УДК: 005.57 316.776

CHEAP TALK, KRIPKE SEMANTICS, COMMON KNOWLEDGE AND TWO EXAMPLES OF BOOM, BUST & ASSET BUBBLES

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Abstract

This paper investigates the Crawford-Sobel model as cheap talk model where communication between players does (not) directly affect the payoffs of the game. Partition equilibrium and equilibrium actions do not converge with bias in beliefs. Green-Stoke model of information transmission showed that BNE differs from Cournot, Nash and Stackelberg and while Cournot is agent and principal best response, BNE is principal best response. Stackelberg equilibrium is outside of principals' or agents' BR. Kripke model of partial separation with mixed strategy equilibrium show that if the world is $w_H m_H$ *beliefs/payoff ratio is higher than that of* $w_L m_H$ *and if the world is* $w_H\ m_L$ *beliefs/payoff ratio is higher than in* $w_L m_L$ *world. In the common knowledge frame communication does not change the structure of knowledge in the model. Deterministic and bursting bubbles differ in the movement of dividend/price and capital gain. Whether investors coordinate for collective optimality Kantian equilibrium or not (Nash equilibrium) matters for asset bubble size.Dot.com bubble and 2008 financial crisis with Case-Shiller measure for housing bubble and S&P movements are investigated as some indicators for asset bubbles and cheap talk.*

Keywords: cheap talk, asset bubbles, common knowledge, Kripke semantics, financial crises

JEL codes: D8, D82, D83

1.Introduction

In the standard cheap talk model informed sender sends message to uninformed receiver. The problem here is that receivers' action affects the payoffs of both parties. Talk is cheap because the payoffs of the players do not depend directly on the Sender's message[1], see Chen, Sobel (2008).Key features of cheap talk are: Costless i.e. Sending a message has no cost to the sender. Non-binding: The sender is not committed to follow through on what they say. Non-verifiable: The receiver can't verify the truth of the message beforehand. Even though it seems "cheap," cheap talk can convey useful information if incentives are aligned or if repeated interactions build reputation. In some equilibriums, truthful communication can emerge even when players could lie. Equilibrium often demands considerable coordination: a problem seen most clearly in games of pure coordination, but present in most games. So, one can expect Nash outcomes due to informal pre-play communication, see Farrel (1998).Like in Aumann (1974), one can suppose that players can talk before choosing their actions, but cannot bind themselves. Aumann suggests that they will reach some agreement on how to play, and that since no external enforcement is available, they can only consider self-enforcing, or Nash, outcomes[2]. Crawford and Sobel (1982) henceforth CS model, show that cheap talk extends the set of outcomes that can be

[1] Cheap talk is a concept in game theory that refers to costless, non-binding, and non-verifiable communication between players before they take any action in a game. It doesn't directly affect the payoffs, but it can influence beliefs and subsequent strategies.

[2] A Nash equilibrium is a strategy profile such that each person's strategy is optimal (for himself) given the inertness of others' strategies, See Roemer(2019).

implemented in equilibrium by allowing for partial revelation of information. Several articles have extended the original CS model. For instance, Morgan and Stocken (2003) extend the CS model to the case where there is uncertainty regarding the difference between the preferences of the sender and receiver. Blume et al. (2007) modify the CS setup where communication errors (or noise) can occur. Andreu Mas-Colell (1987, 659) writes: “The typical starting point [of cooperative game theory] is the hypothesis that, in principle, any subgroup of economic agents (or perhaps some distinguished subgroups) has a clear picture of the possibilities of joint action and that its members can communicate freely before the formal play starts. Obviously, what is left out of cooperative theory is very substantial.”, see Roemer (2019). Cooperation may be the only means of satisfying one’s own self-interested preferences, see Roemer (2019)[3]. In Nash equilibrium, one chooses one’s own strategy to maximize one’s own utility holding others’ strategies fixed at the equilibrium. In contrast, in the Kantian equilibrium, one chooses the common strategy to be adopted by everyone to maximize one’s own utility. Modern economics asks how private information is shared through the market and other mechanisms, see Farrell, Rabin (1996). In previous study Farrell, Rabin (1996), author’s suspected that most information sharing is not done through Spence-style signaling (Spence (1974)),through the price system, nor through carefully crafted Hurwicz-style(see Hurwicz (1973)) incentive-compatible mechanisms: it is done through ordinary, informal talk. Farrell, Rabin (1996) believe that talking is cheap (it does not directly affect payoffs), but, given that people respond to it, talking affects payoffs[4].In Crawford, Sobel model (1982) ,where sender sends costless message to the receiver, receiver cannot verify any of the information sent by the sender and the interests of the sender and the receiver do not necessarily coincide, see Rubinstein, Glaser (2006). The typical question in this literature is whether an informative sequential equilibrium exists. So, in CS model we investigate partitioning equilibrium (also called partial pooling equilibrium) in a cheap talk game which is a type of Bayesian Nash Equilibrium where the sender's type space is divided into intervals (or "partitions"), and each type within a partition sends the same message. So rather than fully revealing their type (as in a separating equilibrium) or sending the same message regardless of type (as in a pooling equilibrium), the sender group types together and communicates coarse information[5].Cheap talk presuppose strategic, non-binding messages affecting beliefs[6].The game with communication always has a subgame perfect equilibrium that replicates any Nash equilibrium without communication: at the stage of actual play the receiver ignores the communication, and rolling back, the sender sends any arbitrary message. This is called a “babbling” equilibrium; the focus of interest is on whether non-babbling equilibria are possible. Kripke Semantics models agents' knowledge and belief about types and messages. Here we assume that the receiver does not ignore the sender’s message, and the sender does convey any information—message is informative[7]. Common Knowledge is necessary for coordination, trust, and interpreting messages. Talk is a useful way to communicate private information in strategic situations, as formalized by Crawford and Sobel (1982) and Green and Stokey (2003). These early papers recognized, however, that their equilibrium analysis is generally indeterminate. Models of cheap talk have multiple equilibria some of them uninformative, see Sidartha et al. (2023).As an assumption there must be some exogenous ordering of messages and it is common knowledge that they will behave in a way that is consistent with this ordering. Furthermore, the sequences are monotonic and converge to equilibria[8]. We will provide two loosely cheap talk real life examples too.

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[3] There is growing attention to the claim that humans are a cooperative species, see Bowles, S. and Gintis, H. (2011), and Henrich, N., and J. Henrich (2007).

[4] A misinformed listener will do something that is not optimal for himself and, if their interests are sufficiently aligned, this is bad for the speaker too.

[5] Coarse information refers to imprecise, aggregated, or simplified information that does not fully reveal the underlying details, but still conveys some useful knowledge.In the context of cheap talk or signaling, coarse information means the receiver learns something about the sender’s type, but not exactly what it is.

[6] See https://www.davidreiley.com/GameTheoryAEA/AEAContEd_I.pdf Avinash Dixit lecture notes on Game theory.

[7] For babbling equilibrium and example see Appendix 1 Babbling equilibrium.

[8] see Appendix 2 Cheap Talk with Noise or Non-Monotonic Strategies

2.Crawford -Sobel model (CS model)

This model formalizes the idea that the amount of information which can be transmitted depends on how well aligned preferences are. In this model, better informed sender S sends a message to a receiver R, who then takes action that determines the wealth of both. S observes the value of random variable m with differentiable CDF $F(m)$ and PDF $f(m)$, supported in range $[0,1]$. Utility function is VNM[9] twice differentiable. In this model signaling rule (chosen by S): $q(n|m)$ such that if m is observed, message n is sent with probability $q(n|m)$,action rule (chosen by R) y(n) such that if n is observed, y is chosen, the rules must satisfy:$R's$ rule is optimal given $S's$,if n' is taken with positive probability in m then:

equation 1

$$n' \in \arg\max_{n} U^S(y(n), m, b)$$

S's rule optimal given R's:

equation 2

$$y(n) \in \arg\max_{y} \int U^R(y, m)p(m|n)$$

Where $p(m|n)$:

equation 3

$$p(m|n) = \frac{q(n|m)f(m)}{\int_0^1 q(n|m)f(m)}$$

in a canonical example these players are same but differ in b:

equation 4

$$U^R(m, y) = -(y - m)^2$$
$$U^S\,(m, y) = -\big(y - (m\ +\ b)\big)^2$$

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[9] The Von Neumann-Morgenstern (VNM) utility function is a way to represent an individual's preferences over risky choices (lotteries) in expected utility theory. It assumes that people make decisions based on the expected utility of different outcomes rather than just the expected value of monetary payoffs. For a utility function u(x)u(x)u(x) to be a VNM utility function, the preferences it represents must satisfy four key axioms: **Completeness**: The decision-maker can always compare two lotteries and state a preference (or indifference).**Transitivity**: If lottery A is preferred to B, and B is preferred to C, then A must be preferred to C. **Independence**: If a decision-maker prefers A to B, then for any probability p, they should also prefer a lottery that gives A with probability p and some other outcome C with probability $1 - p$, over a similar lottery that gives B with probability p and C with probability $1 - p$. **Continuity**: If A is preferred to B, and B is preferred to C, then there exists some probability p where the decision-maker is indifferent between B and a lottery that gives B with probability p and C with probability $1 - p$. If a person's preferences satisfy these four axioms, then their preferences over lotteries can be represented by a **VNM utility function** $u(x)$, such that the expected utility of a lottery $\mathcal{L} = (x_1, p_1, x_2, p_2, \dots, x_n, p_n)$ and $EU(\mathcal{L}) = p_1u(x_1) + p_2u(x_2) + \cdots + p_nu(x_n)$.Where :$x_i$ are possible outcomes,p_i are probabilities of each outcome occurring,$u(x)$ is the utility assigned to outcome x. VNM utility function allows for risk attitudes, if concave $u''(x) < 0$ the agent is risk averse, if $u''(c) = 0$ is risk neutral, and if $u(x)$ is convex $u''(x) > 0$ the agent is risk-seeking. The function is ordinal, meaning it only represents rankings of preferences—any positive affine transformation $a(u)x + b$ where $a > 0$ represents the same preferences, see von Neumann, J., Morgenstern, O., & Rubinstein, A. (1944).

If Sender says his 'bliss point' $n = m + b$, then Receiver would discount that and consider state y = n − b. But the Sender would try to fool him and say n = m... and so on precise messages indeed non-credible. But if we allow for some slack, cheap talk becomes meaningful. Now, let :

equation 5

$$y_i(m;\ b) = \arg\max U^i(y, m, b)$$

Lemma 1 If $y^S(m,b) \neq y^R(m)\ \forall m \rightarrow \exists \epsilon > 0$ such that if u and v are actions induced in equilibrium,$|u - v| > \epsilon$. Further, the set of actions induced in equilibrium is finite.

We can recall here that $\begin{matrix} U^S = -(y-m)^2 \\ U^R = -\big(y - (m + b)\big)^2 \end{matrix}$ then$y^R = m; y^S(m,b) = m + b$ and Lemma holds.

Proof 1 Let $y^S(m,b) > y^R(m)\ \forall m$ (as in canonical example) in particular $\exists \epsilon\ y^S - y^R > \epsilon$. Suppose type mu induce u and type m_v induces v, with $v > u$.Then:

inequality 1

$$\left.\begin{matrix} U^S(u;\ m_u;\ b) \geq U^S(u;\ m_v;\ b \\ U^S(v;\ m_v;\ b) \geq U^S(u;\ m_u;\ b) \end{matrix}\right) \text{(by optimality)}$$

and by continuity

equation 6

$$\exists \bar{m}, U^S(u, \bar{m}, b) = U^S(v, \bar{m}, b)$$

Since $U^S_{yy}(\cdot) < 0, U^i_{ym}(\cdot) > 0$ then:

inequality 2

$$u < y^S(\bar{m}, b) < v$$

u is not induced by any $m > \bar{m}$,v is not induced by any $m < \bar{m}$ with that follows:

inequality 3

$$u < y^R(\bar{m}, b) < v$$

inequality 4

$$u < y^R(\bar{m}) < y^R(\bar{m}) + \epsilon < y^S(\bar{m}, b) < v \Rightarrow v - u > \epsilon$$

We will continue with construction of equilibrium: Let $0 = a_0 < a_1 \dots < a_N = 0$ denote partition$\{0,1\}$.

equation 7

$$\bar{y}(a_i, a_{i+1}) = \arg\max \int_{a_i}^{a_{i+1}} U^R(y, m) f(m) \forall n \in [a_i, a_{i+1}]$$

Theorem 1 suppose b is such that $y^S(m,b) \neq y^R(m), \forall m$.Then $\exists a$ positive integer $N(b)$ such that for any N $1 \leq N \leq N(b)$, $\exists$ and equilibrium with N distinct messages i.e. $n = n_i$ if $n_i \in (a_i, a_{i+1})$ and $n_i \neq n_j$ for $i \neq j$. Moreover, every equilibrium is essentially equivalent to one of this class.

