j,d,t}\frac{N_{k-d,t}}{t}\right)$$

Where in previous $C_{k,j,d,t}$ are bounded in t.Now,by the conditional expectations and linearity of expectations $x_{k,t} = E\left(\frac{N_{k,t}}{t}\right)$:

equation 38

$$x_{k+1,t+1} - x_{k,t} = \frac{m}{c_t(t+1)}\left(k+\frac{c_t}{m}\right)\left(\frac{k-1}{k+\frac{c_t}{m}}x_{k-1,t} + \frac{\frac{c_t}{m}}{k+\frac{c_t}{m}}\delta_{m,k} - x_{k,t}\right) + r_{t+1}$$

Now if $t \to \infty \Rightarrow x_{k,t} \to 0; 0 \leq k \leq m-1$.In the second case $k = m$ we have $\delta_{mm} = 1$now we will consider:

equation 39

$$x_t = x_{m,t} = E\left(\frac{N_{m,t}}{t}\right)$$

$$y_t = \frac{m-1}{m+\frac{c_t}{m}}x_{m-1,t} + \frac{c_t/m}{m+c_t/m}$$

$$\eta_{t+1} = \frac{m}{c_t(t+1)}\left(m+\frac{c_t}{m}\right)$$

All the conditions from lemma have been fulfilled and $y_t \to \frac{2}{m+2}$ so:

equation 40

$$x_{m,t} \to \frac{2}{m+2}$$

If we consider a case $k > m$, we have $\delta_{m,k} = 0$ so that expression for y_t will change:

equation 41

$$y_t = \frac{k-1}{k+\frac{c_t}{m}}x_{k-1,t}$$

Now, we can proceed to obtain by induction:

$$\lim_{t\to\infty} x_{k,t} = \frac{2}{m+2}\prod_{l=m+1}^{k}\frac{l-1}{l-2} = \frac{2m(m+1)}{k(k+1)(k+2)} \blacksquare$$

[13] Simplest definition is : $\delta = \begin{cases}0, for\ i \neq j\\1, for\ i = j\end{cases}$ In other words, the Kronecker delta is equal to 1 if its two arguments are equal, and 0 otherwise. It is commonly used in various branches of mathematics, particularly in linear algebra, analysis, and physics, where it often appears in expressing sums and products involving discrete indices.

3. Erdős-Rényi model

This model was introduced by Erdős,Rényi (1959). This model is the simplest model which used a probabilistic method to generate random graphs. This model provides a tunable expected edge density of graphs. In the model, a graph consists of N nodes, with any pair connected independently with probability p. Thus, an expected number of $pN(N-1) = 2$ edges are connected between these nodes, see Li, Aoxi (2011). For example, we can obtain the most direct feature of Erdős-Rényi model, the expected number of subgraphs of n nodes and l links is given as:

equation 42

$$N(n,l) = \binom{N}{n} p^l \times \frac{n!}{symmetry\ factors}$$

Degree distribution in a random graph with independent connection probability p, the degree k_i of a node i follows a binomial distribution with parameters $N-1$ and p:

equation 43

$$P(k_i = k) = C_{N-1}^{k} p^k (1-p)^{N-1-k}$$

The expected degree of each node is thus to be:

equation 44

$$\langle k \rangle = pN$$

In Erdős-Rényi model, the diameter of graph with parameter p and N is given by :

equation 45

$$d = \ln N / \ln pN = \frac{\ln N}{\ln\langle k \rangle}$$

The average path length is given by:

equation 46

$$l \sim \frac{\ln N}{\ln\langle k \rangle}$$

Clustering coefficient in Erdős-Rényi model is given by :

equation 47

$$C = p = \frac{\langle k \rangle}{N}$$

Clustering coefficient is :

equation 48

$$C = \frac{E(\#\{closed\ paths\ of\ length\ 2\})}{E(\#\{paths\ of\ length\ 2\})}$$

Where$\{closed\ paths\ of\ length\ 2\}$ and $\{paths\ of\ length\ 2\}$ are random variables defined in $\mathcal{G}(n,p)$.And $P(G) = p^m(1-p)^{N-m}$.And $\mathcal{G} = (\Omega, \mathcal{F}, P)$ where Ω is the sample space of all possible graphs on n vertices,$|\Omega| = 2^N = 2^{\binom{n}{2}}$.,$V = \{1,2.,\ldots,n\}$, $N \coloneqq \binom{n}{2}$.Now, if $np \to \lambda$ then degree distribution of Erdős-Rényi graph is Poisson:

equation 49

$$P\{X = k\} = \frac{\lambda^k}{k!} e^{-k}$$

Theorem 2 If p fixed $\mathcal{G}(n,p)$ has diameter 2 a.a.s

Proof: lets consider X_n number of vertex pairs in graph $G \in \mathcal{G}(n,p)$ on n vertices with no common neighbors. We have to show

equation 50

$$P\{X_n = 0\} \to 1, n \to \infty$$

Or switching the complementary events:

equation 51

$$P\{X_n \geq 1\} \to 0, n \to \infty$$

We have that :

equation 52

$$P\{X_n \geq 1\} \leq E(X_n)$$

Previous by Chebyshev inequality[14].We will now consider indicator function:

equation 53

$$X_n = \sum_{u,v \in V} \mathbf{1}_{u,v have no common neighbour}$$

If we apply the expectation:

equation 54

$$E(X_n) = \sum_{u,v \in V} P\{u,v\ have\ no\ common\ neighbour\} = \binom{n}{2}(1-p^2)^{n-2}$$

Previous approaches to zero when $n \to \infty$ ■

Theorem 3 Let $\alpha: \boldsymbol{N} \to \boldsymbol{R}$ be a function such that $\alpha(n) \to 0$; as $n \to \infty$, *let* $p(n) = \frac{\alpha(n)}{n}$; $\forall n \in N$, then $T_{3,n} = 0$ a.a.s.

Proof: The goal is to show $P\{T_{3,n} = 0\} \to 1; n \to \infty$.Now:

equation 55

$$P\{T_{3,n} = 1\} \to 0$$

$$P\{T_{3,n} \geq 1\} \leq E(T_{3,n}) - Markov\ inequality$$ [15]

For each fixed n the random variable $T_{3,n}$ can be represented as:

equation 56

$$T_{3,n} = \mathbf{1}_{\tau_1} + .. + \mathbf{1}_{\tau_k}, k = \binom{n}{3}$$

Where τ_i is an event that the ith triple of vertices from the set of all vertices $\mathcal{G}(n,p)$ forms a triangle, by the linearity of expectation:

equation 57

$$E(T_{3,n}) = E(\mathbf{1}_{\tau_1}) + \cdots . + E(1_{\tau_k}) = P(\mathbf{1}_{\tau_1}) + \cdots . + E(\mathbf{1}_{\tau_k}) = \binom{n}{3}p^3$$

Since $P\{\tau_i\} = p^3$ in Erdős-Rényi graphs $\mathcal{G}(n,p)$ we have:

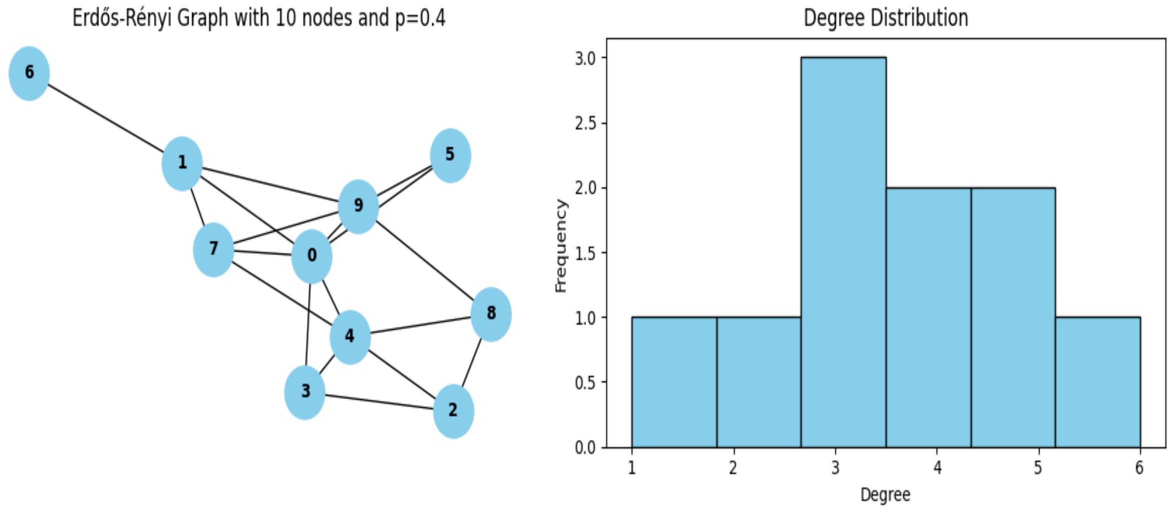
$$E(T_{3,n}) = \binom{n}{3}p^3 = \frac{n!}{(n-3)!\,3!}\frac{\alpha^3(n)}{n^3} \sim \frac{\alpha^3(n)}{6} \to 0 ■$$

Here we will plot Erdős-Rényi graph with 10 nodes and $p = 0.4$

[14] $P(|x-\mu| \geq k) \leq \frac{\sigma^2}{k}$ where k number of standard deviations from the mean

[15] $P(X \geq a) \leq \frac{EX}{a}$; $Ex = \int_0^\infty x f_x(dx) \geq \int_a^\infty a f_X(x)dx = a\int_a^\infty f_X(x)dx = aP(X \geq a)$

Figure 4 Erdős-Rényi graph with with 10 nodes and p=0.4 and degree distribution



Source: Author's own calculations

4. Akerlof (1997) model of social distance

This part here is completely due to Akerlof (1997).
Utility function is given as:
equation 58

$$U = -d(x - \bar{x}) - ax^2 + bx + c$$

Here person loses utility $d(x - \bar{x})$ as she falls behind everyone else in her choice of x and where x-*status producing variable ;* $\bar{x}$-choice of everyone else and :
equation 59

$$x = \frac{b + d}{2a}$$

Here optimum is exceeded by $\frac{d}{2a}$ in the competitive race for status. Status seeking people fail to take full account of the consequences of their own social positioning on the welfare of their friends and relatives, just as fishermen fail to internalize the effect of their behavior on the availability of fish for others ,see Akerlof (1997).Now in the Conformist model (the alternative case-of conformity-in which the individual wants to minimize the social distance between herself and others), here agent losses utility $-d|x - \bar{x}|$ from failing to conform with others,x has additional intrinsic utility $-ax^2 + bx + c$
equation 60

$$U = -d|x - \bar{x}| - ax^2 + bx + c$$

$-d|x - \bar{x}|$*agent loses utility of failing to conform with the others* $-ax^2 + bx + c$*-intrinsic utility*
equation 61

$$x = \bar{x}\ (since\ everyone\ is\ alike)$$

There are multiple values of x as long as $d > 0$,d is parameter describing the taste for conformity

$$eqilibirium\ values\ of\ x\ \in \left(\frac{b - d}{2a}; \frac{b + d}{2a}\right)$$