Proof 2

We model the sender's communication as a **partitioning** of the state space. The sender **groups** their types into N intervals:

inequality 5

$$0 = \theta_0 < \theta_1 M < \cdots < \theta_N = 1$$

$\forall \theta_{i-1}, \theta_i$ S sends message m_i and the receiver chooses action $y^R(m_i)$ based on expectation:

equation 8

$$y^R(m_i) = \mathbb{E}[\theta | \theta \in (\theta_{i-1}, \theta_i)]$$

Using the sender's incentive compatibility, the optimality conditions yield a recurrence relation for partition endpoints:

equation 9

$$\theta_i - \theta_{i-1} = 2b + (\theta_{i-1} - \theta_{i-2})$$

This implies a finite number of partitions bounded by:

equation 10

$$N(b) \approx \frac{1}{\sqrt{b}}$$

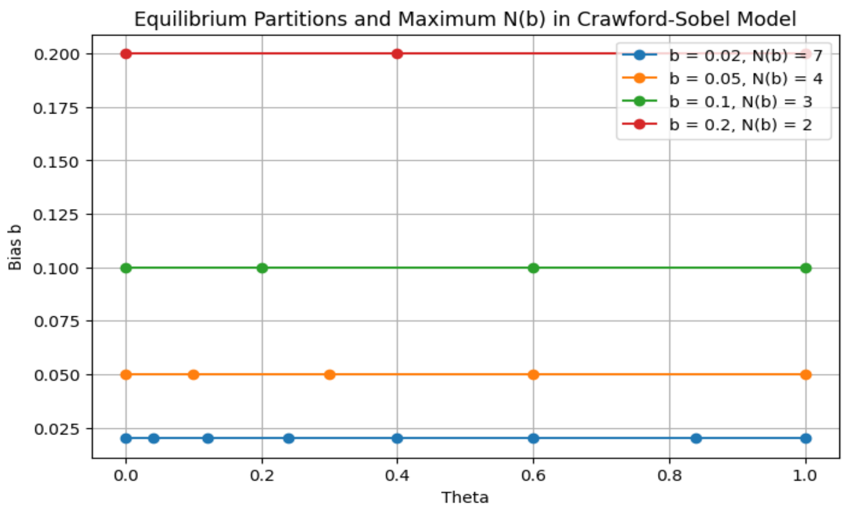
This follows from the fact that if N were too large, the incentive constraints would be violated, as the sender would prefer to deviate. Any equilibrium is essentially equivalent to one of these partition equilibria because any deviation would collapse into a partitioned structure due to the sender's optimal strategy.Given $N(b)$ $\forall N \leq N(b)$, we can construct equilibrium with N messages. Since all equilibria must satisfy the same incentive compatibility conditions, no fundamentally different equilibrium exists beyond these partitions. Thus, the theorem follows:

inequality 6

$$\exists N(b) > 0; \forall\ 1 \leq N \leq N(b); an\ equilibirum\ \ with\ N\ partitions\ exists$$

This will be represented graphically in the following graph. The plot contains four types of bias in range $[0.025 - 2.2]$ with private information spanning from $[0,1]$.

Figure 1 Equilibrium partitions and maximum $N(b)$ in Crawford-Sobel model



Source: Author's own calculations

- X-axis (θ): Represents the sender's private information, which lies in the interval $[0,1]$
- Y-axis (b): Represents different bias values (misalignment between sender and receiver).
- Markers ($\circ$): Represent equilibrium partition points, which are the cutoff points where the sender switches between messages.

Theorem 2 Crawford,Sobel (1982) For any cheap talk game, there exists an integer N such that, for any $p \leq N$, there is a partition equilibrium of the game with p partitions.

Proof 3 :

We consider a cheap talk game where a **sender** (S) with private information $\theta \in [0,1]$ communicates with a **receiver** (R) who takes an action a. The sender and receiver have **aligned but not identical** preferences, modeled as follows:

- Senders' utility:$U^S(a,\theta)$
- Receivers' utility $U^R(a,\theta)$
- Bias: b Captures the misalignment between their preferred actions.

The sender can send messages to the receiver, but these messages are costless and non-binding (cheap talk). The receiver observes the message and selects an action. A **partition equilibrium** means the sender divides the state space [0,1into p intervals, and within each interval, the sender sends the same message. That is, there exist points

inequality 7

$$0 = \theta_0 < \theta_1 \dots \dots < \theta_p = 1$$

$\forall\theta \in (\theta_{i-1}, \theta_i)$ the sender sends the same message, and the receiver responds with an optimal action a_i.The sender's strategy is thus a coarse communication strategy, and the receiver infers the sender's type up to the corresponding partition range. The sender chooses a partition strategy based on incentives. The partition equilibrium satisfies:

✓ **Indifference at Boundaries**: The sender must be **indifferent** between revealing truthfully and deviating within each partition. This yields a **difference equation** that determines the equilibrium partition structure.

✓ **Monotonicity and Refinement**: As the misalignment **bias** $\boldsymbol{b}$ increases, the number of partitions decreases, meaning the equilibrium gets **coarser**.Using these conditions, we derive that the number of partitions satisfies:

equation 11

$$N \sim \frac{1}{b}$$

where N is the maximum number of partitions in equilibrium. There exists a largest integer N beyond which further partitioning is **not incentive-compatible** for the sender. For any $p \leq N$, a corresponding equilibrium exists, where the sender partitions the state space into p regions.

Thus, the theorem follows:
There exists an integer N such that for any $p \leq N$ there is a partition equilibrium with p partitions □.

We will now construct an example of a partition equilibrium for the quadratic case:

equation 12

$$U^S(y,m) = -(y-m)^2$$
$$U^R(y,m) = -(y-m-a)^2$$

y is an action and $m \in (0,1)$ is the state of the world. Signal $n \in [0,1]$. Receiver initially has a prior given by cdf μ. Updates it based on signal to $r(\cdot\,|n)$.We will construct partition equilibrium for $p = 3$. Remember that in equilibrium they (recipients) know 'the strategy of the sender. So, they know upon receiving n_1 that m is uniformly distributed between m_{i-1} and m_i.

equation 13

$$r^*(m|n_i) = U[m_{i-1}, m_i]$$

Objective function is therefore:

equation 14

$$\int_{m_{i-1}}^{m_i} -(y-m-a)^2\left(\frac{1}{m_i - m_{i-1}}\right)dm$$

Taking derivatives with respect to y gives:

equation 15

$$\int_{m_{i-1}}^{m_i} -2(y-m-a)\left(\frac{1}{m_i - m_{i-1}}\right)dm = 0 \Rightarrow \left[-2\left((y-a)m - \frac{m^2}{2}\right)\right]_{m_{i-1}}^{m_i} = 0$$

$$\Rightarrow (y-a)(m_i - m_{i-1}) - \left(\frac{m_i^2 - m_{i-1}^2}{2}\right) = 0 \Rightarrow (y-a)(m_i - m_{i-1})$$

$$= \frac{(m_i - m_{i-1}) - (m_i + m_{i-1})}{2} \Rightarrow y^*(n_i) = \frac{m_i + m_{i-1}}{2} + a$$

And for the sender they prefer to send message n_i to any other message for any m in $[m_{i-1}, m_i]$:

inequality 8

$$U^S(y_i, m) \geq U^S(y_j, m)\ \forall\, m \in [m_{i-1}, m_i]$$

The boundary point at m_i is where sender is indifferent between sending signals n_i and n_{i+1}.This means that for $m > m_i$ then n_{i+1} will be strictly preferred. For $m < m_i$, n_i is strictly preferred. So, the conditions become:

$$U^S(y^*(n_i), m_i) = U^S(y^*(n_{i+1}, m_i)$$

So, if we plug in $U^S(y,m) = -(y-m)^2$ and $y^*(n_i) = \frac{m_i + m_{i-1}}{2} + a$ will give:

equation 16

$$\left(\frac{m_{i-1} + m_i}{2} + a - m_i\right)^2 = \left(\frac{m_{i+1} + m_i}{2} + a - m_i\right)^2$$

as$\frac{m_{i-1}+m_i}{2} < \frac{m_{i+1}+m_i}{2}$ this requires LHS to be negative and RHS to be positive:

equation 17

$$\frac{m_{i-1} + m_i}{2} + a - m_i = m_i - a - \frac{m_{i+1} + m_i}{2} \Rightarrow m_{i+1} = 2m_i - m_{i-1} - 4a$$

The solution for difference equations is :

equation 18

$$m_i = \lambda_i^2 + \mu i + v$$

Plugging into :

equation 19

$$m_3 = 2m_2 - m_1 - 4a \Rightarrow 9\lambda + 3\mu + v = 2(4\lambda + 2\mu + v) - (\lambda + \mu - v) - 4a$$

So $\lambda = -2$.We also know that $m_0 = 0$.Which implies that :

equation 20

$$m_2 = 2m_1 - 4a \Rightarrow 4\lambda + 2\mu + v = 2\lambda + 2\mu + 2v - 4a \Rightarrow -8a + v = -4a + 2v - 4a \Rightarrow v = 0$$

And we know that $m_p = 1$, which implies that $m_p = 1$.And:

equation 21

$$m_i = \lambda_i^2 + \mu i + v \Rightarrow 1 = -2ap^2 + \mu p \Rightarrow \mu = \frac{1}{p} + 2ap$$

And if $p = 3$:

equation 22

$$\begin{gathered} m_0 = 0 \\ m_1 = \frac{1}{3} + 4a \\ m_2 = \frac{2}{3} + 4a \\ m_3 = 0 \end{gathered}$$

So how many partitions can we support? By rewriting $m_{i+1} = 2m_i - m_{i-1} - 4a$:

equation 23

$$m_{i+1} - m_i = m_i - m_{i=1} - 4a$$

We get:

equation 24

$$\begin{gathered} m_2 - m_1 = m_1 - m_0 - 4a \\ m_3 - m_2 = m_1 - m_0 - 8a \\ \vdots \\ m_p - m_{p-1} = m_1 - m_0 - 4a \end{gathered}$$

So, for the sequence to be increasing we need:

inequality 9

$$m_1 - m_0 > (p-1)4a$$

Or, plugging back in:

inequality 10

$$\frac{1}{p} + 2a(1-p) > 0$$

As $\lim_{p\to\infty} = -\infty$ this defines the maximal possible p that can be supported. And it is decreasing in a,we Notice that the actual nature of the signal is meaningless. Formally equilibrium consists of set of signaling rules for$S$denoted $q(n|m)$ and an action rule for R denoted $y(n)$:

equation 25

$$\forall m \in [0,1], \int_N q(n|m)dn = 1$$

Borel set[10] N is the set of feasible signals and if n^* is the support of $q(\cdot\,|m)$ then n^* solves:

equation 26

$$\max_{n\in N} U^S(y(n), mb)$$

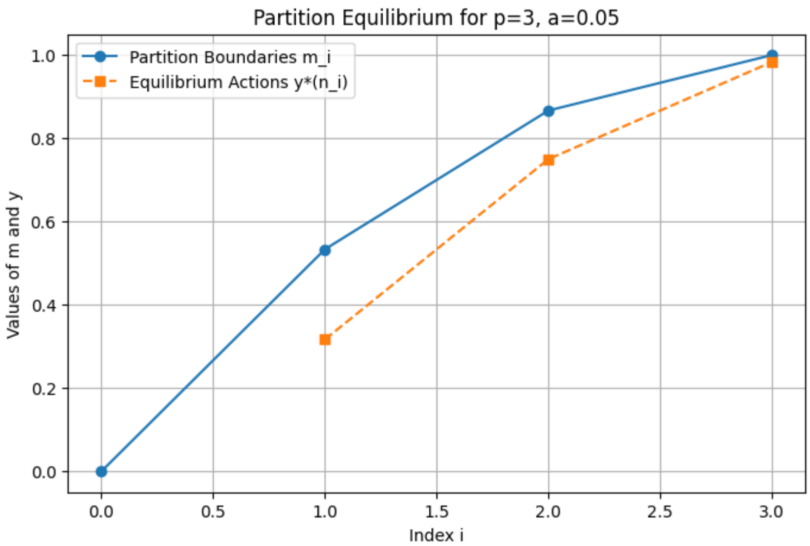
$\forall n, y(n)$ solves:

equation 27

$$\max_y \int_0^1 U^R(y,m)p(m|n)dm$$

Where $p(m|n) = \frac{\frac{q(n|m)}{f(m)}}{\int_0^1 q(n|t)f(t)dt}$. Now we will present graphic partition equilibrium for $p = 3$.

Figure 2 Partition equilibrium for $p = 3, a = 0.05$



Source: Author's own calculations

This code computes and plots the partition equilibrium for $p = 3$ and $a = 0.05$. It follows the recursive equation to determine the boundary points mim_imi and the equilibrium actions $y^*(n_i)$. Now for example:

[10] A Borel set is an element of a Borel sigma-algebra. Roughly speaking, Borel sets are the sets that can be constructed from open or closed sets by repeatedly taking countable unions and intersections. Formally, the class B of Borel sets in Euclidean $\mathbb{R}^n$ is the smallest collection of sets that includes the open and closed sets such that if E, E_1, E_2 ... ..are in B, then so are $\bigcup_{i=1}^{\infty} E_i, \bigcap_{i=1}^{\infty} E_i, \wedge\ \mathbb{R}^n \backslash E$, where $F \backslash E$ is the set difference see, Croft et al.(1991).

equation 28

$$U^S(y,m,b) = -\big(y-(m+b)\big)^2$$
$$U^R(y,m) = -(y-m)^2$$

Optimal choice is given as:

equation 29

$$y^s(m,b) = m+b, y^R(m) = m$$

Best response is given as $n_i \in (a_i, a_{i+1})$:

equation 30

$$\bar{y}(a_i, a_{i+1}) = \frac{a_i + a_{i+1}}{2}$$

In the equilibrium, the partition is constructed so that the Sender is indifferent between $\bar{y}(a_i, a_{i+1})$ and $\bar{y}(a_{i-1}, a_i)$ at a_i :

equation 31

$$\left(\frac{a_i + a_{i+1}}{2} - (m+b)\right)^2 = \left(\frac{a_{i-1} + a_i}{2} - (m+b)\right)^2$$

This only holds if they differ in sign:

equation 32

$$\left(\frac{a_i + a_{i+1}}{2} - (a_i+b)\right)^2 = \left(\frac{a_{i-1} + a_i}{2} - (a_i+b)\right)^2$$

That is:
equation 33

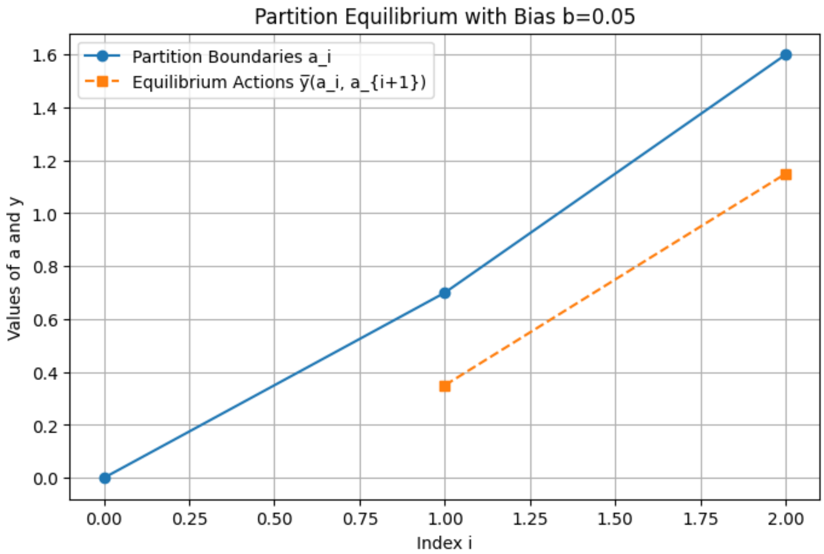
$$a_{i+1} - a_i = a_i - a_{i-1} + 4b$$

That is, each step size must increase by $4b$. As the partition depends on b, it also determines the maximal partition size:
equation 34

$$N(b) = \left\lceil -\frac{1}{2} + \frac{1}{2}\left(1+\frac{2}{b}\right)^{\frac{1}{2}} \right\rceil$$

To at least have some information in (one) equilibrium, $b < \frac{1}{4}$. As $b \to 0$, scope for more and more information transmission if preferences are closer, parties have more meaningful communication, even in cheap talk.