$\frac{\mathrm{b-d}}{\mathrm{2a}} < \mathrm{x} < \frac{\mathrm{b+d}}{\mathrm{2a}}$ a marginal change in one of the parameters that affect the utility a, b, c, d will have no equilibrium effect on x . With conformity, the tendency to mimic the status quo can result in either underproduction or overproduction of x, in amount ranging from$-\frac{\mathrm{d}}{\mathrm{2a}} \to \frac{\mathrm{d}}{\mathrm{2a}}$. As the distance between the representative individual and others goes to zero, the marginal utility of moving closer, in the utility function given by $U = -d|x - \bar{x}| - ax^2 + bx + c$,does not fall to zero. Hence quadratic utility
equation 62

$$U = -ax^2 + bx + c - d(x - \bar{x})^2$$

$x = \frac{b}{2a}$-optimum value. Now to introduce heterogeneity in social interactions, Akerlof (1997) lets individuals occupy different locations in social space. Social interaction, which is represented as mutually beneficial trade between individuals, will increase with proximity in this space. In apure gravity model the trade between two countries is proportional to the GNP's of the respective countries (analogous to their respective mass) and inversely proportional to the square of the distance between them:

equation 63

$$T_{ij} = \frac{Y_i^{\alpha} Y_j^{\beta}}{D_{ij}}$$

In this models of social distance, social exchange depends jointly on the differences between peoples' current positions and also their inherited positions. A formulation that incorporates both of these desirable modifications to the pure gravity model assumes that trade depends on the inverse of the product of a constant plus the inherited social distance and a constant plus the acquired social distance. Now for the benefits from trade and location ,

equation 64

$$expected\ value\ of\ benefits\ of\ trade\ between\ i\ and\ j = \frac{e}{(f + d_{0,i,j})(g + d_{1,i,j}^{e})}$$

Where :
$d_{0,i,j}$-initial social distance between i and j.
$d_{1,i,j}^{e}$- expected final social distance between i and j.
Intrinsic value of $x = -ax^2 + bx + c$ (eg. Education)
x_{1i} is contingent upon x_{0i} .
The problem confronting each individual i is to choose x_{1i} contingent on her initial social position, x_{0i}. In order to make this decision the individual must form expectations about the position of her potential trading partners in social exchange. Many outcomes are possible depending upon how these expectations are formed. The simplest assumption is static expectations that the acquired social position of all the other individuals will coincide with their initial position. With such static expectations about social position, $d_{1,i,j}^{e}$, i's expected acquired distance between herself and j will be $|x_{1i} - x_{0j}|$.[16]Each respective agent i chooses the respective value of x_{1i} to maximize.

equation 65

$$U_i = \sum_{j \neq i} \frac{e}{[(f + |x_{0,i} - x_{0,j}|) \cdot (g + |x_{1,i} - x_{1,j}|)] + [-ax^2 + bx + c]}$$

Many possible equilibria are attainable. There person game is described as follows: Person 1 chooses x at the initial position of person 2 and person 2 chooses x at the initial position of person 1 if $x_{01} \sim x_{02}$, if x_{03} is sufficiently distant, and if the value of social exchange relative to the marginal intrinsic value of x is sufficiently high. And, if person 3, who is socially distant from persons 1 and 2, does not much value trade with persons 1 or 2 she will choose a value of x that is close to the economic optimum value of $\frac{b}{2a}$. Consider person 1's choice of x_{11}. We shall show that under the appropriate conditions it will be chosen at x_{02}. This variable will be chosen at the point where the derivative of U_1 turns from positive to negative. The derivative is well-defined at all but the two points, $x_{11} = x_{02}$ and $x_{11} = x_{03}$, where instead there are left-hand and right-hand derivatives, but of different magnitudes, see Akerlof (1997).Now according to $U_i = \sum_{j \neq i} \frac{e}{[(f+|x_{0,i}-x_{0,j}|)\cdot(g+|x_{1,i}-x_{1,j}|)]+[-ax^2+bx+c]}$ in the interval $x_{11} < x_{02}$

[16] Jones' (1984) model of tradition,had a similar assumption-that half the population (of workers) is new in each generation-plays a similar role, as each new generation finds itself conforming to the traditions of the older, inflexible half of the population.

equation 66

$$U_1 = [e/(f + (x_{02} - x_{01})]\left[\frac{1}{(g - (x_{11} - x_{02}))}\right] + [e/(f + (x_{03} - x_{01})]\left[\frac{1}{(g - (x_{11} - x_{03}))}\right] - ax_{11}^2 + bx_{11} + c$$

Note the negative quantities $x_{11} - x_{02} < 0$; $x_{11} - x_{03} < 0 \quad x_{11} < x_{02}; x_{11} < x_{03}$.Now differentiating previous eq. we find:

equation 67

$$\frac{\partial U_1}{\partial x_{11}} = [e/(f + (x_{02} - x_{01})]\left[\frac{1}{(g - (x_{11} - x_{02}))^2}\right] + [e/(f + (x_{03} - x_{01})]\left[\frac{1}{(g - (x_{11} - x_{03}))^2}\right] + [-2ax_{11} + b]$$

In this range each bracket terms is positive. Now, $[-2ax_{11} + b] > 0$ because is the description of x_{01}, x_{02} , 1 and 2 persons were underinvesting in x.In consequence $x_{11} \geq x_{02}$.So , $x_{01}; x_{02}$ are underinvesting in x .For the range $x_{02} < x_{11} < x_{03}$ the value of U_1 is :

equation 68

$$U_1 = [e/(f + (x_{02} - x_{01})]\left[\frac{1}{(g - (x_{11} - x_{02}))}\right] + [e/(f + (x_{03} - x_{01})]\left[\frac{1}{(g - (x_{11} - x_{03}))}\right] - ax_{11}^2 + bx_{11} + c$$

And in this range:

equation 69

$$\frac{\partial U_1}{\partial x_{11}} = -[e/(f + (x_{02} - x_{01})]\left[\frac{1}{(g + (x_{11} - x_{02}))^2}\right] + [e/(f + (x_{03} - x_{01})]\left[\frac{1}{(g - (x_{11} - x_{03}))^2}\right] + [-2ax_{11} + b]$$

Let’s note that the sign of the first square-bracketed term changes at $x_{11} = x_{02}$ from positive at x_{02}^{-} negative at x_{02}^{+}. If the distance between x_{03} and x_{01}, is sufficiently large, and if the intrinsic value of x is sufficiently small relative to the value of the exchange, then the first term dominates the sign of $\frac{\partial U_1}{\partial x_{11}} = -[e/(f + (x_{02} - x_{01})]\left[\frac{1}{(g+(x_{11}-x_{02}))^2}\right] + [e/(f + (x_{03} - x_{01})]\left[\frac{1}{(g-(x_{11}-x_{03}))^2}\right] + [-2ax_{11} + b]$ and:

equation 70

$$\frac{\partial U_1}{\partial x_{11}} < 0 ; x_{11} = x_{02}^{+}$$

$$\frac{\partial U_1}{\partial x_{11}} > 0 ; x_{11} = x_{02}^{-}$$

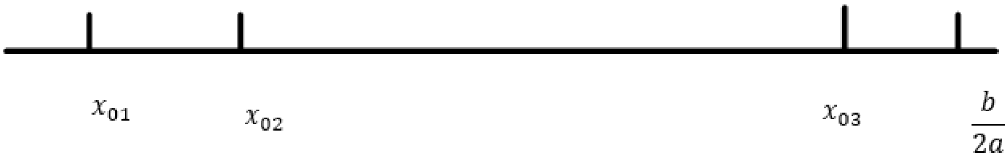
Previous results apply for $x_{02} < x_{11} < x_{03}$ and $x_{11} > x_{03}$ respectively. For $x_{11} > x_{03}$ the value of utility is :

equation 71

$$U_1 = [e/(f + (x_{02} - x_{01})]\left[\frac{1}{(g + (x_{11} - x_{02}))}\right] + [e/(f + (x_{03} - x_{01})]\left[\frac{1}{(g + (x_{11} - x_{03}))}\right] - ax_{11}^2 + bx_{11} + c$$

There is no guarantee that $\frac{\partial U_1}{\partial x_{11}}$is negative in this entire range $x_{11} > x_{03}$ when it is negative for $x_{02} < x_{11} < x_{03}$; nevertheless, if the marginal value of intrinsic utility, $-2ax_{11} + b$, is sufficiently small, $\frac{\partial U_1}{\partial x_{11}}$will be negative throughout this region. It was found here that if the intrinsic value of x is sufficiently small relative to the value of social exchange, and if 1 and 2 are sufficiently distant from 3 and also from $\mathrm{b}/2\mathrm{a}$, the optimal value of $\mathrm{x}_{11} = \mathrm{x}_{02}$.

Figure 5 Three person model inherited social positions of three persons, person three 3 initial position is close to optimum $\frac{b}{2a}$



Source: Akerlof (1997)
There is one stable solution in which 1 and 2 will exchange each other's positions while 3 will choose a point that is close to the economic optimum, only slightly influenced by the possibilities of trade with 1 and 2, because they are socially distant. This was proven previously.