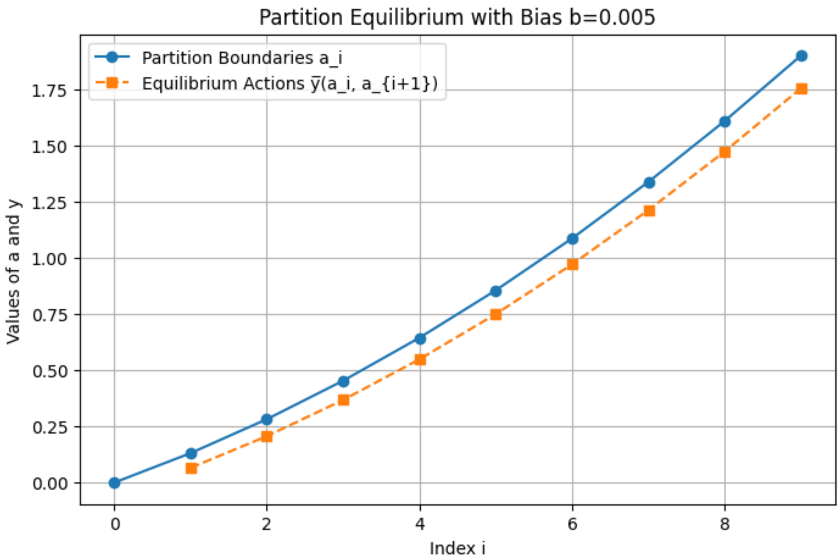
Figure 3 partition equilibrium with bias $b = 0.05$



Source: Authors' own calculations

(array([0. , 0.7, 1.6]), [np.float64(0.35), np.float64(1.15)], 2)

Figure 4 partition equilibrium with bias $b = 0.005$



Source: Authors' own calculations

3.Green,Stokey (2003) framework :A two-person game of information transmission

Here the two individuals will be called the *agent* and the *principal*. Their joint decision problem is to choose an *action*, $a_k k$, from the set $A = \{a_1, \ldots, a_K\}$. The von Neumann–Morgenstern utility levels of the two participants depend upon the chosen action and the realization of the *state of nature*, θ_m, from the set $\Theta = \{\theta_1, \ldots, \theta_M\}$. These utilities can be represented by $K \times M$ matrices $U = [u_{km}]$ and $U' = u'_{km}$ for the principal and the agent, respectively, where the elements are the utilities realized if a_k is chosen and θ_m occurs. The agent receives an *observation* which is statistically related to the true state in Θ, and transmits the observation to the principal. He might not do so truthfully. There are N possible observations, y_n, in the set $Y = \{y1,\dots,y_N\}$. Allowing randomizations, his strategies can be represented by an $N \times N$ Markov matrix $R = [r_{nn'}]$, where $r_{nn'}$ is the probability that $y_{n'}$ is transmitted given that the actual observation is y_n. Now, the principal chooses action $a_k \in a$ given that the observation $y_{n'}$ has been transmitted to him. Again, allowing randomization, his strategy is an $N \times K$ Markov matrix $Z = [z_{n'k}]$,where $z_{n'k}$ is the probability that $a_k$is chosen given that $y_{n'}$ was transmitted. The statistical relationship between states and observations is called the information structure. It is represented by an $M \times N$ Markov matrix $\Lambda = [\lambda_{mn}]$, where λ_{mn} is the probability that y_n is observed if the true state is θ_m. The interpretation of y_n depends on the prior beliefs of the individual in question. We allow different beliefs, $\pi = (\pi_1,\dots,\pi_M) \in \Delta^M$ and $\pi' = (\pi'_1,\dots,\pi'_M) \in \Delta^M$for the principal and agent, respectively, where Δ^M is the set of all M-dimensional probability vectors. The principal's posterior probabilities, given an observation, can be derived from π and Λ by Bayes rule.These posteriors are denoted$p_1,\dots,p_N$ where $p_n^P \in \Delta^M$ is his posterior if y_n is observed, for $n = 1,\dots,N$. The probability of observing each y_n is also implied by π and Λ. Thus, we have a distribution of the posterior which is simply the measure over Δ^M assigning the corresponding weight to each of the p_n^P . A similar argument applies for the agent. If the strategy choices are Z and R, the expected utilities for the principal and agent, respectively are[11]:

equation 35

$$tr\ U\Pi\Lambda RZ$$
$$tr\ U'\Pi'\Lambda RZ$$

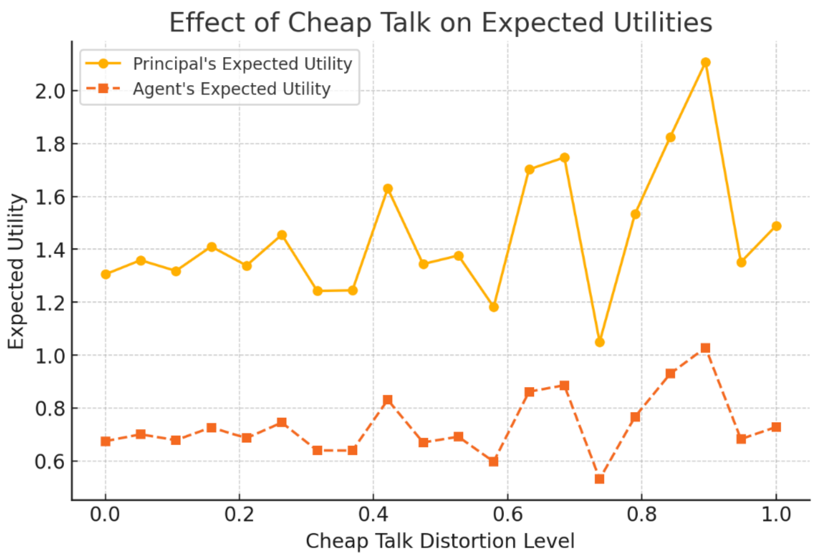
An information structure (Y,Λ) is said to be more informative than (Y',Λ') if, for any U and any π, the decision problem under the former has at least as high a value as that under the latter. Using the notation developed above, this can be restated as

inequality 11

$$\max_Z trU\Pi\Lambda Z \geqq \max_{Z'} trU\Pi\Lambda' Z'$$

Λ is more informative than Λ' if and only if there exists a Markov matrix B such that $\Lambda' = \Lambda B$.

Figure 5 Effect of cheap talk on expected utilities

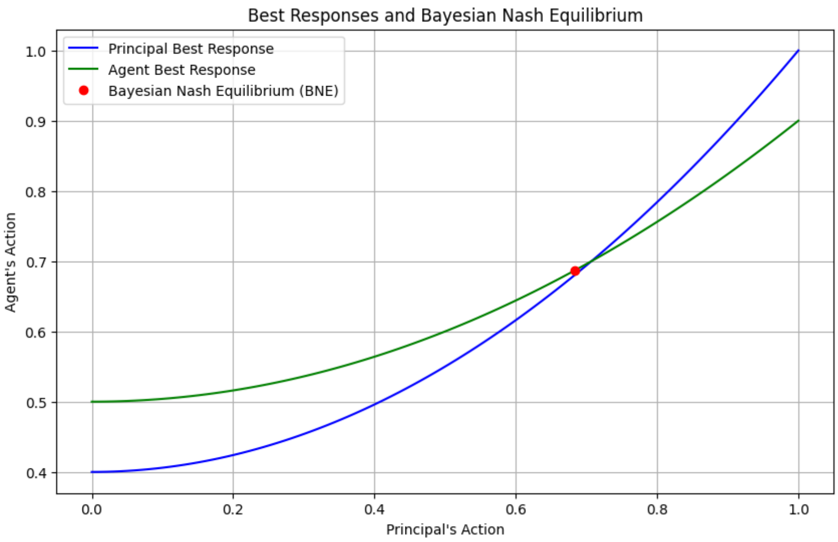


Source: Authors' own calculations

[11] Trace Operator $(tr(\cdot))$: The trace of a matrix is the sum of its diagonal elements.

The plot illustrates how increasing cheap talk distortion affects the expected utilities of both the principal and the agent. As the distortion increases, the principal's expected utility generally decreases due to misinformation, while the agent's utility fluctuates depending on the strategic advantage gained from misrepresentation.

Figure 6 BR and BNE



Source: Authors' own calculations

The blue curve represents the best response for the principal and will be a non-linear curve. The green curve represents the best response for the agent and will also be a distinct non-linear curve. The red dot will show the Bayesian Nash Equilibrium (BNE), which is where the two curves intersect, ensuring that both the principal's and the agent's strategies are mutually best responses. Mathematical representation of previous starts with expected utilities:

- ✓ $U \in \mathbb{R}^{K\times M}$: the utility matrix for principal
- ✓ $U' \in \mathbb{R}^{K\times M}$: the utility matrix for agent
- ✓ $\Pi \in \mathbb{R}^{M\times 1}$: the principal's belief over states the world
- ✓ $\Pi' \in \mathbb{R}^{M\times 1}$: the agents' belief over states the world
- ✓ $\Lambda \in \mathbb{R}^{M\times N}$: the information structure (mapping states to observation).
- ✓ $\mathrm{R} \in \mathbb{R}^{N\times N}$: The agent's observation strategy.
- ✓ $\mathrm{Z} \in \mathbb{R}^{N\times K}$: The principal's action strategy.

The principal's expected utility can be written as:

equation 36

$$EU_{principal} = \mathbb{E}[U] = \Pi^T \cdot (\Lambda \cdot R \cdot Z)$$

- ✓ Π^T is the principal's belief distribution over states of the world, transposed to a row vector.
- ✓ $\Lambda \cdot R \cdot Z$ is the transformation of the information structure (states to observations) combined with the agent's actions and the principal's strategies.

Since the utility matrix U is $K \times M$ and $\Pi^T \cdot (\Lambda \cdot R \cdot Z)$ will be $1 \times K$, the expected utility is computed by summing element wise multiplication:

equation 37

$$EU_{principal} = \sum_k U(k) \cdot \Pi^T \cdot (\Lambda \cdot R \cdot Z)$$

On the other hand, the agent's expected utility is:

equation 38

$$EU_{agent} = \mathbb{E}[U'] = \Pi'^T \cdot (\Lambda \cdot R \cdot Z)$$

- ✓ Π'^T is the agent's belief distribution.
- ✓ $\Pi'^T \cdot (\Lambda \cdot R \cdot Z)$ is the transformation for the agent's observation strategy and the principal's actions.

The best response functions for both the **principal** and the **agent** can be derived from their expected utilities. These functions determine the optimal action for one agent, given the action taken by the other agent.

Principal's Best Response Function:

- ✓ We define the best response of the principal $BR_{principal}(a_{agent})$

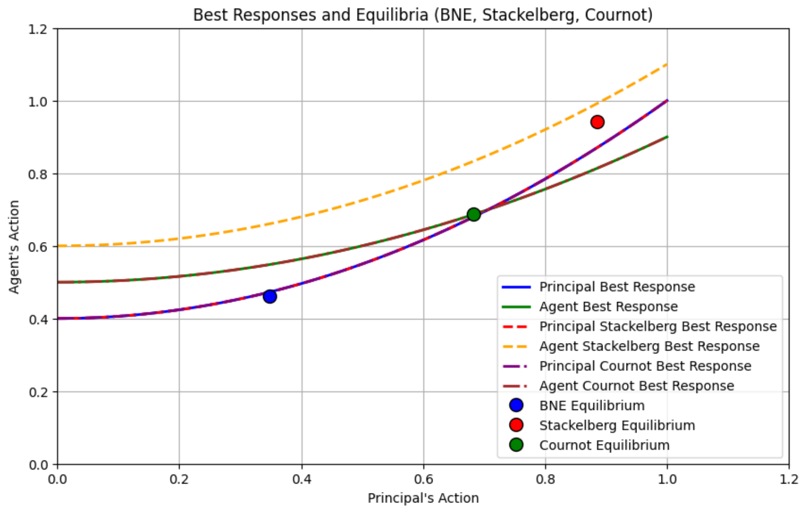
equation 39

$$BR_{principal}(a_{agent}) = 0.6 \cdot a_{agent}^2 + 0.4$$

- ✓ We define the best response of the principal $BR_{agent}\,(a_{principal})$

$$BR_{agent}\,(a_{principal}) = 0.4 \cdot a_{principal}^2 + 0.5$$

Figure 7 BNE,Cournot,Stackelberg equilibria



Source: Authors' own calculations

The BNE is found by solving the system of best response functions for both agents simultaneously. This is done by setting the principal's and agent's best response functions equal to each other.

BNE equilibrium:

equation 40

$$BR_{principal}(a^*_{agent}) = a^*_{principal}$$
$$BR_{agent}\,(a^*_{principal}) = a^*_{agent}$$

Using the best response functions:

equation 41

$$0.6(a^*_{agent})^2 + 0.4 = a^*_{principal}$$
$$0.6(a^*_{principal})^2 + 0.5 = a^*_{agent}$$

Cournot Equilibrium

The Cournot equilibrium is characterized by the intersection of the best response functions for the principal and the agent, where each agent's action is optimal given the other’s action. It can be found by solving the system of equations that sets the best responses equal to each other. To find the Cournot equilibrium, solve the system of best responses:

equation 42

$$BR_{principal}(a_{agent}) = a_{principal}$$
$$BR_{agent}\,(a_{principal}) = a_{agent}$$

Or solve:

$$0.6a^2_{agent} + 0.4 = a^*_{principal}$$
$$0.4a^2_{principal} + 0.5 = a^*_{agent}$$

Stackelberg Equilibrium

In the Stackelberg equilibrium, the principal moves first and the agent responds to the principal’s action. The principal’s best response is determined by the agent’s reaction to the principal’s actions. To find the Stackelberg equilibrium, solve the system where the principal chooses $a_{principal}$ first, and the agent reacts to this choice:

equation 43

$$a_{principal} = BR_{principal}(a^*_{agent})$$
$$a^*_{agent} = BR_{agent}(a^*_{principal})$$

Or solve:

$$a^*_{principal} = 0.6(a^*_{agent})^2 + 0.4$$
$$a^*_{agent} = 0.4(a^*_{principal})^2 + 0.5$$

See Dixon,H(2001) on these types of duopolies. The BNE, Cournot, and Stackelberg equilibria are derived from the best response functions, and their solutions represent the optimal actions for the principal and agent in the game. The BNE and Cournot equilibria are solved simultaneously, while in the Stackelberg model, the principal chooses first, and the agent responds accordingly.

4.Kripke semantics,classical and intuitionistic Kripke model, and Zermelo-Fraenkel set theory

Kripke semantics is a formal system used to model modal logic—logics that involve modalities like necessity (□) and possibility (◊). These modalities express statements about what is necessarily true or possibly true, often in the context of knowledge, belief, or time. Kripke semantics provides a structure to interpret these modalities using possible worlds. The concept of Kripke model is due to Kripke (1959), Kripke(1962),Kripke (1963),Kripke (1965).

Definition 1 A classical Kripke model (due to Ilik et al.(2010) is given by a quintuple $(K, \leq, D, \Vdash_s, \Vdash_\perp)$, K inhabited, such that $(K, \leq)$is a poset[12]of possible worlds,D is the domain function assigning sets to the elements of K such that:

Well-formed formula 1

$$\forall w, w' \in K, (w \leq w' \Rightarrow D(w) \subseteq D(w'))$$

i.e., D is monotone. Let the language be extended with constant symbols for each element of $\mathcal{D} := \cup \{D(w) : w \in K\}$.And,$(-):(-) \Vdash_s$ is a binary relation of "strong refutation" between worlds and atomic sentences in the extended language such that:

Well-formed formula 2

$$w: X(d_1, \dots, d_n) \Vdash_s \Rightarrow d_i \in D\,(w), \forall i \in \{1, \dots, n\}$$
$$w: X(d_1, \dots, d_n) \Vdash_s, w \leq w' : X(d_1, \dots, d_n) \Vdash_s$$

The relation $\Vdash$ is called the *satisfaction relation*, *evaluation*, or *forcing relation*[13] Given a model M (usually a transitive model of ZFC-Zermelo–Fraenkel set theory,see Zermelo (1930)), any poset $(P, <)$in it is a notion of forcing and its elements forcing conditions.

Definition 2 For $\mathbb{P} \in V$ a poset and $p \in \mathbb{P}$,we say *p forces* φ and write $p \Vdash \varphi$ iffor every generic over V filter X containing, p, $V[X] = \varphi$.

Definition 3 The relation $(-):(-) \Vdash_s$ of strong refutation is extended to the relation between worlds w and composite sentences A in the extended language with constants in $D(w)$, inductively, together with the two new relations. A sentence A is forced in the world w (notation $w : A \Vdash$) if any world $w' \geq w$,which strongly refutes A, is exploding. A sentence A is forced in the world w (notation $w : A \Vdash$) if any world $w' \geq w$,which forces A, is exploding

Definition 4 A modal logic Λ is a set of modal formulas that contains all propositional tautologies and has the following closure conditions:

Well-formed formula 3

(modus ponens)[14] $\phi \in \Lambda \,; \wedge (\phi \rightarrow \psi) \in \Lambda, \rightarrow \psi \in \Lambda$

(uniform substitution) if $\phi \in \Lambda$ then any complete substitution of propositional variables of ϕ is also a formula in Λ.If $\phi \in \Lambda$ we may say ϕ is a theorem of Λ ,or equivalently, $\vdash_\Lambda \phi$ .Otherwise we have $\nvdash_\Lambda \phi$.

---

[12] A partially ordered set (normally, poset) is a set, L, together with a relation, $\leq$, that obeys, $\forall\, a, b, c \in L$: (reflexivity) $a \leq a$; (anti-symmetry) if $a \leq b$ and $b \leq a$ then $a = b$; and (transitivity) if $a \leq b$ and $b \leq c$ then $a \leq c$. The relation $\leq$ is called a partial order on L. See, Dickson (2007).

[13] In mathematics or set theory forcing is a technique for proving consistency and independence results.

[14] A valid form of argument in which the antecedent of a conditional proposition is affirmed, thereby entailing the affirmation of the consequent

Well-formed formula 4

$$w{:}\,A \wedge B \Vdash_s \ if\ w{:}\,A \Vdash \bigvee w{:}\,B \Vdash$$
$$w{:}\,A \vee B \Vdash_s \ if\ w{:}\,A \Vdash \bigwedge w{:}\,B \Vdash$$
$$w{:}\,A \rightarrow B \Vdash_s \ if\ w{:} \Vdash A \bigwedge w{:}\,B \Vdash$$
$$w{:}\,\forall x A(x) if\ w{:}\,A(d) \Vdash for\ some\ d \in D(w)$$
$$w{:}\,\exists x.\,A(x) \Vdash_s, \forall\, w' \geq w \wedge d \in D(w'), w'{:}\,A(d) \Vdash;$$
$$\bot \text{ is always srongly refuted}$$
$$\top \text{ is never strongly refuted}$$

Lemma 2 Let $\mathcal{M}_0, \Gamma \Vdash B \Leftrightarrow B \in \Gamma$

This is called *Main Semantic Lemma*. Or the Main Semantic Lemma states:

1. If a formula ϕ is provable in the modal logic i.e. $\vdash \phi$, then ϕ is true in all models i.e. $M \vDash \phi$.
2. If ϕ is true in all models, then ϕ is provable i.e. $M \vDash \phi$ *implies* $\vdash \phi$.

Lemma 3 This holds in classical Kripke semantics

1. $w{:} \Vdash A \Leftrightarrow \neg A \Vdash_s$
2. $w{:}\,A \Vdash \Leftrightarrow w{:} \Vdash \neg A$
3. $w{:}\,\neg A \Vdash \Leftrightarrow w{:} \Vdash A$
4. $w{:}\,\neg A \Vdash \Leftrightarrow w{:}\,\neg A \Vdash_s$
5. $w{:} \Vdash A \Leftrightarrow w{:} \Vdash \neg\neg A$
6. $w{:}\,A \Vdash \Leftrightarrow w{:} \Vdash \neg\neg A \Vdash$
7. $w{:}\,\neg A \Vdash_s \Leftrightarrow w{:} \Vdash \neg\neg A \Vdash \Leftrightarrow w{:} \Vdash A$

Proof: under number 1 obvious because $w{:}\,\bot \Vdash$; under second it is obvious because: $w{:} \Vdash A \rightarrow B \Leftrightarrow \forall w' \geq w, w'{:} \Vdash A \Rightarrow w'{:} \Vdash B$, $w{:} \Vdash A \wedge B \Leftrightarrow w{:} \Vdash A \wedge w{:} \Vdash B$, $w{:} \Vdash A \vee B \Leftarrow w{:} \Vdash A \vee w{:} \Vdash A(d)$, $w{:} \Vdash A \vee B \Leftarrow w{:} \Vdash A \vee w{:} \Vdash B$, $w{:} \Vdash \exists x A(x) \Leftarrow \forall d \in D(w), w{:} \Vdash A(d)$■,

The Zermelo-Fraenkel axioms are the basis for Zermelo-Fraenkel set theory.

1.*Axiom of Extensionality*: If X and Y have the same elements, then $X = Y$.

Well-formed formula 5

$$\forall u(u \in X \equiv u \in Y) \Rightarrow X = Y$$

2. *Axiom of the Unordered Pair* (axiom of pairing): For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.

Well-formed formula 6

$$\forall a \forall b \exists c \forall x \left(x \in c \equiv \left(x = a \bigvee x = b \right) \right)$$

3. *Axiom of subsets(Axiom of Separation or Axiom of Comprehension)*: If φ is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X{:}\, \varphi(u, p)\}$ that contains all those that have the property φ .

4. *Axiom of the sum of set (Axiom of Union):* For any X there exists a set $Y = \bigcup X$, the union of all elements of X.

Well-formed formula 7

$$\forall X \exists T \forall u (u \in Y \equiv \exists z (z \in X \bigwedge u \in z))$$

5. *Axiom of the power set*: For any X there exists a set $Y = P(X)$, the set of all subsets of X.

Well-formed formula 8

$$\forall X \exists Y \forall u (u \in Y \equiv u \subseteq X)$$

6. *Axiom of Infinity*: There exists an infinite set.

Well-formed formula 9

$$\exists S \left[\emptyset \in S \bigwedge (\forall x \in S) \left[x \bigcup \{x\} \in S \right] \right]$$

7. *Axiom of Replacement*: If F is a function, then for any X there exists a set $Y = F[X] = \{F(x): x\ in\ X\}$.

Well-formed formula 10

$$\forall x \forall y \forall z \left[\varphi(x, y, p) \bigwedge \varphi(x, z, p) \Rightarrow y = z \right] \Rightarrow \forall X \exists Y \forall y [y \in Y \equiv (\exists x \in X) \varphi(x, y, p)]$$

8. *Axiom of Foundation*: Every nonempty set has an ∈ in -minimal element.

Well-formed formula 11

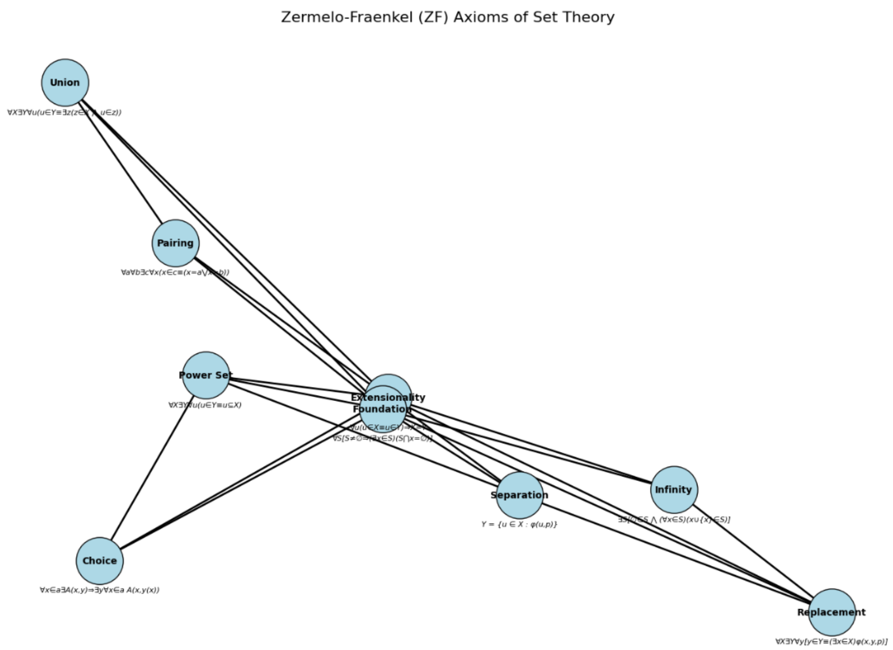
$$\forall S \left[S \neq \emptyset \Rightarrow (\exists x \in S\ ) S \bigcap x = \emptyset\ \right]$$

9. *Axiom of Choice*: Every family of nonempty sets has a choice function.

Well-formed formula 12

$$\forall x \in a \exists A(x, y) \Rightarrow \exists y \forall x \in a A\big(x, y(x)\big)$$

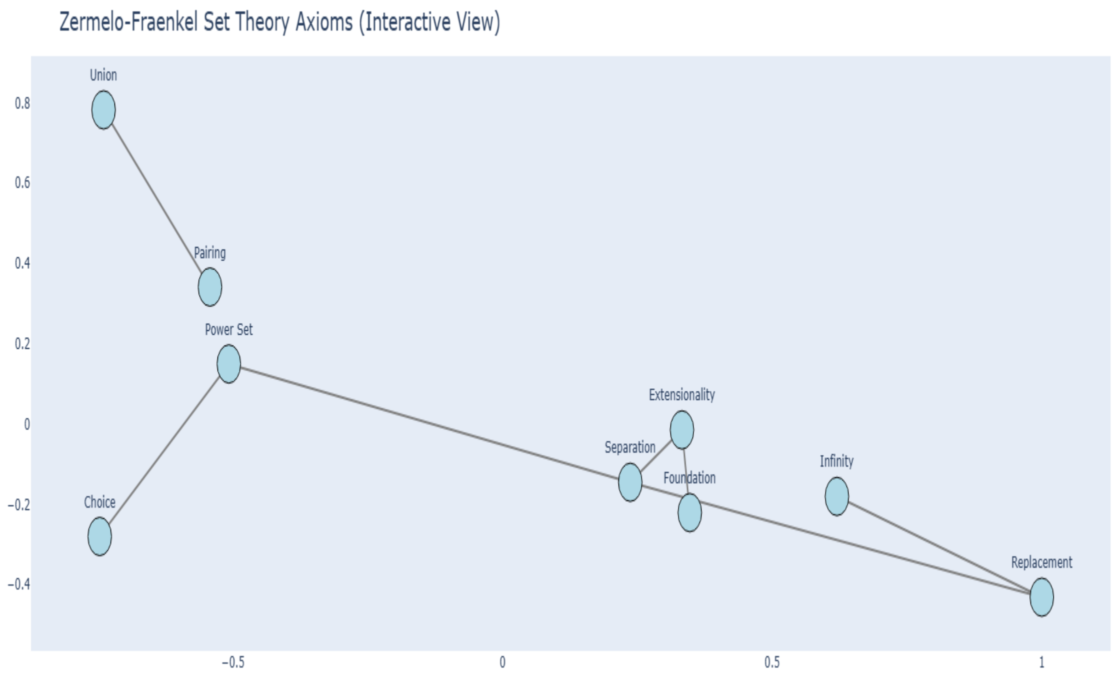
Figure 8 Zermelo -Fraenkel Axioms of set theory



Source: Author’s own calculation

Each node is an axiom. Each edge shows a rough logical dependency or usefulness in building further axioms. Mathematical formulas (WFFs) are displayed under each axiom name.