5. Model of new veracity in social networks (Kranton (2020))

In this model by Kranton,McAdams (2020), consumer choose whether to pass or not the information unlike much of the network literature on information diffusion (Acemoglu, Ozdaglar,Parandeh,Gheibi (2010) ; Banerjee et al.(2013)). The market for decision-relevant information, which we refer to as news, consists of a large finite number N of consumers, of whom M generate revenue for producers. Producers are modeled as a unit-mass continuum of agents, but the analysis applies equally to a setting with finitely-many producers or even a single identifiable producer, as long a s producers lack commitment power, see Kranton,McAdams (2020). Low quality stories are costless to produce and are false with probability 1; high-quality stories entail a reporting cost $c_R > 0$ and are true with probability one[17].Each news consumer (i,j) follows d others and d is what is referred a social connectedness. Timing of game in this network are three phases:$t = \{0,1,2\}$. At $t = 0$ dependent on cost each producer decides whether to produce high- or low-quality story. Broadcast or probability that the news would be seen is: $b \in \{0,1\}$.Now, p_0 is the probability that the news is true which will be referred as news veracity. At $t = 1$, each consumer who saw a story's broadcast decides whether to share the story with her neighbors. Consumers have personal experience in evaluating stories whether they are true or false $s_i \in \{T,F\}$ signal may be true or false. These signals are informative:

equation 72

$$\Pr(s_i = T|true) = \Pr(s_i = F|False) = \rho \in \left(\frac{1}{2},1\right)$$

Consumer payoff for sharing story is $\pi_T^s > 0$; and from sharing false story payoff is given as: $-\pi_F^s < 0$, and zero payoff from not sharing a story. Sharing threshold is given as:

equation 73

$$p^s = \frac{\pi_F^s}{\pi_F^s + \pi_T^s} \in (0,1)$$

$$\pi_T^s = \pi_F^s \Rightarrow p^s = \frac{1}{2}$$

At $t = 2$, consumers view the stories shared by their neighbors and each consumer who has seen a story decides whether to take an action based on it, earning $\pi_T^A > 0$ when acting on a true story, $-\pi_F^A < 0$ when acting on a false story, and zero payoff when not acting. Action threshold is given as:

equation 74

$$p^A = \frac{\pi_F^A}{\pi_T^A + \pi_F^A} \in (0,1)$$

$$\pi_T^A = \pi_F^A \Rightarrow p^A = \frac{1}{2}$$

Producers' incentive to invest depends on the extra visibility of true news, denoted:

[17] The cost c_R is an i.i.d variable with continuous distribution $F(c_R)$ and support $(0,\infty)$.

equation 75

$$\Delta V \equiv V_T - V_F .$$

$V_T; V_F$ denotes visibility of news. Optimal consumer sharing is given as:

equation 76

$$p_1(T: p_0) = \frac{p_0\rho}{p_0\rho + (1-p_0)(1-\rho)}$$
$$p_1(F: p_0) = \frac{p_0(1-\rho)}{p_0(1-\rho) + (1-p_0)\rho}$$

In previous given private signal $s_i = T; s_i = F$ updated beliefs $p_1(s_i: p_0)$ are given as previous. Now let z_T, z_F denote each consumer likelihood of sharing after private signal $s_i \in (T, F)$ and $z \in (z_T, z_F)$.

Lemma 2 If $p_0 < 1-\rho \Rightarrow Z(p_0) = (0,0)$, if $p_0 > p$ then $Z(p_0) = (1,1)$, if $p_0 \in (1-\rho, \rho)$ then $Z(p_0) = \tilde{z} \equiv (1,0)$,$Z(1-\rho) = \{(z_T, 0: z_T \in [0,1]\}$,$Z(\rho) = \{(1, z_F): z_F \in [0,1]\}$

visibility of false and true story is given as:

equation 77

$$V_T(z) = 1 - (1-b)(1 - b(\rho z_T + (1-\rho)z_F)^d$$
$$V_F(z) = 1 - (1-b)(1 - b\big((1-\rho z_T) + \rho z_F\big)^d$$

each neighbor shares true stories with probability $\sqrt[d]{(\rho z_T + (1-\rho)z_F)^d}$;and false stories with probability: $b\big((1-\rho z_T) + \rho z_F\big)$. Next, we will show in table four regions: Always share, never share, filtering and threshold region[18].

Table 1 News-veracity regions and optimal consumer sharing

	Prior p_0	Explanation
Always share region	$p_0 \in (\rho, 1)$	If news veracity is high enough, consumers find it optimal to share news after a good and after a bad signal ;$p_1(T: p_0) = p_1(F; p_0) = \frac{1}{2}$;sharing is uninformative
Never share region	$p_0 \in (0,1-\rho)$	If news veracity is low enough, consumers find it optimal never to share news, since both$p_1(T: p_0) < \frac{1}{2}$; $p_1(F; p_0) = \frac{1}{2}$
Filtering region	$p_0 \in (1-\rho, \rho)$	if news veracity is in this intermediate range, consumers find it optimal to share after a good signal because $p_1(T: p_0) > \frac{1}{2}$; but find it optimal not to share after a bad signal because $p_1(F; p_0) > \frac{1}{2}$. Sharing here is informative and we say that consumers \filter" the news
Threshold region	$p_0 \in (1-\rho, \rho)$	If news veracity is exactly $p_0 = \rho$, what we call the always-share threshold, consumers are indifferent whether to share after seeing a bad signal ($p_1(F ; p_0) = \frac{1}{2}$) and hence use a sharing rule of the form $\boldsymbol{z} = (1; z_F)$. if news veracity is $p_0 = 1 - \rho$, the never-share threshold, consumers are indifferent after seeing a good signal ($p_1(T ; p_0) = \frac{1}{2}$) and use a sharing rule of the

[18] ρ denotes signal precision

		form $\boldsymbol{z} = (z_T\ ;\ 0)$

Source: Kranton,McAdams (2020)
With M revenue-generating consumers, expected revenue from true and false stories is given as:
equation 78

$$R_T(z) = MV_T(z)$$
$$R_F(z) = MV_F(z)$$

True stories earn a revenue premium:
equation 79

$$\Delta R(z) = M\Delta V(z)$$
$$\Delta V\ (z) \equiv V_T(z) - V_F(z)$$

Producers maximize expected profit by producing high quality whenever $c_R < M\Delta V(z)$ which occurs with ex ante CDF $F\big(MV(z)\big)$,and best nes veracity $p_0(z)$:
equation 80

$$p_0(z) = F\big(MV(z)\big)$$

$c_R = M\Delta V(z)$ occurs with $F\big(MV(z)\big) = 0$[19].Filtering of news veracity means:
equation 81

$$\tilde{p}_0 \equiv p_0(1,0) = F\big(M\Delta V(1,0)\big)$$

There are two types of equilibrium on this market. In a dysfunctional equilibrium, producers never invest, all stories are false, and consumers never share. In a functional equilibrium, producers sometimes invest, consumers sometimes share, and some stories are true. Dysfunctional equilibrium always exists, and functional equilibrium exist only if :
equation 82

$$\bar{p}_0 > 1 - \rho$$

In functional equilibrium:
equation 83

$$p_0^* = \max\{1 - \rho, \min\{\tilde{p}_0, \rho\}\}$$

distribution of producers' reporting costs when scaled by a parameter γ is given as:
equation 84

$$F(c_R;\gamma) = F\left(\frac{c_R}{\gamma}\right); \forall c_R > 0$$

filtering news veracity and maximal news veracity are given as:

equation 85

$$\tilde{p}_0(\gamma) \equiv F\left(\frac{M\Delta V(1,0)}{\gamma}\right)$$
$$\bar{p}_0(\gamma) \equiv F\left(\frac{M\Delta V(\bar{z}_T,0)}{\gamma}\right)$$
$$\tilde{p}_0(\gamma) < \bar{p}_0(\gamma); \forall\gamma$$
$$0 < \gamma_1 < \gamma_2 \leq \gamma_3$$
$$\tilde{p}_0(\gamma_1) = \rho\ ; \tilde{p}(\gamma_2) = 1 - \rho, \bar{p}(\gamma_3) = 1 - \rho$$

About the misinformation: From distribution $F(\cdot)$we suppose that there are $m \geq 0$ mass of misinformation agents. Now, let a previously:$\tilde{p}_0(m)$, $\bar{p}_0(m)$, $p_0^*(m)$ denote filtering news veracity,

[19] If the cumulative density function (CDF) is equal to zero at a particular point, it means that the probability of the random variable being less than or equal to that point is zero. In other words, the random variable cannot take on a value less than or equal to that point.For continuous random variables, such as in the case of probability distributions like the normal distribution or the exponential distribution, the probability of a specific point is technically zero. Therefore, having a CDF equal to zero at a specific point is not uncommon and simply indicates that the probability of the random variable being less than or equal to that point is extremely low.

maximal news veracity, and equilibrium news veracity, as functions of the quantity of misinformation, The share of news that is true, the filtering veracity and maximal veracity are given as :
equation 86

$$p_0(z,m) = \frac{p_0(z)}{1+m} - share\ of\ true\ news$$
$$\tilde{p}_0(m) = \frac{\tilde{p}_0}{1+m} - filtering\ veracity$$
$$\bar{p}_0(m) = \frac{\bar{p}_0}{1+m} - maximal\ veracity$$

We define three thresholds here :
equation 87

$$m_1 = \frac{\tilde{p}_0}{\rho} - 1$$
$$m_2 = \frac{\tilde{p}_0}{1-\rho}$$
$$\bar{m} = \frac{\bar{p}_0}{1-\rho} - 1$$

Now;

$$\tilde{p}_0(m_1) = \rho$$
$$\tilde{p}_0(m_2) = 1-\rho$$
$$\bar{p}_0(\bar{m}) = 1-\rho$$

So , $\tilde{p}_0(m) > \rho$ if and only if $m < m_1$, $\tilde{p}_0(m) > 1-\rho$ if and only if $m < m_2$, and $\bar{p}_0(m) > 1-\rho$ if and only if $m < \bar{m}$.
Proposition 1 A functional equilibrium ∃ if and only if $m < \bar{m}$,equilibrium news veracity $p^*\_0(m)$ is non-decreasing over the range $m < \bar{m}$ and strictly decreasing over range $m \in (m_1, m_2)$,the quantity of true news $(1+m)p_0^*(m)$ is nondecreasing over the range $m < \bar{m}$ and strictly increasing over the ranges $m \in (0, m_1)$ and $m \in (m_2, \bar{m})$.
Next, this model makes proposition about deep fake technology that gaslights the consumers[20].This is done by decreasing the signal precision $\rho \to \rho'$ .First, consumers switch form always sharing to filtering if $p_0 \in (\rho, \rho')$ or switch from filtering to never sharing if $p_0 \in (1-\rho, 1-\rho')$. Second, holding fixed p_0, a consumer who only shares after a good signal will share more false stories and fewer true stories, third dropping signal quality $\rho \to \rho'$ leads to less true news shared and viewed, giving producers less incentive to produce true news.
Proposition 2 Signal-precision thresholds $\frac{1}{2} < \underline{\rho} \leq \bar{\rho} \leq 1$ exist such that if $\rho \in \left(\frac{1}{2}, \underline{\rho}\right)$, then the dysfunctional equilibrium is the unique equilibrium, now if $\rho \in \left(\underline{\rho}, \bar{\rho}\right)$ then there is a unique functional equilibrium, $p_0^*(\rho) = 1-\rho$, and $p_0^*(\rho)$ is decreasing in ρ over that range, if $\rho \in (\bar{\rho}, 1)$ then there is a unique functional equilibrium, $p_0^*(\rho) = min\{\tilde{p}_0(\rho), \rho\} > 1-\rho$ and $p_0^*(\rho)$ is increasing over this range.
So, if equilibrium news veracity is high enough that consumers always share after a good private signal, i.e., $p_0^*(\rho) > 1-\rho$, then reducing consumers' ability to discern which stories are true causes news veracity to fall. On the other hand, if $p_0^*(\rho) = 1-\rho$, so that consumers are indifferent whether to share after a good signal, slightly reducing consumers' ability to discern the truth causes news veracity to increase. "Finitely dense networks" would likely refer to networks that have a finite number of nodes and edges and exhibit a certain level of density, where the density could vary depending on the specific context or definition used. Social connectedness we remember was d.If $d = 0$ all stories are seen with broadcast probability b; true stories have no extra visibility. Producers therefore have no incentive to invest, and $\tilde{p}_0(0) = 0$. When consumer follows one person $d = 1$ this link increases the consumer's

[20] To gaslight someone means to manipulate another person into doubting their own perceptions, experiences or understanding of events, according to the American Psychological Association

likelihood of viewing any given story by $(1-b)b\rho$ if the story is true or by $(1-b)b(1-\rho)$ if the story is false. The extra visibility of true stories therefore increases from $0 \rightarrow (1-b)b(2\rho-1)$. Given that $\tilde{p}_0(d)$ is single peaked in d, there exist thresholds $\underline{d}$ and $\bar{d}$ such that $\tilde{p}_0(d) > 1-\rho$ if and only if $d \in \{\underline{d}, \bar{d}\}$.