Figure 9



Source: Author's own calculation

Theorem 3 Kripke's S5 system: $N, \Omega, \{T_i\} \in N$ is a knowledge space. And $A \in 2^{\Omega}$ is an event. Where, N is a set of players, A is a finite set of actions,Ω is the space of the states of the world. We will assume here that Ω is finite.T_i is the space of possible types of player i.And, $t_i: \Omega \to T_i$ is player i's private signal or type. Information partitions is :$P_i(\omega) = \{\omega': t_i(\omega') = t_i(\omega)\}$, that is $P_i(\omega)$ is the set of states of the world for which player i has the same type as he/she does in ω. And$\omega_i \in P_i(\omega)$,the set $\{P_i(\omega)\}_{\omega\in\Omega}$is easily seen to be a partition Ω, and s called i's information partition. A knowledge space can thus be given as:$(N, \Omega, \{P_i\} \in N)$,see Tamuz (2024).

1. $K_i\Omega = \Omega$ A player knows that some state of the world has occurred. And given K_iA a set of states of world in which i knows A and $A \in 2^{\Omega}$:

equation 44

$$K_iA = \{\omega: P(\omega) \subseteq A\} \equiv K_iA = \bigcup \{\omega: P(\omega) \subseteq A\}$$

2. $K_iA \cap K_iB = K_i(A \cap B)$. A player knows A and a player knows B if and only if he knows A and B.
3. Axiom of knowledge:$K_iA \subseteq A$ a player knows A then A has indeed occurred.
4. Axiom of positive introspection: $K_iK_iA = K_iA$.If a player knows A then he/she knows that he/she knows A.
5. Axiom of negative introspection: $(K_iA)^c = K_i((K_iA)^c)$.If a player does not know A then she knows that she does not know A.

Proof:

1. This follows from the definition
2. $K_iA \cap K_iB = \{\omega: P_i(\omega) \subseteq A\} \cap \big\{\omega: P_i(\omega) \subseteq B = \{\omega: P_i(\omega) \subseteq A, P_i(\omega) \subseteq B\}\big\} =$ $\{\omega: P_i(\omega) \subseteq A \cap B\} = K_i(A \cap B)$

3. If $\omega \in K_iA$,so that $P_i(\omega) \subseteq A$,since $\omega_i \in P_i(\omega)$,it follows that $\omega \in A$ and so$K_iA \subseteq A$.
4. By the previous we have that $K_iK_iA \subseteq K_iA$. Now, let $\omega \in K_iA$ so that $P_i(\omega') = P_i(\omega)$,so it follows that $\omega' \in K_iA$,and since ω' is an arbitrary element of $P_i(\omega)$ it was shown that $P_i(\omega) \subseteq K_iA$, and hence by definition $\omega \in K_iK_iA$
5. The left-hand side $(K_iA)^c$ represents the event that agent i does *not* know A. The right side, $K_i((K_iA)^c)$ represents the event that agent i knows that they do not know A.In modal logic we apply positive introspection i.e. if an agent knows something, they know that they know it.Formally, $K_iA \Rightarrow K_iK_iA$. We also assume the negative introspection axiom i.e., if an agent does not know something, they know that they do not know it: $(K_iA)^c = K_i((K_iA)^c)$■

Intuitionistic propositional logic is sound w.r.t. Kripke semantics:

Theorem 4 if $\vdash_{Int} A$,then for every Kripke model $\mathcal{M} = \langle W, R, v\rangle$ and for every possible world $x \in W$ of this model $\mathcal{M}, x \Vdash A$.

Proof: in order to prove soundness, one needs to prove that if A is an axiomof Int,then $\mathcal{M}, x \Vdash A$ and second if $\mathcal{M}, x \Vdash A \to B \Rightarrow \mathcal{M}, x \Vdash B$.The second part is easy: If $x \Vdash A \to B$.then for every world $y \in R(x)$ we have either $y \nVdash A$ or $y \nVdash B$.Since $y = x$by reflexivity of R ,then given $x \Vdash A$,we obtain $x \Vdash B$. Here, we need to prove $x \Vdash A \to (B \to C) \to \big((A \to B) \to (A \to C)\big)$.In order to establish that a formula of the form $E \to F$ is true in x ,one needs o check that $\forall y \in R(x)$ if $y \Vdash E$, then $y \Vdash F$.Again ,lets consider arbitrary $z \in R(y)$, such that $z \Vdash A \to B$.On this turn we need to show that $z \Vdash A \to C$. Let w be a world from $R(z)$, such that $w \Vdash A$ and finally we need $w \Vdash C$. So, now:

Well-formed formula 13

$$\begin{array}{c} w \Vdash A \\ \uparrow \\ z \Vdash A \to B \\ \uparrow \\ y \Vdash A \to (B \to C) \\ \uparrow \\ x \end{array}$$

By monotonicity, since yRw, zRw , the formulae $A \to (B \to C)$ and $A \to B$ are also true in w. Since modus ponens[15] is applicable for $\Vdash$,we have $w \Vdash B \to C$,$w \Vdash B, w \Vdash C$ which is our goal ■ .

5.Kripke model and cheap talk with partial and separating equilibrium

Agent A has a private type: $\theta \in \{H, L\}$.Sender sends messages: $m \in \{m_H, m_L\}$ to receiver Agent A. Cheap talk: message is costless and non-binding. Receivers observe messages, not type, and chooses action a. in Kripke semantics each world corresponds to a full specification of: type θ and message m.Let's define:

equation 45

$$W = \{w_H, m_H, w_H, m_L, w_L, m_H, w_L, m_L\}$$

Each world $w \in W$ has a structure:

equation 46

$$w = \big(\theta(w), m(w)\big)$$

A Kripke frame $i \in \{A, B\}$ for the two agents is:

---

[15] It can be summarized as "P implies Q. P is true. Therefore, Q must also be true." Or $\frac{P \to Q,\ \ P}{Q}$ see Stone (1996).

equation 47

$$\mathcal{F}_i = (W, R_i)$$

Agent A (Sender):Always knows their own type and message.Thus:

equation 48

$$R_A = \{(w, w) | w \in W\}$$

Agent B receiver observes only the message $m(w)$, not the type. Hence, their epistemic accessibility is:

equation 49

$$R_B = \{(w, w') \in W \times W | m(w) = m(w')\}$$

Receiver cannot distinguish between worlds with the same message. Propositional atoms p_θ are true if world has a type θ .M_m is true if message is m.$B_{i\phi}$ agent i believes formula φ.So in a world $w_{L_{m_H}}$,$p_L \wedge M_{m_H}$. Receiver's belief:

Well-formed formula 14

$$B_B(p_L) \vDash \forall w' \in R_B(w), p_L(w') = \vDash$$

But in partial separation: p_L may be **false in some** $R_B(w)$ worlds.So instead we attach probabilistic beliefs:

Well-formed formula 15

$$\mu_B(p_H | m_H) = 0.7$$
$$\mu_B(p_L | m_H) = 0.3$$

These are posterior beliefs based on Bayes' Rule (if prior and sender's strategy are known). Now for Bayesian Semantics: $\mu(B)$ be a prior probability over types, $\sigma_A(m|\theta)$. Then the posterior of Receiver after message m is:

equation 50

$$\mu_B(\theta | m) = \frac{\mu(\theta) \cdot \sigma_A(m|\theta)}{\sum_{\theta'} \mu(\theta') \cdot \sigma_A(m|\theta)}$$

Each world $w = \{\theta, m\}$,gives payoff: to receiver $u_B(\theta, a(m))$.

Separating equilibrium:

inequality 12

$$\forall w \neq w' \in W$$
$$\theta(w) \neq \theta(w') \Rightarrow m(w) \neq m(w')$$
$$R_B(w) = \{w\}$$

Pooling equilibrium:All types send the same message:

equation 51

$$m(w) = m^* \forall w \in W$$

Receiver's belief spread:

equation 52

$$R_B(w) = \{w_{H,m^*}, w_{L,m^*}\}R_B(w)$$

Partial Separation (Mixed Strategy):

- ✓ Types send different messages with probability.
- ✓ Receiver updates probabilistically
- ✓ Receiver's accessibility relation reflects epistemic uncertainty, i.e.:

equation 53

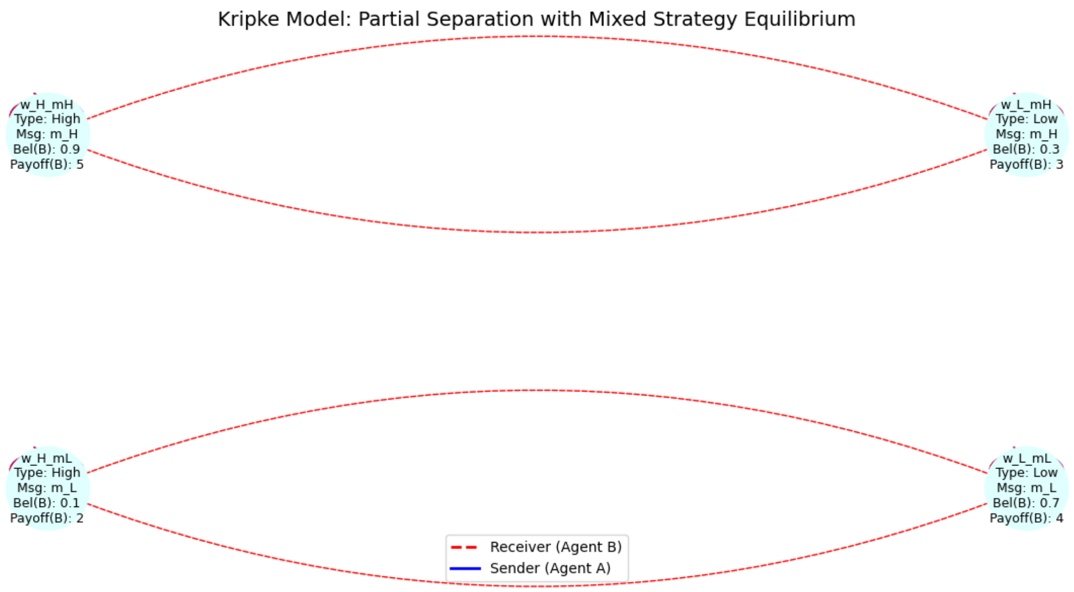
$$R_B(w_{H,m_L}) = \{w_{H,m_L}; w_{L,m_L}\}$$
$$\mu_B(H|m_L) < 1$$

Table 1 summary table

World w	Type θ	Message m	Accessible $R_B(w)$	$\mu_B(H\|m)$
w_{H,m_H}	H	m_H	$\{w_{H,m_H}, w_{L,m_H}\}$	0.7
w_{L,m_H}	L	m_L	$\{w_{H,m_H}, w_{L,m_H}\}$	0.7
w_{H,m_L}	H	m_H	$\{w_{H,m_L}, w_{L,m_L}\}$	0.3
w_{L,m_L}	L	m_L	$\{w_{H,m_L}, w_{L \cdot m_L}\}$	0.3

Source: Author's own calculation

Figure 10 Kripke model: Partial separation with mixed strategy equilibrium



Source: Author's own calculation

{'m_H': {'posterior': {'H': 0.75, 'L': 0.25}, 'best_action': 'a_H', 'expected_utilities': {'a_H': 0.75, 'a_L': 0.25}}, 'm_L': {'posterior': {'H': 0.12500000000000003, 'L': 0.875}, 'best_action': 'a_L', 'expected_utilities': {'a_H': 0.12500000000000003, 'a_L': 0.875}}}

6.Common knowledge

Let the state of the world be represented by Ω, which is a set of possible states $\omega \in \Omega$. Each agent i is associated with an information partition $\mathcal{P}_i$ which is a partition of Ω . This partition represents the agent's knowledge, i.e., what states of the world the agent can distinguish. If two states ω, and ω' are in the same element of $\mathcal{P}_i$, then agent i cannot distinguish between these two states. Now, knowledge can be represented as set theoretic concept, for each agent i, the information partition $\mathcal{P}_i$ induces a **knowledge operator** K_i, where for any event $E \subseteq \Omega, K_i(E)$ is the set of states in which agent i knows that E has occurred. Formally we define this as:

equation 54

$$K_i(E) = \{\omega \in \Omega |\ \forall\omega' \in \mathcal{P}_i(\omega), \omega' \in E\}$$

In words agent i knows that event E occurs if, at state ω , all the states indistinguishable from ω i.e. those in same partition cell are also in E. Common knowledge among all agents can be derived using set theory. This is Geanakopolos (1992) model of common knowledge.

Definition 5 We can define common knowledge of an event E as the event where everyone knows E, everyone knows that everyone knows E, and so on ad infinitum.

This is captured by the common knowledge operator $K^*(E)$ which is the intersection of all iterated knowledge operators:

equation 55

$$K^*(E) = \bigcap_{n=1}^{\infty}\left(\bigcap_{i_1,i_2,\dots,i_n \in N} K_{i_1}, K_{i_2}, \dots, K_{i_n}(E)\right)$$

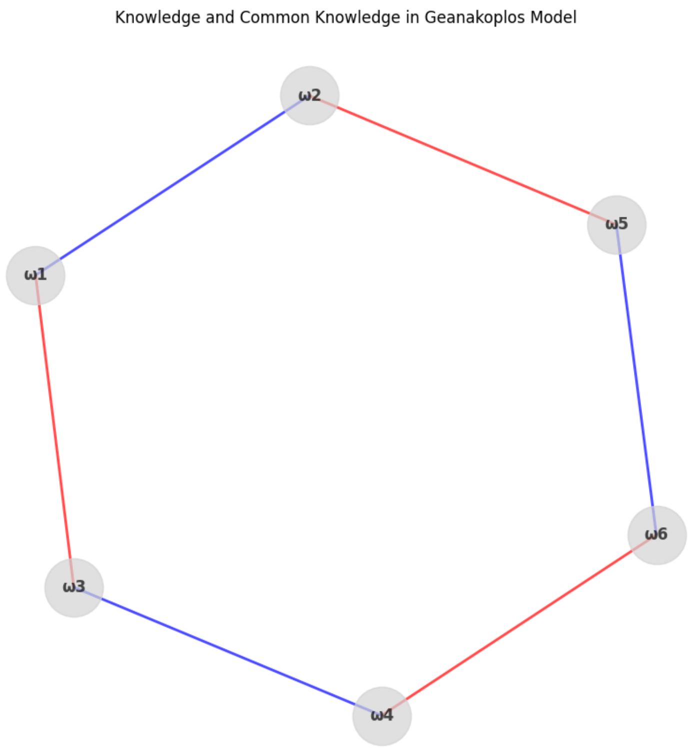
This intersection represents the set of states where E is common knowledge—i.e., where all agents know E, all agents know that all agents know E, and so on. Geanakoplos's work often involves Bayesian updating, where agents revise their beliefs based on new information. In a set-theoretic framework, we can model this as follows. Each agent i has a prior belief which is represented by probability distribution $\mu_i \in \Omega$. When agent i observes an event E, they update their belief using Bayes' rule[16]. The updated belief $\mu_i(E|\omega)$ is defined as :

equation 56

$$\mu_i(E|\omega) = \frac{\mu_i\big(E \cap \mathcal{P}_i(\omega)\big)}{\mu_i(\mathcal{P}_i|\omega)}$$

[16] $p(A_i|A) = \frac{P(A_i)P(A|A_i)}{\sum_{j=1}^{N} P(A_j)P(A|A_j)}$ see Papoulis (1984).Previous: $P(B_j|A) = \frac{P(B_j)P(A|B_j)}{P(A)}$, and $S = \bigcup_{i=1}^{N} A_i$.