Proposition 3 $\exists\ 0 \leq \underline{d} \leq \bar{d} < \infty$ such that: $p_0^*(d) > 1-\rho$ if and only if $d \in \{\underline{d}, \bar{d}\}$; $p_0^*(d)$ is single peaked in d over this range, and $p_0^*(d)$ is maximized at $d = \bar{d}$,$\forall d \leq \underline{d}$ and $d \geq \bar{d}$ either the dysfunctional equilibrium is the unique equilibrium or a unique functional equilibrium exists with news veracity equal to $1-\rho$.

Now, about infinitely dense networks. Consider next the limit of a sequence of news markets, each having the same number M of revenue-generating consumers but with social connectedness $d \rightarrow \infty$, this is limit market. Let :

equation 88

$$\underline{\rho}^{\infty} \equiv \lim_{d\rightarrow\infty} \underline{\rho}(d)$$

Previous is lower signal precision. If $p_0^*(d)$ is news veracity in the unique functional equilibrium $\forall d$,if it exists or $p_0^*(d) = 0$ if no functional equilibrium exists, let $p_0^{*\infty} = \lim_{d\rightarrow\infty} p_0^*(d)$

Proposition 4 If $\rho < \underline{\rho}^{\infty}$ then $p_0^{*\infty} = 0$ and the limit-market is dysfunctional. Now if $\rho > \underline{\rho}^{\infty}$ then the limit market has a unique functional equilibrium $p_0^{*\infty} = 1-\rho$

Cost parameter thresholds are given as:

equation 89

$$F\left(\frac{M}{\bar{\gamma}}\right) = 1-\rho$$
$$F\left(\frac{M}{\underline{\gamma}}\right) = \rho$$

he thresholds $\bar{\gamma}$ and $\underline{\gamma}$ capture how high news veracity could conceivably be: greater than ρ if $\gamma < \underline{\gamma}$ the case of low costs);in the interval $(1-\rho;\ \rho]\ if\ \gamma \in (\underline{\gamma}; \bar{\gamma})$,intermediate costs case ; or $\leq 1-\rho$ if $\gamma \geq \bar{\gamma}$ high cost.Now about the impact of misinformation, the actual news veracity falls:

equation 90

$$F\left(\frac{M}{\gamma}\right) \rightarrow \frac{F\left(\frac{M}{\gamma}\right)}{1+m}$$

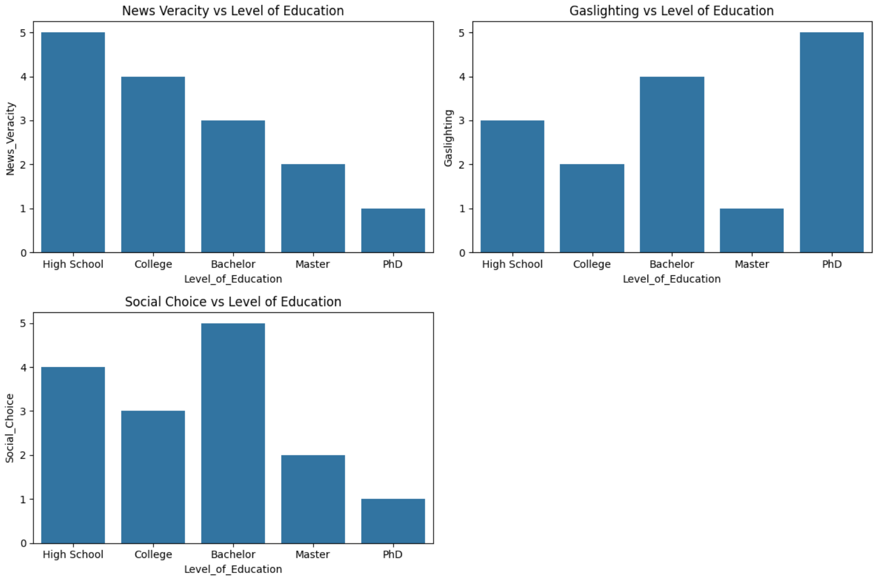
Now the thresholds are given as:

equation 91

$$\frac{F\left(\frac{M}{\underline{\gamma(m)}}\right)}{1+m} = 1-\rho$$
$$\frac{F\left(\frac{M}{\bar{\gamma}(m)}\right)}{1+m} = \rho$$

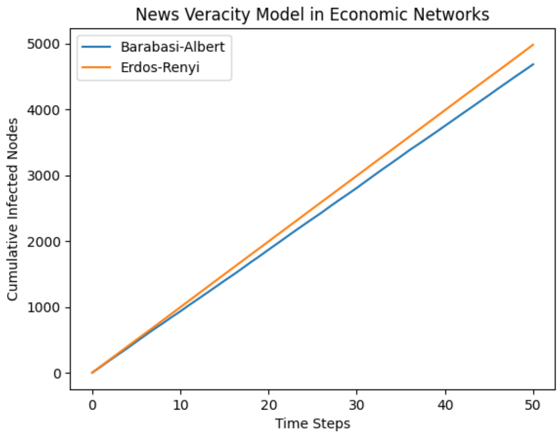
Theorem 4 In an action supported limit market first if $\gamma \geq \bar{\gamma}$ the unique equilibrium is dysfunctional ,if $\gamma \in \left[\underline{\gamma}, \bar{\gamma}\right]$ then there is unique functional equilibrium $p_0^{*A\infty} = F\left(\frac{M}{\gamma}\right)$ and there is perfect learning by consumers and if $\gamma < \underline{\gamma}$ then there exists a functional equilibrium with news veracity $p_0^{*A\infty} = \rho$ and some imperfect learning by consumers. Next will plot the results of our code in Python for news veracity vs level of education, gaslighting vs level of education and social choice vs level of education.

Figure 6 news veracity vs level of education, gaslighting vs level of education and social choice vs level of education



Source: Author’s own calculation

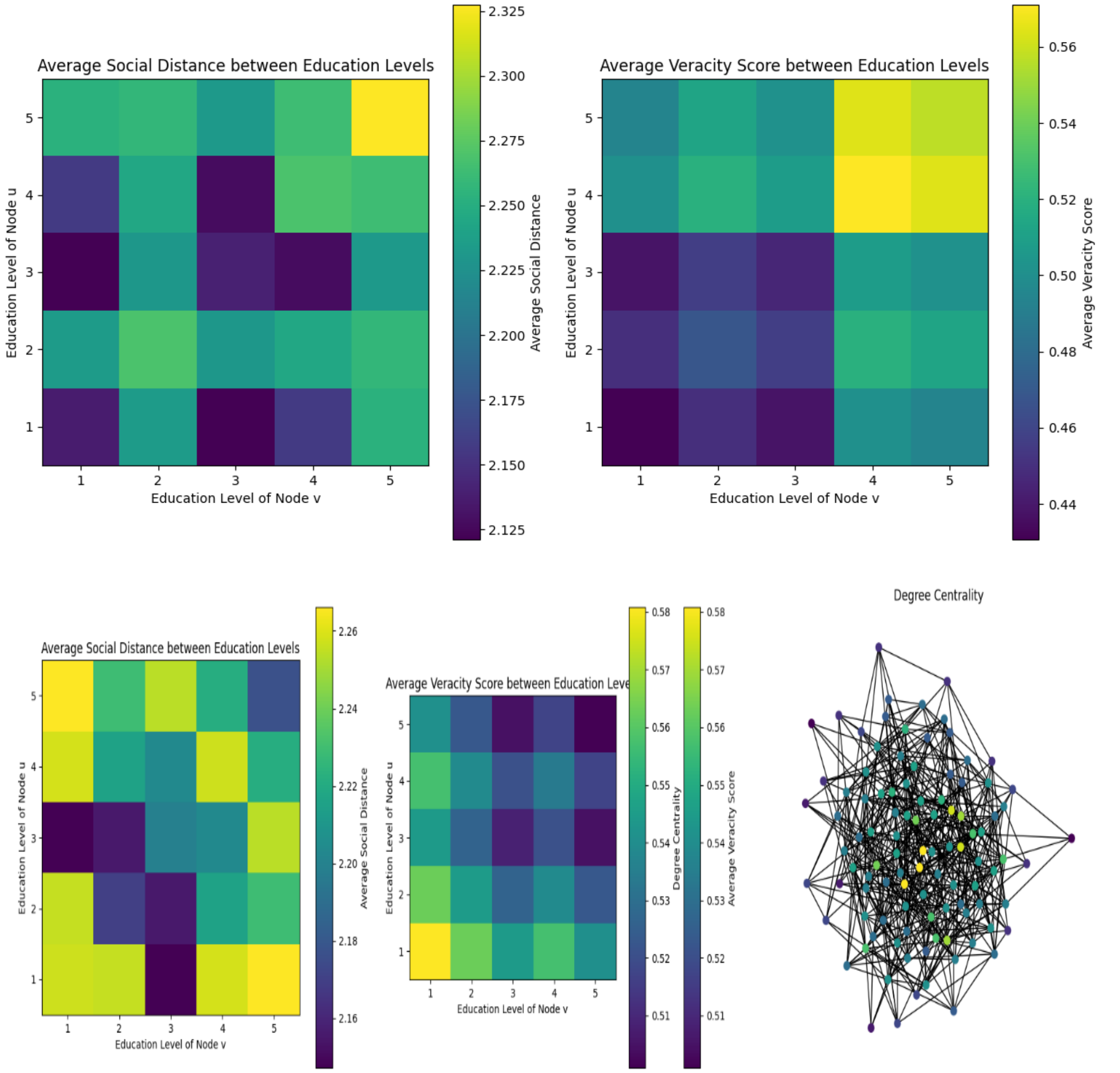
Figure 7 new veracity in Barabási–Albert and Erdös- Rényi(1959) model



Source: Author’s own calculation

Next, will be shown a heatmaps of social distance between education levels and average veracity between education levels, and degree centrality.

Figure 8 Heatmap of social distance between education levels and average veracity between education levels



Source: Author's own calculation

This code calculates and plots the relationship between education level, social distance, news veracity, and centrality in a social news network. It plots heatmaps for average social distance between education levels, average veracity score between education levels, and degree centrality of nodes in the network. The color intensity in each heatmap represents the corresponding metric's value for each node in the network.

6. Prospect theory due to Kahneman,Tversky (1979)

Prospect theory is a critique of the expected utility theory as a decision-making model under risk and was introduced in a paper published in Econometrica 1979 by Daniel Kahneman and Amos Tversky titled "Prospect Theory: An Analysis of Decision under Risk". Decision-maker has a reference point x_0, weights gains and losses relative to x_0 differently. Now the reference dependent utility function is given as:

equation 92

$$u(x|x_0) = v(x - x_0)$$

Where in previous expression v satisfies:

- Concavity on $\mathbb{R}_+$ (risk aversion towards gains)

- Convexity on $\mathbb{R}_-$ (risk lowing toward losses)
- Kink at 0 (loss-aversion)

Lets consider a game with two possible outcomes : x with probability p and y with probability $1-p$,where $x \geq 0 \geq y$. The prospect theory value of the game is :

equation 93

$$V = \pi(p)u(x) + \pi(1-p)u(y)$$

In prospect theory the probability of weighting π is concave and first order convex ,e.g.

equation 94

$$\pi^\beta = \frac{p^\beta}{p^\beta + (1-p)^\beta}$$

For some $\exists \beta \in (0,1)$.A useful parametrization of the prospect theory value function is a power law function

equation 95

$$u(x) = |x|^\alpha \,; x \geq 0$$
$$u(x) = -\lambda |x|^\alpha ; x \leq 0$$

Expected-utility theory predicts that people are not confused by the frame of wealth. Prospect theory predicts that people are regularly confused. Consider gambles with two outcomes: x with probability p , and y with probability $1-p$ where $x \geq 0 \geq y$. Expected utility (EU) theory says that if you start with wealth W then the (EU) value of the gamble is given as:

equation 96

$$V = pu(W+x) + (1-p)u(W+y)$$

Prospect theory (PT) says that the (PT) value of the game is as:

equation 97

$$V = \pi(p)u(x) + \pi(1-p)u(y)$$

Where π is probability weighting function, defined as previous $\pi^\beta = \frac{p^\beta}{p^\beta+(1-p)^\beta}$,and $\beta \in (0,1)$ .And now $\pi(p) > p$ for small p. Small probabilities are overweighted, and too salient. E.g. people play a lottery. Empirically, poor people and less educated people are more likely to play lottery. Extreme risk aversion. Second, $\pi(p) < p$ for $p \sim 1$, large probabilities are underweight. In economics $\pi(p) = p$ is concept often used in insurance and lotteries. About the utility function here, we assume that $u(x)$ is increasing in x , is convex for losses, concave for gains, and first order concave at 0 that is :

equation 98

$$\lim_{x \to 0+} \frac{-u(-x)}{u(x)} = \lambda > 1$$

A useful parametrization would be:

equation 99

$$u(x) = x^\beta \,; x \geq 0$$
$$u(x) = -\lambda |x|^\beta \,; x \leq 0$$

Cumulative prospect theory (cumulative PT) , for continuous gambles with distribution $f(x)$
Expected utility (EU) gives:

equation 100

$$V = \int_{-\infty}^{+\infty} u(x) f(x) dx$$

PT gives:

equation 101

$$V = \int_{0}^{+\infty} u(x)f(x)\pi'(x)\big(P(g \geq x)\big)dx + \int_{-\infty}^{0} u(x)f(x)\pi'\big(P(g \leq x)\big)dx$$ [21]

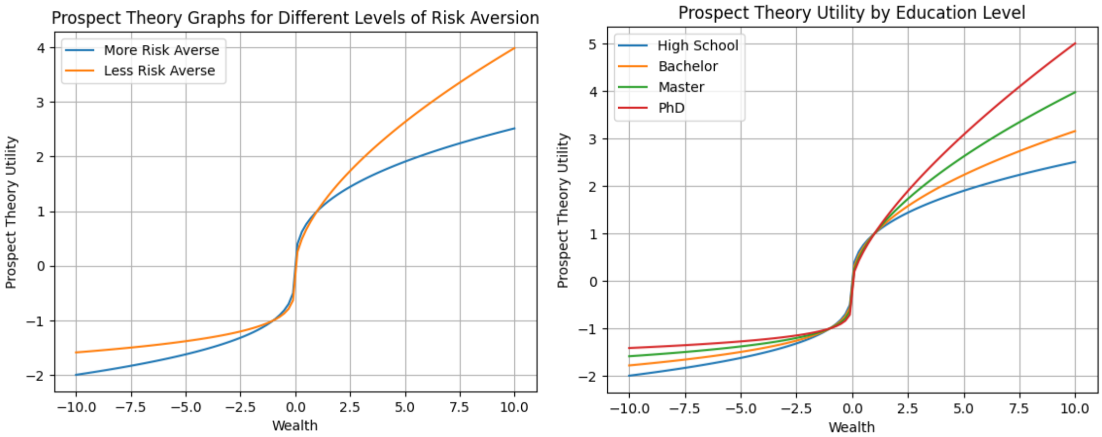
Kahneman, Knetsch, Thaler, (1991), show that in expected utility $WTP = WTA$ or willingness to pay is equal to willingness to accept . Or as in Horowitz, Mcconnell (2003):

equation 102

$$\frac{\partial WTP}{\partial y} \approx 1 - \frac{WTP}{WTA}$$

$\frac{\partial WTP}{\partial y}$ is labeled as the income effect. Horowitz and McConnell (2002) found that WTA is about seven times higher than WTP. Hanemann, (1991), showed that the difference between WTP and WTA depends on the ratio of ordinary income elasticity of demand for the good with respect to Allen-Uzawa elasticity [22]of substitution between the good and a composite commodity. Since $\frac{\partial x_i}{\partial p_j} = \frac{\partial^2 \lambda}{\partial p_i \partial p_j}$ then $\sigma_{ij} = \frac{\lambda^2 \frac{\partial^2 \lambda}{\partial p_i \partial p_j}}{\frac{\partial \lambda}{\partial p_i}\frac{\partial \lambda}{\partial p_j}}$.And now aggregate Allen-Uzawa elasticity of substitution between consumption denoted by q and the Hicksian composite commodity[23] $x_0 \equiv \sum \overline{p}_i x_i$ will be denoted σ_0 . Following Diewert (1974) , a formula that relates ξ which is the income elasticity and σ_0 compensated own price elasticity for the commodity consumption q or $\varepsilon = -\sigma_0(1-\alpha)$ which is price demand elasticity. Next, we will plot level of education with different levels of risk aversion and social distance versus social choice.

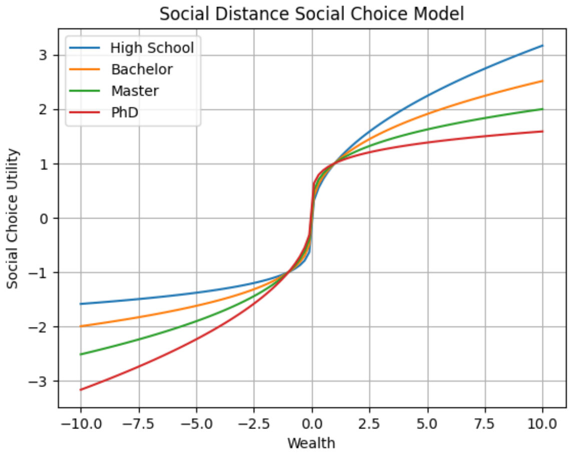
Figure 9 Prospect theory and level of education, social distance and social choice.



[21] Previous can be written as Riemann-Stieltjes integral: $V = \int_{0}^{+\infty} u(x)d\pi\big(1 - P(g < x)\big) + \int_{-\infty}^{0} u(x)d\pi\big(P(g \leq x)\big)$

[22] The elasticity of substitution can be defined as: $\sigma = \frac{\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}}{f(x_1,x_2)\frac{\partial^2 f}{\partial x_1 \partial x_2}}$

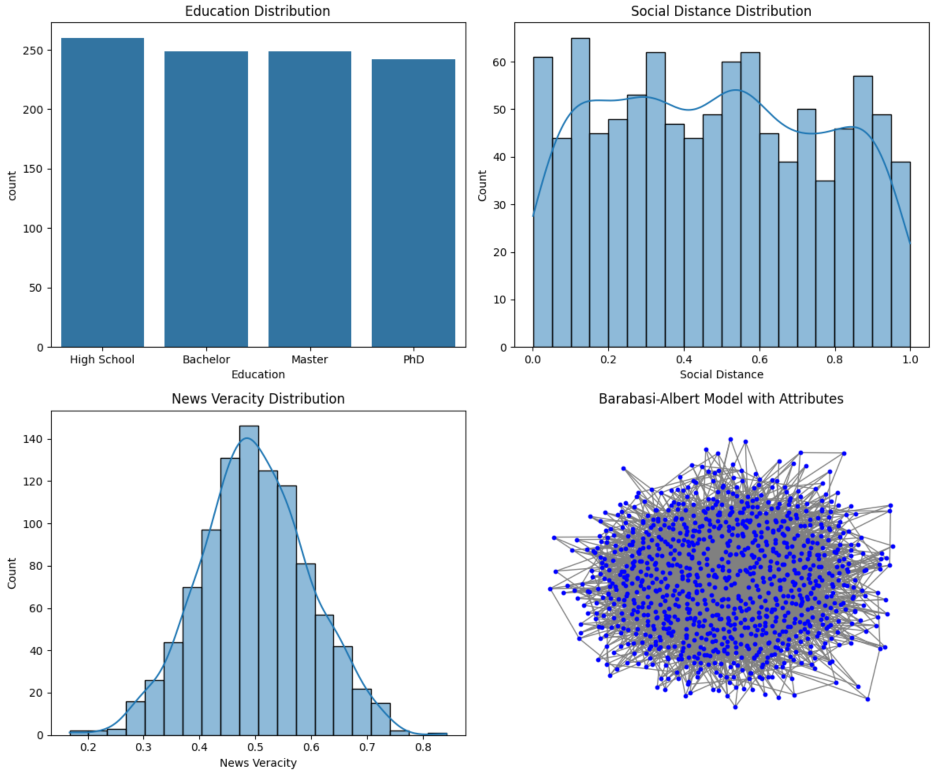
[23] Hicks' Composite Commodity Theorem states that "if the prices of a group of goods change in the same proportion, that group of goods behaves just as if it were a single commodity" or ‘’A set of goods whose relative prices do not change, so that they can be treated as a single commodity’’.