Figure 11 knowledge and common knowledge in Geanakoplos model



Source: Authors' own calculations

- ✓ Blue edges representing Agent 1's knowledge.
- ✓ Red edges representing Agent 2's knowledge.
- ✓ Green edges representing common knowledge (where both agents know the event).

7.Common knowledge in Kripke frame

Kripke frame with valuation function is:

equation 57

$$M = \{W, (R_i)_{i \in N}, V\}$$

$V\colon Prop \to 2^W$ is a valuation function that assigns a set of worlds to each proposition $p \in Prop$,$\forall p, V(p) \subseteq W$ set of worlds where $p \to true$. Knowledge operator that we should define is K_i for each agent i.Now, given a Kripke model $M = \{W, (R_i)_{i \in N}, V\}$ and a world $w \in W$,agent i knows p at world w, if for $\forall w' \Rightarrow wR_i w'$ i.e. w' is possible according to agent $i's$ knowledge in world w,p holds in w' or formally:

Well-formed formula 16

$$M, w \vDash K_i p \Leftrightarrow \forall w' \in W, (wR, w') \Rightarrow M, w' \vDash p$$

This means that agent i knows p at world w if in all worlds they consider possible $p \rightarrow true$.We can define common knowledge[17] here by using iterated knowledge operator over the agents. Now, let K_i denote the knowledge operator for agent i and let N be the set of all agents. The common knowledge operator can be defined recursively:

equation 58

$$C_p = \bigcap_{n=1}^{\infty} K_1, K_2, .., K_n p$$

Alternatively, we can define common knowledge by creating a new relation $R_C$called common knowledge relation, which is transitive closure of the union of the individual relations R_i:

equation 59

$$R_C = \bigcap_{i \in N} R_i$$

Then the common knowledge operator C can be defined as:

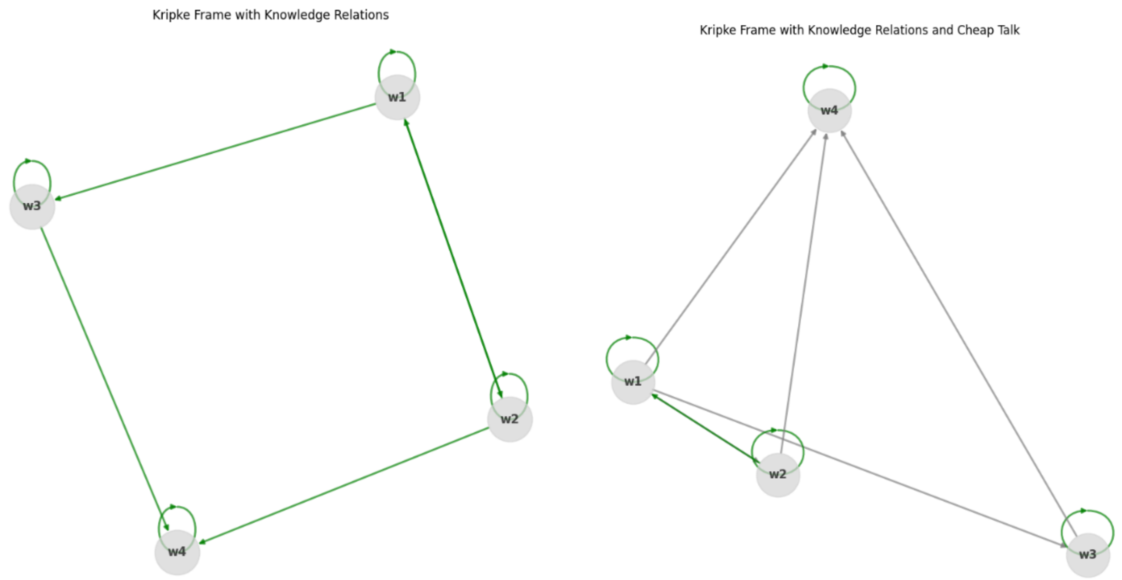
Well-formed formula 17

$$M, w \vDash C_p \Leftrightarrow \forall w' \in W, (w R_C w') \Rightarrow M, w' \vDash p$$

This means that common knowledge of p hold at world w if, $\forall w'$ worlds that are reachable through the common knowledge relation R_C, p holds:

Figure 12 Kripke frame with knowledge relations without and with cheap talk

[17] A very basic assumption of studies in game theory is that the game is common knowledge, see Rubinstein (1989).Situations without common knowledge are labeled as games with incomplete information see Harsanyi (1967) part I, Harsanyi (1968)part II, Harsanyi (1968)part III.



Source:Author's own calculations

In the current example, agents in worlds w_1, w_2, w_3 can communicate with agents in $w_4$through cheap talk. This allows us to model a scenario where agents can talk to each other and share information, but this communication does not change the structure of knowledge in the model.

8.Aumann's agreement theorem

Aumann (1976) posed the following question: could two individuals who share the same prior ever agree to disagree? See Levin (2016). That means if i, j share common previous beliefs over states of the world, could it be that state arise at which it was commonly known that i assigns probability of some evet $\wp_i$,and j assigned probability of $\wp_j$ and$\wp_i \neq \wp_j$. Aumann concluded that this sort of disagreement is impossible. Now, formally let p be a probability measure on Ω which are agents' prior belief. For any state ω and event E , let $p\left(E\middle|\wp_i(\omega)\right)$ denote $i's$ posterior belief, so that $p\left(E\middle|\wp_i(\omega)\right)$ is obtained under Bayes' rule.The event that agent i assigns probability $\wp_i$ to E is $\{\omega \in \Omega : p(E|\wp_i(\omega) = \wp_i)\}$

Proposition 1 Suppose two agents have the same prior belief over a finite set of states Ω. If each agent's infomation function is partitional and it is common knowledge in some state $\omega \in \Omega$ that agent 1 assigns probability$\wp_1$ to some event E and agent 2 assigns probability $\wp_2$ to E, then $\wp_1 = \wp_2$

Proof: If the assumptions are satisfied then there is some self-evident event F and $\omega \in F$:

equation 60

$$F \subset \left\{(\omega' \in \Omega : p(E|\wp_1(\omega') = \wp_1) \cap \{(\omega' \in \Omega : p(E|\wp_2(\omega') = \wp_2)\}\right\}$$

Since Ω is finite, so is the number of sets in each union and let $F = \cup_k A_k = \cup_k B_k$ and for a nonempty disjoint sets C, D with with $p(E|C) = \wp_i$ and $p(E|D) = \wp_i$ we have that $p(E|C \cup D) = \wp_i$,and $\forall k, p(E|A_k) = \wp_1$,then $p(E|F) = \wp_1$ and similarly $p(E|F) = p(E|B_k) = \wp_2$ ■

9.Cheap talks and Asset bubbles

Some endogenous variable y obeys the following expectational difference equation:

equation 61

$$y_t = aE_t y_{t+1} + bx_t$$

Where $E_t y_{t+1} \equiv E(y_{t+1}|\Omega)$, $\Omega = \{y_{t-i}, x_{t-i}, i = 0, \dots, \infty\}$.

Asset–pricing model: if p_t be a price of stock, d_t is dividend, and r is the rate of return of riskless asset. assumed to be held constant over time. Standard theory of finance teaches us that if agents are risk neutral, then the arbitrage between holding stocks and the riskless asset should be such that the expected return on the stock — given by the expected rate of capital gain plus the dividend/price ratio — should equal the riskless interest rate:

equation 62

$$\frac{E_t p_{t+1} - p_t}{p_t} + \frac{d_t}{p_t} = r$$

or equivalently :

equation 63

$$p_t = aE_t p_{t+1} + ad_t$$

Where $a \equiv \frac{1}{1+r} < 1$ .Now we will assume that :

equation 64

$$\tilde{y} = y_t + b_t$$

Where y_t is the solution and b_t is bubble. Now if $\tilde{y} = y_t + b_t$ it has to be the case:

equation 65

$$E_t \tilde{y}_{t+1} = E_t y_{t+1} + E_t b_{t+1}$$

Or :

equation 66

$$b_t = aE_t b_{t+1} \Rightarrow E_t b_{t+1} = a^{-1} b_t$$

The ever–expanding bubble: b_t then simply follows a deterministic trend of the form:

equation 67

$$b_t = b_0 a^{-t}$$

In the asset price equation $\frac{E_t p_{t+1} - p_t}{p_t} + \frac{d_t}{p_t} = r$ for simplicity we take $d_t = d^*$no–bubble solution (the fundamental solution) takes the form:

equation 68

$$p_t = p^* = \frac{d^*}{r}$$

which sticks to the standard solution that states that the price of an asset should be the discounted sum of expected dividends:

equation 69

$$\frac{d^*}{r} = \sum_{i=0}^{\infty}(1+r)^{-i}d^*$$

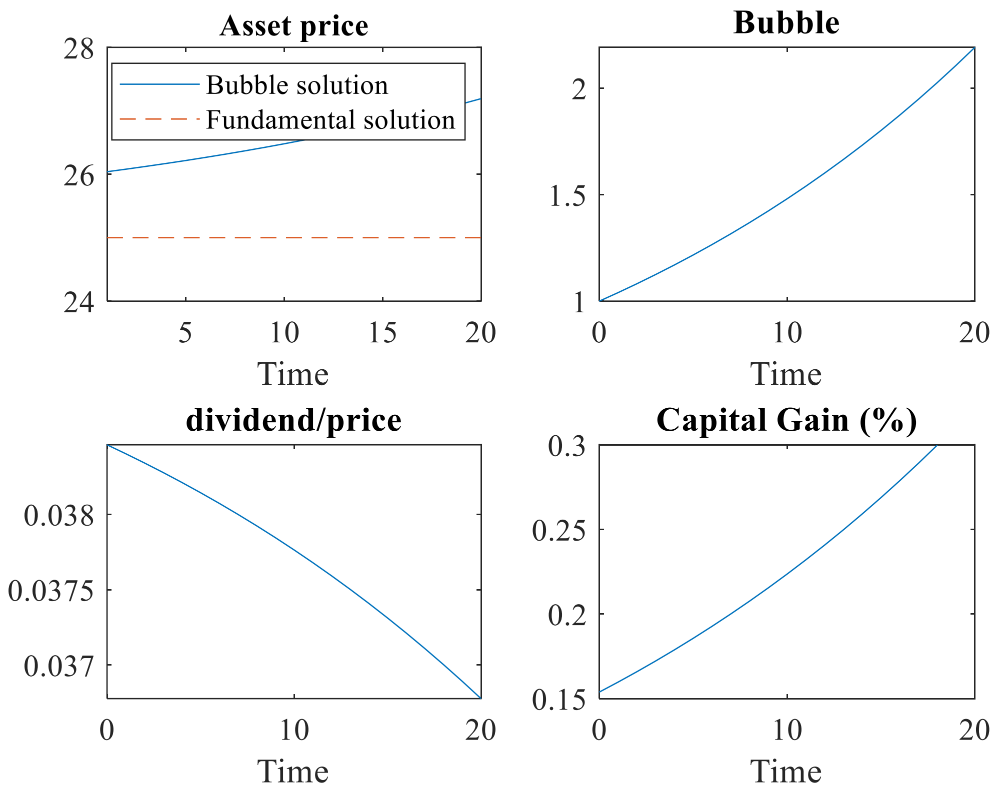
we now add a bubble of the kind we consider:

equation 70

$$b_t = b_0 a^{-t} = b_0(1+r)^t, b_0 > 0$$

the price of the asset will increase exponentially though the dividends are constant[18]. Next, we will plot and outline deterministic bubble.

Figure 13 Deterministic asset bubble



Source:Author's own calculations

Next, we will show the bursting–bubble: A problem with the previous example is that the bubble is ever–expanding whereas observation and common sense suggests that sometimes the bubble bursts. We may therefore define the following bubble:

equation 71

$$b_{t+1} = \begin{cases} (a\pi)^{-1} b_t + \zeta_{t+1} \\ \zeta_{t+1} \end{cases}$$

[18] Individuals are ready to pay a price for the asset greater than expected dividends because they expect the price to be higher in future periods, which implies that expected capital gains will be able to compensate for the low price to dividend ratio.