Source: Author’s ow calculations

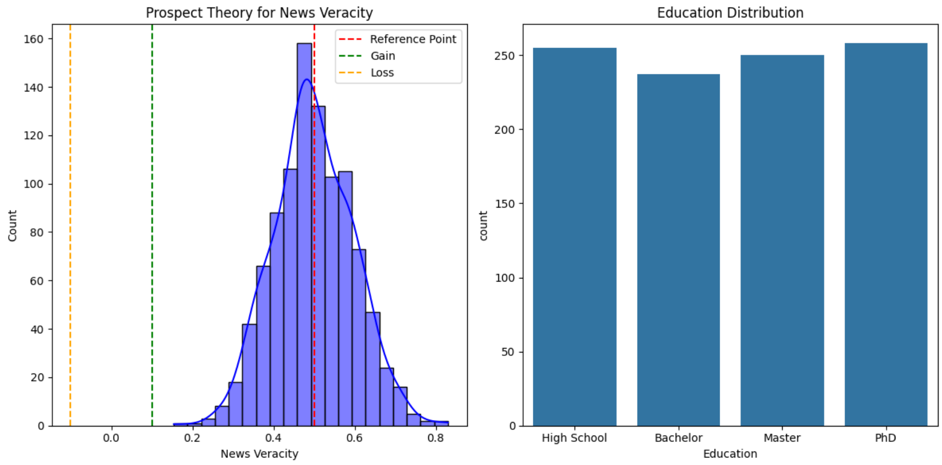
The previous plot shows the Prospect theory for different levels of education and that when it comes to wealth those who have high school degrees are most risk averse in terms of losses and gains. Individuals with lower levels of education, such as those who have completed high school but not pursued further education, may have limited financial literacy and a less sophisticated understanding of investment strategies and financial markets. This lack of knowledge and experience could contribute to a more risk-averse attitude towards wealth management, as they may perceive financial decisions as more uncertain and riskier. While when it comes to social distance and social choice model PhD’s re most risk averse. One reason for the last is income and stability: While completing a PhD can involve financial sacrifices and uncertain job prospects during the program, individuals who successfully obtain a PhD often have access to relatively stable and well-compensated careers in academia, research, or other fields. This financial stability may reduce the need or inclination to take risks compared to individuals with less secure employment prospects. Next, we will show plots of education distribution with social distance distribution, news veracity distribution and Barabási–Albert model with attributes, and a plot for prospect theory for news veracity and education distribution in this economy.

Figure 10 education distribution with social distance distribution, news veracity distribution and Barabási–Albert model with attributes



Source: Author's own calculations

Figure 11 prospect theory for news veracity and education distribution in this economy



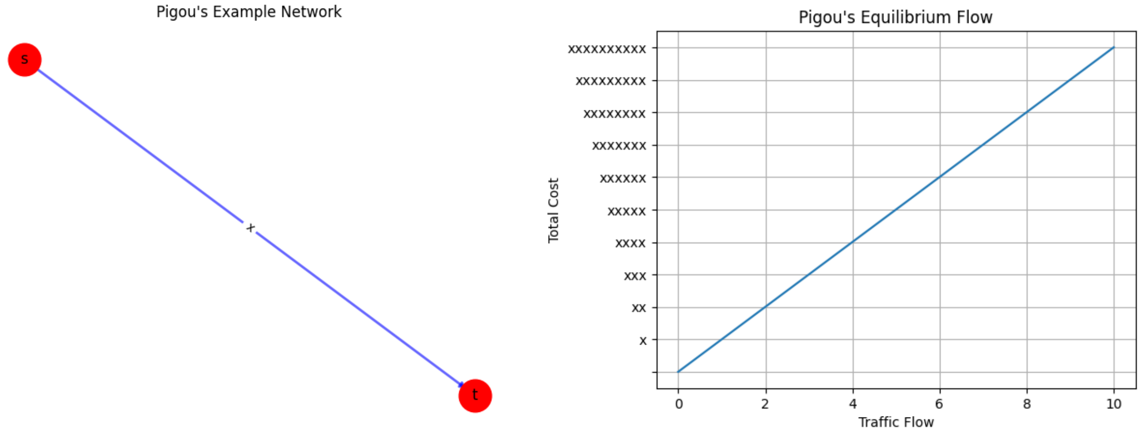
Source: Author's own calculations

Previous plots show that fro the most count population is distributed among the ranks of high news veracity and reference point is high above any gain and loss that comes from news veracity. In this economy there is almost equal number of individuals with PhD degrees and high school diplomas, meaning that first are risk averse towards social distance and second are risk averse towards wealth.

7.Wardrop equilibrium and Braess paradox

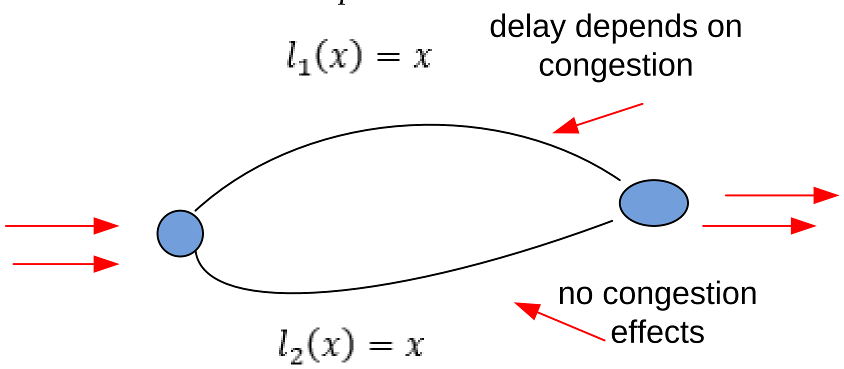
This equilibrium is due to Wardrop (1952),To get to Wardrop equilibrium we will explain Pigou's example. The Pigou's example is a basic network composed by two parallel routes, each a single edge, that connects a source vertex s to a destination vertex t. Each edge has a cost that is a function of the amount of traffic, i.e the flow that uses the edge, and which corresponds to the travel time. The upper edge has a constant cost function $c_1(x) = 1$ (it can be 1 hour for example). Note that it is immune to congestion. The lower edge has a variable cost $c_2(x) = x$, which increases as the edge gets more congested. Below is one plot as a graphic presentation of previous.

Figure 12 Pigou'(1920) network and Pigou's equilibrium flow



Source: Author's own calculations

Figure 13 A Paradoxical Network Example



Source: Lectures 9,12 of Acemoglu, Ozdaglar (2009) lecture notes on Networks
System optimum (minimizing aggregate delay) can be found by solving:

equation 103

$$\min_{x_1+x_2\leq 1} C_{system} x^S = \sum_i l_i(x_i^S)x_i^S$$

FOC:
equation 104

$$I_1(x_1) + x_1 I_1'(x_1) = I_2(1-x_1) + (1-x_1)I_2'(1-x_1)$$
$$2x_1 = 1$$

System optimum is to split traffic equally between routes:
equation 105

$$\min_{x_1+x_2\leq 1} C_{(system)}(x^S) = \sum_i I_i(x_i^S)x_i^S = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

Now, let's suppose instead that there is selfish routing so that each motorist chooses the path with the lowest delay taking aggregate traffic pattern as given. This gives: $x_1 = 1; x_2 = 0 ; \forall x_1 < 1 . I_1(x_1) < 1 = I_2(1-x_1)$,so aggregate delay is given as:

equation 106

$$C_{eq}(x^{WE}) = \sum_i I_i(x_i^{WE})x_i^{WE} = 1 + 0 = 1 > \frac{3}{4}$$

Instead, the Nash equilibrium of this large (non-atomic) game, also referred to as Wardrop equilibrium, is $x_1 = 1$ and $x_2 = 0$. The outcome is socially suboptimal which is common occurrence in game theory. This inefficiency is sometimes quantified by PoA or price of anarchy[24] :

equation 107

$$\frac{C_{system}(x^S)}{C_{eq}(x^{WE})} = \frac{3}{4}$$

For Wardrop equilibrium: It is nothing but a Nash equilibrium in this game, in view of the fact that it is non-atomic| each player is infinitesimal. Thus, taking the strategies of others as given is equivalent to taking aggregates, here total traffic on different routes, as given. so far we often took the set of players, $\mathcal{J}$, to be a finite set. But in fact, nothing depends on this, and in non-atomic games, $\mathcal{J}$ is typically taken to be some interval in $\mathbb{R} \in [0;\ 1]$.Now we will take on more general traffic network.Dircted network $N = (V, E)$,$\mathcal{P}$ are set of paths between origin and destination,x_P denotes flow on path $p \in \mathcal{P}$,each link $i \in E$ has a latency function[25] $I_i(x_i)$:

equation 108

$$x_i = \sum_{(p \in \mathcal{P} | i \in p)} x_p$$

Here notation $(p \in \mathcal{P} | i \in p)$ denotes the paths p that traverse link $i \in E$. The latency function captures congestion effects. Let us assume for simplicity that $l_i(x_i)$ is nonnegative, differentiable, and nondecreasing. The total delay (latency) cost of a routing pattern x is:

equation 109

$$C(x) = \sum_{i \in E} x_i l_i(x_i)$$

that is, it is the sum of latencies $l_i(x_i)$ for each link $i \in E$ multiplied by the flow over this link, x_i, summed over all links E.Now, the socially optimal routing is defined as: defined as the routing pattern minimizing aggregate delay, is given by x^S that is a solution to the following problem:

equation 110

$$\min_{i \in E} x_i l_i(x_i)$$

$$s.t. \begin{array}{l} \sum_{(p \in \mathcal{P} | i \in p)} x_p = x_i, \forall i \in E \\ \sum_{p \in \mathcal{P}} x_p = 1; x_p \geq 0, \forall p \in \mathcal{P} \end{array}$$

What is Wardrop eq? Since it is a Nash equilibrium, it has to be the case that for each motorist their routing choice must be optimal. This implies that if a motorist $k \in \mathcal{J}$ is using path p, then there does not exist path p_0 such that :

equation 111

$$\sum_{i \in p} l_i(x_i) < \sum_{i \in p'} l_i(x_i)$$

x must be such that :

[24] In a game with negative payoffs (“costs”or “losses”that we want to minimize), the price of anarchy is the ratio of the total cost borne by all agents in the worst equilibrium to the total cost at the social optimum.

[25] Latency, from a general point of view, is a time delay between the cause and the effect of some physical change in the system being observed or lag.

equation 112

$$\forall p, p' \in \mathcal{P}\, ; x_p, x'_p > 0, \sum_{i \in p'} l_i(x_i) = \sum_{i \in p} l_i(x_i)$$
$$\forall p \in \mathcal{P}, p' \notin \mathcal{P}\, ; x_p > 0, x'_p = 0$$
$$\sum_{i \in p'} l_i(x_i) \geq \sum_{i \in p} l_i(x_i)$$

Theorem 5
A feasible routing pattern x^{WE} is a Wardrop equilibrium if and only if it is solution to:
equation 113

$$\min \sum_{i \in E} \int_0^{x_i} l_i(z)dz$$
$$s.t. \sum_{(p \in \mathcal{P} | i \in p)} x_p = x_i, \forall i \in E$$
$$\sum_{p \in \mathcal{P}} x_p = 1; x_p > 0, \forall p \in \mathcal{P}$$

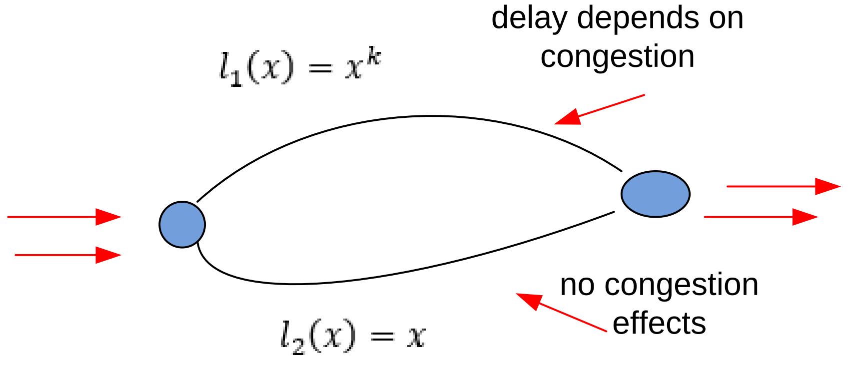
If $x_i \gg 0$ or strictly increasing, then x^{WE} is unique.
Proof: Rewrite the minimization problem;
equation 114

$$\min \sum_{i \in E} \int_0^{\sum_{i \in p} x_p} l_i(z)dz$$
$$s.t. \sum_{p \in \mathcal{P}} x_p = 1; x_p \geq 0, \forall p \in \mathcal{P}$$

Since each l_i is nondecreasing, this is a convex program, FOC's are necessary and sufficient, and with respect to x_p are:
equation 115

$$\sum_{i \in p} l_i\left(x_i^{WE}\right) \geq \lambda$$

With the complementary slackness , i.e. with equality whenever $x_p^{WE} > 0$ .In previous λ is Lagrange multiplier on the constraint $\sum_{p \in \mathcal{P}} x_p = 1$, the Lagrange multiplier will be equal to lowest cost path, which then implies the result $\forall p \in \mathcal{P}, p' \in \mathcal{P}$ with $x_p^{WE}, x_p'^{WE} > 0$,$\sum_{i \in p'} l_i\left(x_i^{WE}\right) = \sum_{i \in p} l_i\left(x_i^{WE}\right)$ and paths with $x_p^{WE} = 0$ the cost can be higher. Since $l_i \gg 0$ strictly increasing, then the set of equalities $\sum_{i \in p'} l_i\left(x_i^{WE}\right) = \sum_{i \in p} l_i\left(x_i^{WE}\right)$ admits to unique solution, which establishes uniqueness ∎.
A related consequence of selfish behavior in networks is captured through the celebrated Braess' Paradox Braess (1969), which demonstrates that the addition of an intuitively helpful route negatively impacts network users at equilibrium.
Figure 14 Pigou example, except with a different latency on path 1



Source: see Menache, Ozdaglar. (2011).
In this example social routing involves:
equation 116

$$l_1(x_1) + x_1 l_1'(x_1) = l_2(1 - x_1) + (1 - x_2) l_2'(1 - x_1)$$
$$x_1^k + k x_1^k = 1$$

So, the system optimum sets $x_1 = (1+k)^{-\frac{1}{k}}$ and $x_2 = 1 - (1+k)^{-\frac{1}{k}}$ so that actually:
equation 117

$$\min_{x_1 + x_2 \le 1} C_{system}(x^S) = \sum_i l_i(x_i^S) x_i^S = (1+k)^{-\frac{k+1}{k}} + 1 - (1+k)^{-\frac{1}{k}}$$

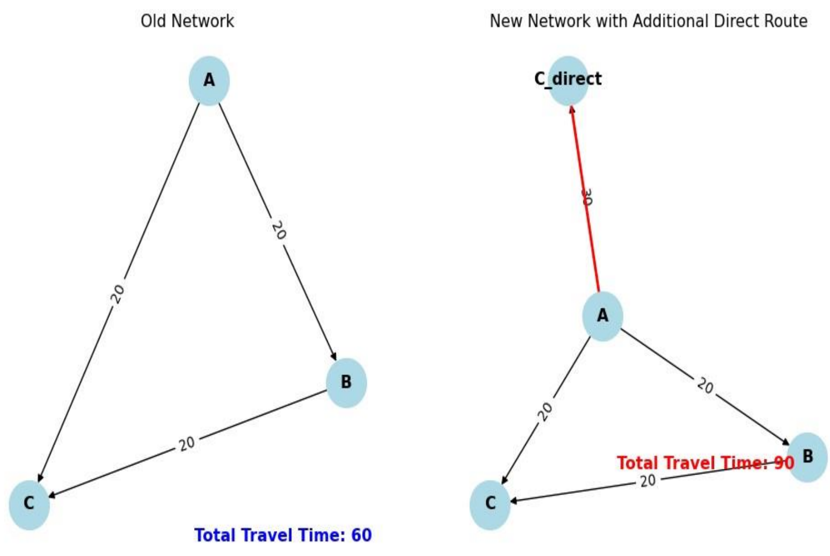
The Wardrop equilibrium has $x_1 = 1, x_2 = 0, \forall x_1 < 1, l_1(x_1) < 1 = l_2(1 - x_1)$:
equation 118

$$C_{eq}(x^{WE}) = \sum_i l_i(x_i^{WE}) x_i^{WE} = 1 + 0 = 1$$

Therefore, the PoA now is given as:
equation 119

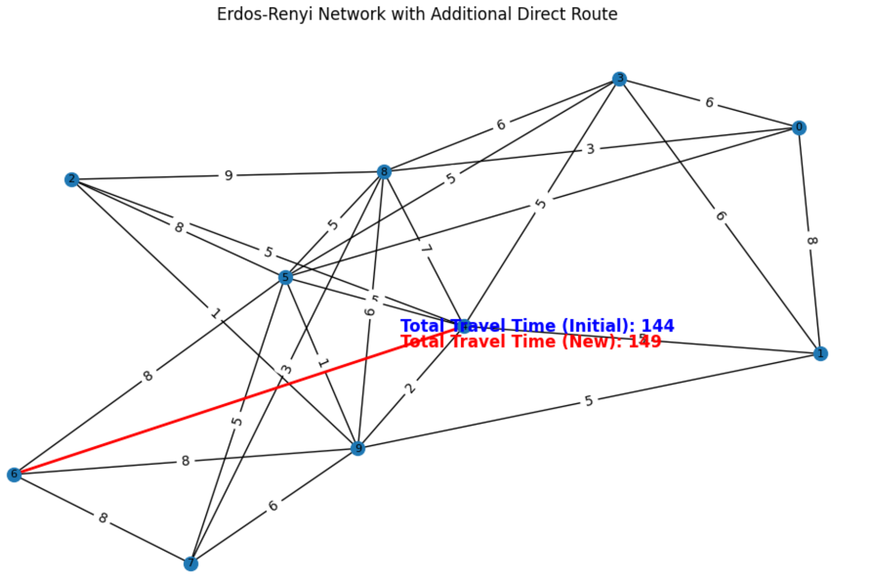
$$\frac{C_{system}(x^S)}{C_{eq}(x^{WE})} = (1+k)^{-\frac{k+1}{k}} + 1 - (1+k)^{-\frac{1}{k}}$$

Now $\lim_{k\to\infty} (1+k)^{-\frac{k+1}{k}} + 1 - (1+k)^{-\frac{1}{k}} = 0$,since $\lim_{k\to\infty}(1+k)^{-\frac{k+1}{k}} = 0$, $\lim_{k\to\infty} +1 = 1$; $\lim_{k\to\infty}(1+k)^{-\frac{1}{k}} = 1$ so $0 + 1 - 1 = 0$. This limits to 0 as $k \to \infty$ (the first term tends to zero, while the lastterm limits to 1). Now wwe will introduce Braess paradox in next plot.
Figure 15 Old network and new network with additional route



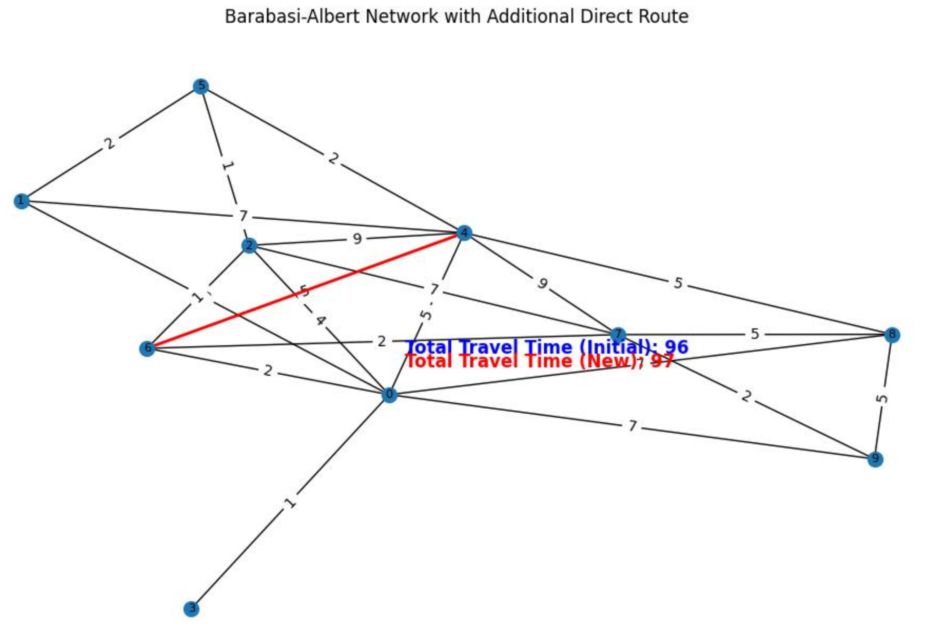
Source: Author’s own calculations
In this modified version, we introduce a direct route from node A to node C in the updated network, which can potentially lead to Braess's Paradox. The direct route is highlighted in red to distinguish it from the existing routes. The visualization demonstrates how adding this new route affects the total travel time and illustrates the paradoxical increase in travel time despite the additional capacity. Next, wewill show Erdős-Rényi (ER) network with additional direct route and Barabási–Albert (BA) with additional direct route and we will calculate the price of anarchy for both models.