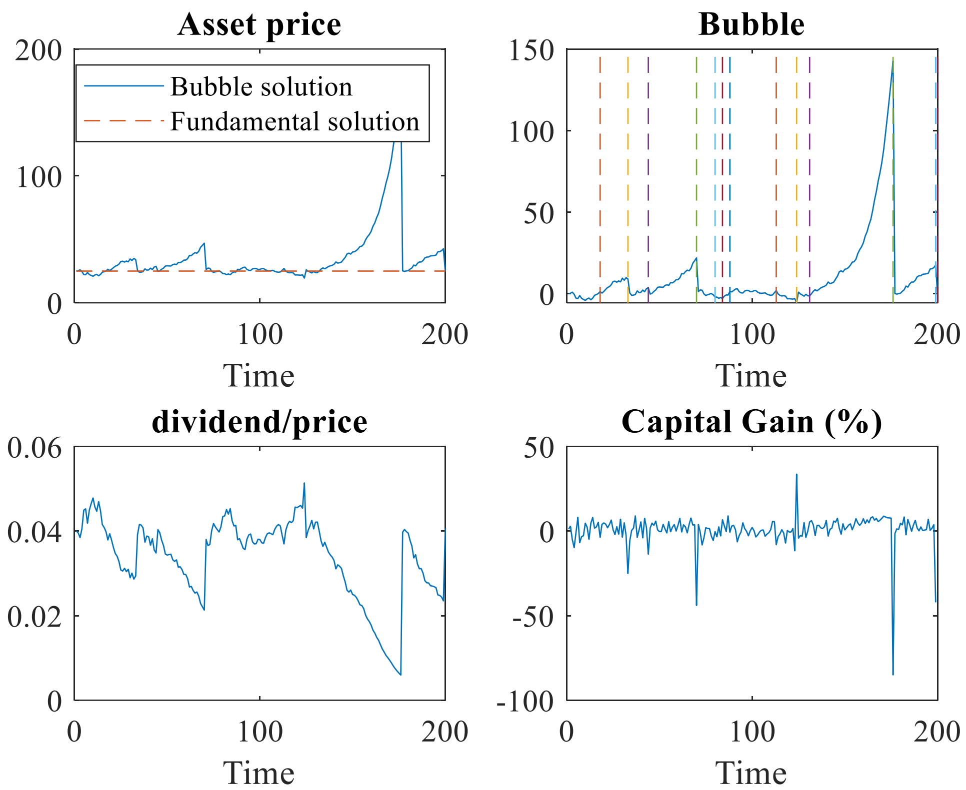
$(a\pi)^{-1}b_t + \zeta_{t+1}$ happens with probability π , and ζ_{t+1} happens with probability $1 - \pi$.Now, with $E_t\zeta_{t+1} = 0$. So defined, the bubble keeps on inflating with probability π and bursts with probability $(1 - \pi)$.Now:

equation 72

$$b_t = aE_t(\pi((a\pi)^{-1}b_t + \zeta_{t+1}) + (1-\pi)\zeta_{t-1}) = aE_t\pi((a\pi)^{-1}b_t + \zeta_{t+1}) + \zeta_{t+1}) = aE_t(a^{-1}b_t) = b_t$$

Next, we will plot an example of such bubble. The vertical lines in the upper right panel of the figure corresponds to time when the bubble bursts.

Figure 14 Example of a bursting bubble : The case of asset pricing (constant dividends)



Source:Author's own calculations

Now we will include cheap talk to see the impact it has on asset price bubble. We can model the impact of cheap talk on asset bubbles using a simple rational expectations framework with noise traders and strategic speculators. The key idea is that cheap talk shifts beliefs and influences demand, leading to price deviations from fundamentals. Let the fundamental value of the asset be:

equation 73

$$V_t = E_t[D_{t+1}] + \frac{1}{1+r}E_t[P_{t+1}]$$

D_{t+1}-is the future dividend, P_{t+1} is the future price, r is the risk free interest rate. The price without speculation is simply:

equation 74

$$p_t = \frac{E_t[D_{t+1}]}{r}$$

Now, let there be cheap talk signals C_t, which shift investors' expectations:

equation 75

$$E_t[P_{t+1}] = (1 + \lambda C_t)V_t$$

Where $\lambda > 0$ captures how strong investors respond to cheap talk:

equation 76

$$p_t = \frac{E_t[D_{t+1}]}{r} + \lambda C_t$$

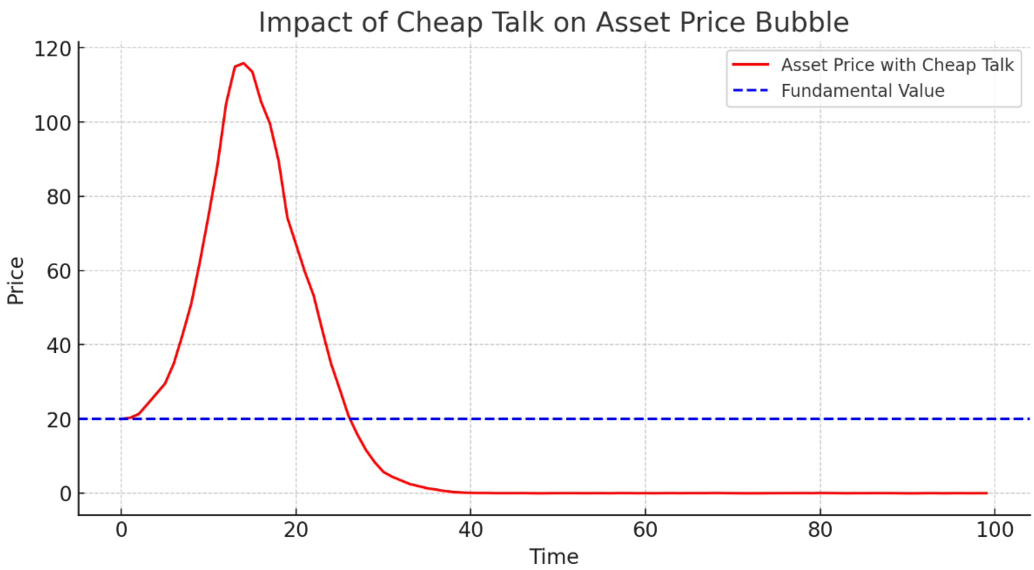
If C_t is purely speculative (without fundamental backing), this extra term represents a bubble component**.** If investors believe that others will keep buying based on CtC_tCt, we get a self-fulfilling loop:

equation 77

$$P_t = (1 + \lambda C_t)P_{t-1} + \epsilon_t$$

his recursive equation generates bubbles when λC_t is persistent. This will be illustrated on the following plot.

Figure 15 Impact of cheap talk on asset price bubble



Source:Author's own calculations

The plot illustrates how cheap talk (random but persistent signals) inflates asset prices beyond their fundamental value. Over time, speculative effects create deviations from fundamentals, forming a bubble-like pattern. The fundamental value remains stable, while the speculative price fluctuates and rises due to self-reinforcing expectations.

9.1 Sender-Receiver Game (Crawford-Sobel Setup)

Now, we will investigate the Sender-Receiver Game (Crawford-Sobel Setup): A sender (S) (e.g., analyst, media, or influential investor) observes a private signal about an asset's fundamental value V_t.

The sender sends a cheap talk message m_t to the market (receiver). The receiver (R) (e.g., investors) updates beliefs and sets a price P_t. Misaligned preferences: The sender may want to exaggerate information to boost asset demand. Let the fundamental value of the asset be:

equation 78

$$V_t = \theta + \epsilon_t$$

Where $\theta \sim N(\mu, \sigma^2)$ is the true fundamental and $\epsilon_t \sim N(0, \sigma_\epsilon^2)$. The sender sends a message:

equation 79

$$m_t = V_t + b$$

where b represents bias (possibly strategic). The receiver interprets this and forms expectations:

equation 80

$$E_t[V_t|m_t] = \alpha m_t + (1 - \alpha)E_t[V_t]$$

where α is the weight given to the signal. Thus, the asset price is updated as:

equation 81

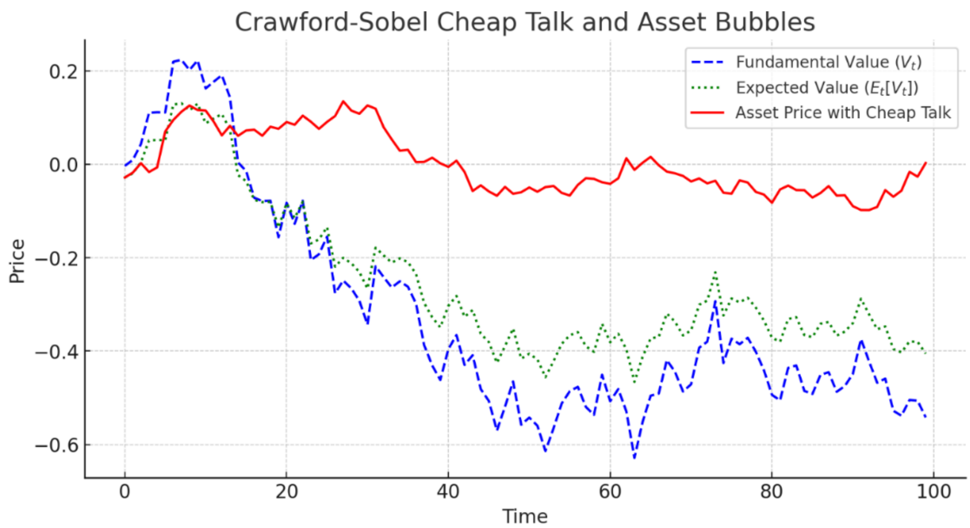
$$P_t = (1 + \lambda C_t)P_{t-1} + v_t$$

v_t is a noise term.

- ✓ $\lambda C_t > 0$ inflates the bubble.
- ✓ $\lambda C_t < 0$ market corrects.

Next, we will simulate this model and visualize how strategic cheap talk influences asset bubbles.

Figure 16 Crawford-Sobel model cheap talk and asset bubbles



Source: Author's own calculations

The plot shows how **cheap talk (biased signals)** influences investor expectations and inflates asset prices beyond their fundamental value. While fundamentals V_t evolve smoothly, expectations $E_t(V_t)$ and speculative pricing P_t diverge, creating a bubble. The misalignment between sender and receiver amplifies deviations, making prices more volatile

УДК: 005.57 316.776

9.2 Crawford-Sobel (CS) Cheap Talk Equilibrium, Aumann's Common Knowledge Model, Nash Equilibrium, and Kantian Equilibrium in the context of asset bubbles

Crawford-Sobel model is again as previous: $V_t = \theta + \epsilon_t$ Where $\theta \sim N(\mu, \sigma^2)$ is the true fundamental and $\epsilon_t \sim N(0, \sigma_\epsilon^2)$. The sender sends a message: $m_t = V_t + b$ where b represents bias (possibly strategic). The receiver interprets this and forms expectations:

equation 82

$$E_t[V_t|m_t] = \alpha m_t + (1-\alpha)E_t[V_t]$$

where α is the weight given to the signal. Thus, the asset price is updated as:

equation 83

$$P_t = (1 + \lambda C_t)E_t[V_t|m_t] + v_t$$

Aumann's Aumann (1976) framework involves multiple investors who receive private signals and engage in cheap talk, but if common knowledge is not reached, mispricing emerges. Each trader receives private signal:

equation 84

$$s_i = V_t + \eta_i$$

Where $\eta_i \sim N\left(0, \sigma_\eta^2\right)$. Traders communicate:

equation 85

$$m_t = s_i + b$$

If traders fail to reach common knowledge, expectations diverge:

equation 86

$$P_t = (1 + \lambda C_t)E_t[V_t|m_t] + v_t$$

Nash Equilibrium (NE) assumes each investor optimizes based on others' fixed strategies:

equation 87

$$P_t^{NE} = \max E_t[V_t|market\ signals]$$

Leading to speculative mispricing. Now about Kantian equilibrium first we would have to define it:

Definition 6 :

A vector of strategies $\mathcal{L} = (\mathcal{L}^1, \dots, \mathcal{L}^n)$ is a multiplicative Kantian equilibrium of the game $G = S(V^1, \dots, V^n)$ for $\forall i = 1, \dots, n$

equation 88

$$\arg_{\alpha \in \mathbb{R}_+} \max V^i(\alpha \mathcal{L}) = 1$$

Proposition 2:

$\mathcal{L} = (\mathcal{L}^1, .., \mathcal{L}^n) \in S^n$ is a multiplicative Kantian equilibrium of the game $G = S(V^1, \dots, V^n)$ if :

inequality 13

$$(\forall i = 1,..,n)(\forall \alpha \in \mathbb{R}_+)(V^i(\mathcal{L}) \geqq V^i(\alpha\mathcal{L})$$

In previous S is a common strategy space.

Definition 7

$\mathcal{L}$ is G -efficient if $\nexists \mathcal{L}' \in S^n$ that Pareto dominates $\mathcal{L} \in G$.

Definition 8

Some game $G = S(V^1, \dots, V^n)$ is monotone increasing (decreasing) if $(\forall i = 1,..,n)(V^i(\cdot)$ is strictly increasing (decreasing) in $\mathcal{L}^{-i}$.

Theorem 5 :

Suppose that $G = S(V^1, \dots, V^n)$ is monotone increasing (decreasing).And let $\mathcal{L}^*$ be Kantian equilibrium of G with $\mathcal{L}^i > 0, \forall i = 1,..,n$,then $\mathcal{L}^*$ is G efficient.

Proof: Now, let V^i be monotone increasing. Suppose now that $\mathcal{L}^*$ is Kantian but is not G -efficient and is Pareto dominated by allocation $\hat{\mathcal{L}}$ then:

equation 89

$$r = \max_i \frac{\hat{\mathcal{L}}_i}{\mathcal{L}_i^*}$$

$\exists j, r > \frac{\hat{\mathcal{L}}_j}{\mathcal{L}_j^*}$; for if not then $\mathcal{L}^*$ is not Kantian equilibrium, because all agents would weakly prefer to change to $r\,\mathcal{L}^*$ and some would prefer the change ,and let i^* be an agent for whom $r\mathcal{L}_i^* = \hat{\mathcal{L}}_i$ .So now $r \neq 1$ or else agent i^* would be worse of at $\hat{\mathcal{L}}_i$ than $\mathcal{L}_i^*$ by V^i as monotone increasing. Now by vector $\mathcal{L} = r\mathcal{L}^*$ and we have:

equation 90

$$V^{i*}(r\mathcal{L}^{i*}) > V^{i*}(\hat{\mathcal{L}}) \geq V^{i*}(\mathcal{L}^{i*})$$

The first inequality follows from the fact that in $r\,\mathcal{L}^*, i^*$ expends the same labor as the agent does in $\hat{\mathcal{L}}$, while some other agents expend strictly more labor, and none expends less labor than in $\hat{\mathcal{L}}$; the second inequality follows by Pareto domination. Previous This contradicts the assumption that $\alpha^{i*}(\mathcal{L}) = 1$, which proves the claim. About the effort similar argument applies for $V^i(\cdot; e)$ and is monotone decreasing, $r = \min \frac{\hat{\mathcal{L}}_i}{\mathcal{L}_i^*}$ ■

For this part see more in Roemer (2010), Roemer (2019).