Figure 16 Erdős-Rényi (ER) network with additional direct route



Source: Author's own calculations; Price of Anarchy: 1.0347222222222223

Figure 17 Barabási–Albert (BA) with additional direct route



Source: Author's own calculations: Price of Anarchy: 1.0104166666666667
We calculate Price of Anarchy as $poa = \frac{total\ travel\ time\ new}{total\ travel\ time\ initial}$

Now, let's take directed network $G = (V, A)$, with node set V , link (or edge) set A, and w source-destination node pairs $\{s_1, t_1\}, \dots, \{s_w, t_w\}$.Let $W = \{1, \dots, w\}$. Let P_i denote the set of paths available from s_i to t_i using the edges in A; and we view each path $p \in P_i$ as a subset of $A, p \subset A$. Here we define $P = \cup_{i \in W} P_i$. Each link $j \in A$ has a strictly increasing, nonnegative latency function

$l_j(x_j)$ as a function of the flow on link j. We assume that X_i units of flow are to be routed from s_i to t_i $\forall\, i \in W$, and here it is defined define $X = [X_1, \dots, X_w]$. This tuple $R = (V, A, P, s, t, \mathbf{X}, \mathbf{l})$ is called a routing instance, see Acemoglu, D., Johari,R. A. Ozdaglar, A.(2007).
Socially Optimal Routing : given $R = (V, A, P, s, t, \mathbf{X}, \mathbf{l})$social optimum $x^{SO}(R)$ is and optimal solution of the folowing problem:
equation 120

$$\min \sum_{j \in A} x_j l_j(x_j)$$

$$s.t. \sum_{p \in P, j \in p} y_p = x_j, j \in A$$

$$\sum_{p \in P_i} y_p = X_i, i \in W$$

$$y_p \geq 0. p \in P$$

The total latency cost at a social optimum is given by:
equation 121

$$C\big(x^{SO}(R)\big) = \sum_{j \in A} x_j^{SO}(R) l_j\left(x_j^{SO}(R)\right)$$

Selfish routing: When traffic routes selfishly or when sources choose minimum delay end-to-end paths—all paths with nonzero flow must have the same total delay. A flow configuration with this property is called a Wardrop equilibrium. So $x^{WE}(R)$ is an unique solution to:
equation 122

$$\min \int_{j \in A}^{x_j} l_j(z) dz$$

$$s.t. \sum_{p \in P\,; j \in p} y_p = x_j; j \in A$$

$$\sum_{p \in P_i} y_p = X_i, i \in W$$

$$y_p \geq 0. p \in P$$

The latency costs are given as:
equation 123

$$C(x^{WE}(R)) = \sum_{j \in A} x_j^{WE}(R) l_j\left(x_j^{WE}(R)\right)$$

Equivalently a feasible solution x^{WE} for a routing instance R is a Wardrop equilibrium if and only if it satisfies:
equation 124

$$\sum_{j \in A} l_j\big(x_j^{WE}\big)\big(x_j^{WE} - x_j\big) \leq 0$$

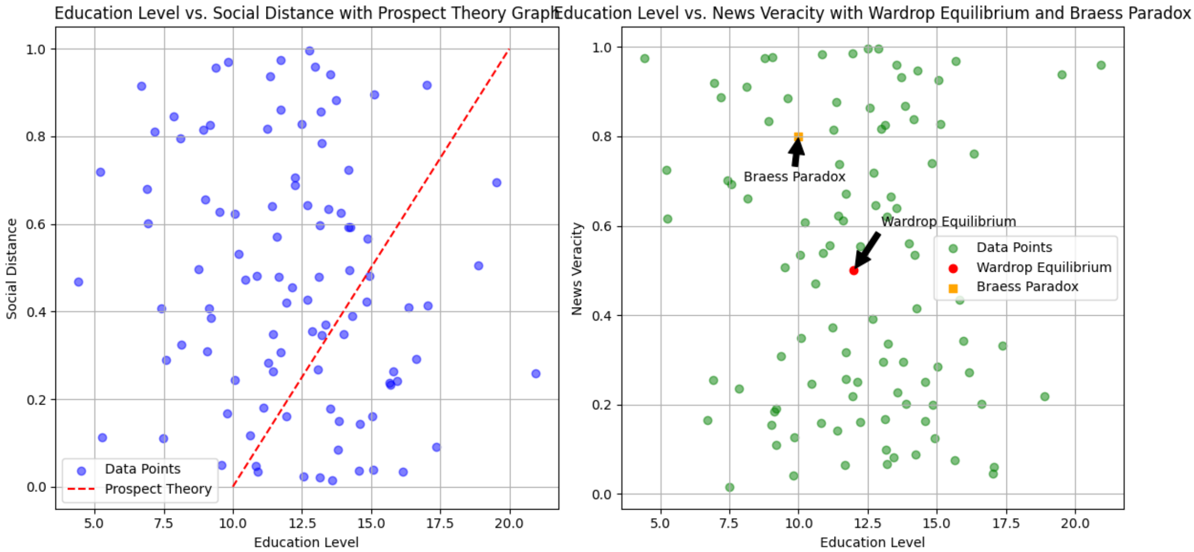
Proposition 5 Definition (Braess' paradox): Consider a routing instance $R = (V, A, P, s, t, \boldsymbol{X}, \boldsymbol{l})$ and a subnetwork $R_0 = (V_0, A_0, P_0, s_0, t_0) \subset R$. We say that Braess' paradox occurs in R centered at R_0 if there exists another routing instance $R_m = (V, A, P, s, t, X, m)$, with a vector of strictly increasing, nonnegative latency functions, $m = (m_j, j \in A)$, such that for $\forall x_j \geq 0$.
equation 125

$$m_j(x_j) \leq l_j(x_j), \forall j \in A_0, m_j(x_j) = l_j(x_j), \forall j \notin A_0$$

$$C\big(x^{WE}(R_m)\big) > C\big(x^{WE}(R)\big)$$

Next we will show Braes paradox and Wardrop equilibrium with Prospect theory ,nexs veracity and social distance with education level model.

Figure 18 Education level, Prospect theory, Social distance, News veracity ,Wardrop equilibrium, Braess paradox



Source: Author's own calculations

Braes paradox makes more individuals with lower level of education to be moved further away from equilibrium. Prospect theory reference line here is nearly on the medium level of education.

8.Conclusion

This paper reviewed network economics topics. In the Barabási–Albert (BA) network adding a new node degree distribution becomes normal. In Erdős-Rényi graph degree distribution is Poisson. However, in practical scenarios, when the number of nodes is sufficiently large and the edge probability is not extremely small or extremely large, the degree distribution of an ER graph can sometimes exhibit behaviors that appear approximately normal. News veracity is higher in Erdős-Rényi graph vs Barabási–Albert (BA) network perhaps because of degree distribution which is binomial (true or false in Erdős-Rényi vs continuous distribution in Barabási–Albert (BA) network). In an Erdős-Rényi graph, where nodes have similar degrees and there's less variation in connectivity, it might be easier for misinformation or rumors to spread uniformly across the network. However, the lack of highly connected hubs could also limit the reach of such misinformation. In a Barabási–Albert graph, with its presence of hubs, influential nodes might have a significant impact on the spread of information, including news veracity. Misinformation could potentially spread rapidly if it originates from or is propagated by these highly connected hubs. Conversely, if accurate information is disseminated by influential nodes, it could reach a large portion of the network quickly. In the average social distance and education level model there is higher news veracity between most educated individuals (PhD's) more than when two nodes have less than highest education. Reasons are let say: critical thinking skills for those with highest education, media literacy, skepticism and verification, and awareness of Cognitive Biases between individuals with highest level of education (such as: confirmation bias (or the tendency of people to favor information that confirms or strengthens their beliefs or values and is difficult to dislodge once affirmed).Probability of news being true (news veracity) is lowest for PhD's but gaslighting is highest meaning that because of skills for critical thinking individuals with PhD degree are more aware of gaslighting second to them are those with bachelor's degrees. PhD holders also may work in environments where gaslighting is more prevalent, such as academia, research, or leadership roles where power dynamics are significant. Higher levels of education may also correlate with greater confidence and assertiveness, enabling individuals to speak out against gaslighting behaviors when they encounter them. In terms of social choice, a bachelor's degree seems that is optimal level of education when comes to social choice and social distance model. When degree centrality (the degree centrality of a node is simply its degree—the number of edges it has) of network is included in the calculations of the model in the average social distance and education levels model, well news veracity is higher even when nodes (v, u) have lower levels of education, so now again news veracity (probability of news being true is lowest for higher levels of education when networks' centrality degree is included).In prospect theory analysis when compared by education level and wealth (social choice variable) those with high school education are more risk averse towards wealth while when it comes to social distance social choice and wealth model now PhD's are more risk averse. One explanation is: High Opportunity Cost: Pursuing a Ph.D. typically requires a significant investment of time, effort, and resources. Ph.D. holders may perceive the opportunity cost of taking risks as higher compared to individuals with lower levels of education, who may have fewer alternative opportunities to consider. Second, is the perception of Social Distance: Ph.D. holders may perceive themselves as socially distant from individuals with different educational backgrounds or socioeconomic statuses. This perception of social distance could influence their risk preferences, as they may prioritize maintaining their perceived status or avoiding potential social stigma associated with failure. With Prospect theory social distance and news veracity follow a normal distribution. Reference point of Prospect theory mean in the news veracity distribution and gain and loss lines are further way on the left from reference point. Reference point in prospect theory determines how an outcome is perceived. So, with news veracity neither gain nor loss enter the area of news veracity, but again loss is further left away meaning risk averse demand for news. In Pigou's equilibrium flow traffic costs are higher with higher traffic flow. Price of anarchy (The Price of Anarchy (PoA) is a concept from game theory and network analysis that measures the inefficiency of a system when individuals make decisions selfishly, without considering the overall social welfare.) is higher in Erdős-Rényi (ER) vs Barabási–Albert graph. In ER graphs, where the network structure is relatively homogeneous, the spread of inefficiency may be more evenly distributed across the network. Since nodes have similar degrees, there may be less variation in the impact of selfish behavior on overall system performance. Therefore, the Price of Anarchy in ER graphs may be relatively lower compared to BA graphs. In BA graphs, the presence of hubs introduces significant heterogeneity in node connectivity. Highly connected nodes (hubs) have a disproportionate influence on network dynamics. Selfish behavior by these highly connected nodes or their neighbors may lead to greater inefficiency, as the impact of their decisions can propagate more extensively through the network. Consequently, the Price of Anarchy in BA graphs may be higher compared to ER graphs due to the potential for greater systemic inefficiencies caused by selfish behavior concentrated around hubs. In the prospect theory model of network Braess paradox is on lower level of education with higher news veracity and Wardrop equilibrium is on higher level of education with lower level of news veracity. One implication of this finding is that by adding additional news agencies, news websites etc. (congestion of the news market) more and more agents are further away from equilibrium state: In a hypothetical scenario approaching Wardrop equilibrium, individuals across different education levels would converge on a subset of news sources that are widely recognized for their accuracy and reliability. While personal biases and preferences may still influence individual choices to some extent, the overall distribution of news consumption would reflect a collective preference for trustworthy sources.In the context of a news veracity model and education level, we can adapt the concept of Wardrop equilibrium to describe a scenario where individuals with different levels of education seek out news sources based on their perceived accuracy and reliability, aiming to minimize the cost associated with misinformation or biased reporting.

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