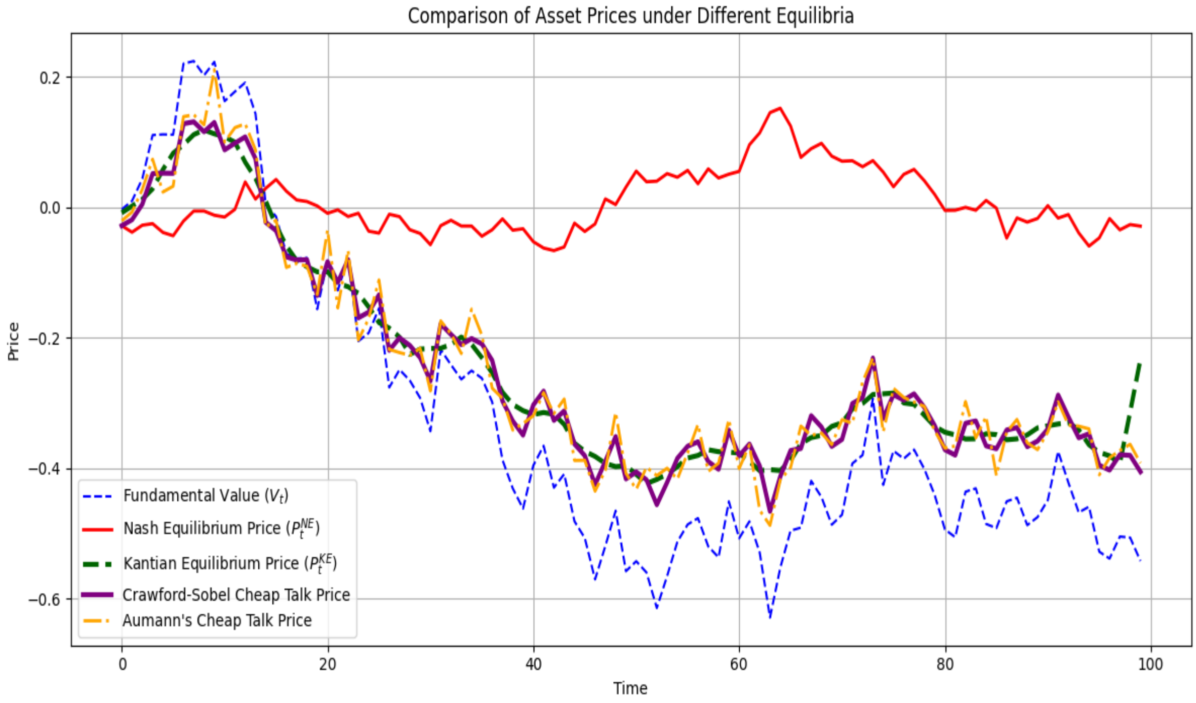
So, in our model Kantian equilibrium (Kantian Equilibrium (KE) assumes investors coordinate for collective optimality) is defined as:

equation 91

$$P_t^{KE} = E_t[V_t | social\ optimal\ strategy]$$

which reduces bubbles.

Figure 17comparison of asset prices under different equilibria



Source: Author's own calculations

9.3 Dot-com bubble

During the late 1990s, the rapid rise of the internet and technology companies led to excessive speculation in the stock market. Many executives, analysts, and media figures hyped internet startups with little to no profits, using buzzwords like "new economy" and "paradigm shift." Mechanism of Cheap Talk in the Bubble:

1. Tech CEOs and Venture Capitalists
   - ✓ Startup founders overstated growth potential of their companies in media interviews and press releases.
   - ✓ Venture capitalists and investors talked up startups to drive hype and increase valuations.
2. Stock Market Analysts and Investment Banks
   - ✓ Many financial analysts from top investment banks issued "buy" recommendations on stocks they privately doubted.
   - ✓ Henry Blodget of Merrill Lynch was caught in 2003 emails calling some dot-com stocks "junk"
3. Media and Public Sentiment
   - ✓ Financial news outlets like CNBC and newspapers constantly hyped internet stocks.
   - ✓ Investors, influenced by cheap talk rather than fundamentals, bought into the hype, driving prices higher. while publicly promoting them.

Outcome:

- ✓ The NASDAQ index **quadrupled from 1995 to 2000** before **crashing by nearly 78%** from its peak.
- ✓ Many hyped companies, like **Pets.com**, went bankrupt when reality set in.
- ✓ Retail investors who bought into the hype **suffered massive losses**.

Why This Is an Example of Cheap Talk?

- ✓ None of the claims were **binding**—executives, analysts, and media figures **were not directly penalized** for exaggerating growth potential.
- ✓ Investors **relied on signals** from influential people rather than **hard fundamentals**, leading to an unsustainable price surge.

This case illustrates how costless talk, even when misleading, can drive irrational exuberance and asset bubbles.

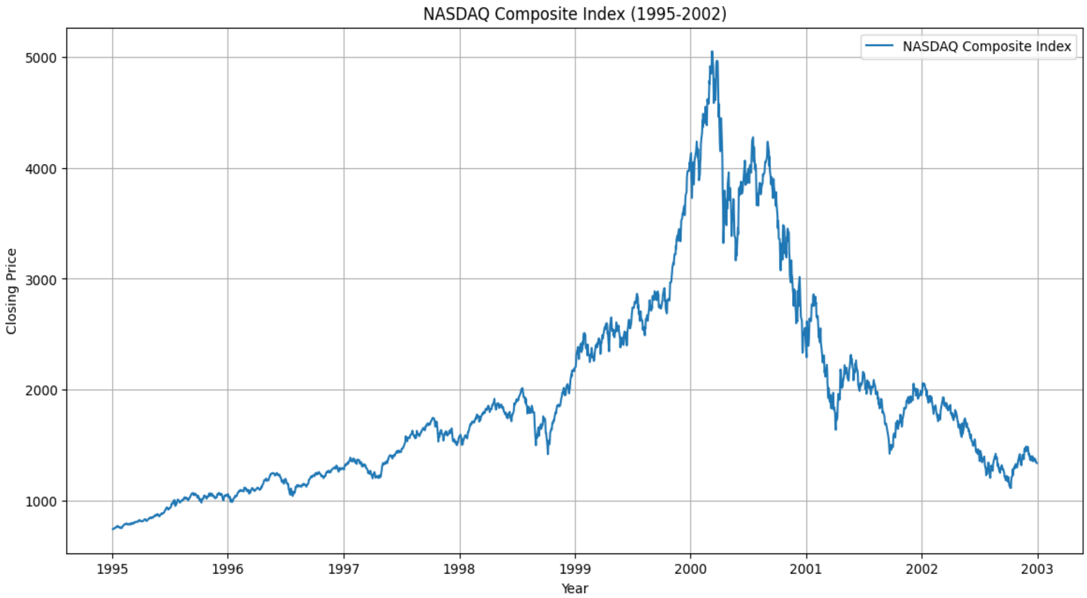
Table 2 Cheap talk examples and their impact on NASDAQ [19]

Actor	Cheap Talk Examples	Impact on NASDAQ
Startup CEOs	“We're the next Amazon.” “This will revolutionize everything.”	Raised expectations and stock prices of new IPOs
Analysts	“Buy! Strong growth ahead!” (even without profits)	Boosted retail/institutional investment in NASDAQ firms
Media	“Tech is the future!” “Don’t miss out on the next Microsoft.”	Created a narrative that tech stocks could only go up
Politicians/Fed (indirectly)	Alan Greenspan’s “irrational exuberance” speech (1996) was one of few dampening voices, but was mostly ignored	Market still rallied as others hyped tech without accountability

Source: Author’s own calculations

[19] $NASDAQ_{index} = \frac{\sum_{i=1}^{N} P_i \cdot Q_i}{D}$; where P_i-Price of stock i, Q_i-number of outstanding shares, D- divisor adjusted over time. Divisor is a value set by NASDAQ to normalize the index. It is adjusted for stock splits, dividends, or changes in the index components to keep the index consistent over time. The divisor in the NASDAQ index (and other market-cap weighted indexes) is a normalization factor that ensures continuity of the index value over time, especially when there are structural changes. t does not have a fixed mathematical formula but is adjusted to account for: Stock Splits and Reverse Splits, Dividends (Special Cash Dividends), Additions or Removals of Companies, Mergers, Acquisitions, and Spinoffs.

Figure 18 NASDAQ composite index 1995-2003



Source: Author's own calculations based on yfinance data

The NASDAQ Composite Index rose from around 700 in 1995 to over 5,000 by March 2000 — an over 600% increase in five years.Much of this growth was not backed by profits, but by beliefs and narratives — classic results of cheap talk.When reality hit (i.e., when companies failed to deliver profits), the NASDAQ crashed — by 78% from peak to bottom between 2000–2002.The total losses from the Dot-com Bubble are estimated to be in the trillions of dollars. Market capitalization of tech stocks on NASDAQ fell from about $6.7 trillion to $1.6 trillion, meaning: Roughly $5 trillion in paper wealth was wiped out from the NASDAQ market alone.

Table 3 different losses from Dot.com bubble

Sector	Example	Losses
Retail Investors	Many bought into IPOs like Pets.com or Webvan	Lost 100% of investment as companies went bankrupt
Venture Capital Firms	Funded startups with no profits	Billions in write-offs
Institutional Investors	Pension funds, mutual funds, etc.	Huge losses on “next big thing” tech bets
Employment	Tech and finance sectors	Hundreds of thousands of jobs lost
Real Economy	Slower growth in 2001–2002	U.S. recession in 2001

Source: Different sources across internet

See, Akerlof, Shiller (2009) for more on this topic.

9.4 Cheap talk and financial crisis 2008

Cheap talk was everywhere before — and it contributed to mispricing of risk, delayed corrective actions, and a general sense of complacency before the collapse. Here is a table of some of more notable examples of cheap talk pre crisis

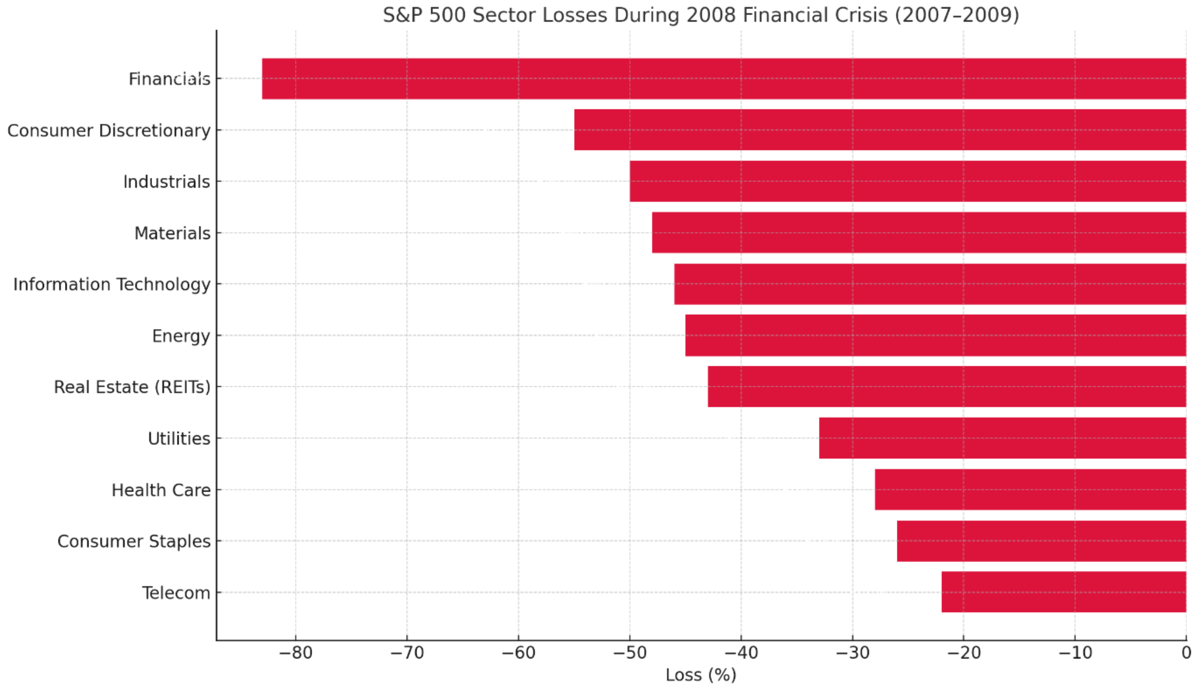
Table 4 Cheap talk type and S&P in 2008

Actor	Event / Statement	Cheap Talk Type	Impact on S&P 500
Alan Greenspan (Fed Chair)	Reassured public that housing prices are "unlikely to decline nationally" (2005)	Downplaying systemic risk	Fueled investor and consumer confidence → housing boom → rising S&P via financials and construction stocks
Credit Rating Agencies	Rated subprime mortgage-backed securities (MBS) as AAA despite known risks	Misleading cheap talk via inflated ratings	Encouraged massive investment in toxic assets → contributed to boom in financial stocks (e.g., Lehman, AIG) on S&P
Investment Banks	Promoted MBS as safe, low-risk products despite internal doubts (2003–2007)	Strategic misrepresentation	Attracted capital inflows → inflated balance sheets → boosted S&P 500 financial sector performance
Ben Bernanke (Fed Chair)	“The impact on the broader economy and financial markets of the problems in the subprime market seems likely to be contained” (2007)	False reassurance	Delayed market correction → prolonged overvaluation → greater crash impact on S&P 500 when truth emerged
Bush Administration	Publicly stated the housing market was “strong” and “under control” (2006)	Political optimism	Helped sustain investor and consumer confidence → delayed policy responses → intensified the eventual market fall
Financial Media	CNBC, WSJ, etc. highlighted bullish analyst reports and downplayed concerns	Herd-reinforcing commentary	Amplified bubble sentiment → investors chased returns → overexposure to risky assets within S&P index
Mortgage Brokers / Borrowers	Widespread use of “liar loans” with no income verification (2004–2006)	Micro-level cheap talk (false signaling of creditworthiness)	Allowed unqualified buyers to enter market → drove demand and speculation → fed into stock prices of S&P 500 financials

Sources: Federal Reserve communications, Credit rating agencies and MBS misrepresentation, Bush Administration and public statements,

- ✓ The **boom** phase (2002–2007) was inflated by **optimistic but unfounded cheap talk** from regulators, rating agencies, and banks.
- ✓ The **bust** (late 2007–2009) was worsened when it became clear that **prior communication was deceptive or uninformed**.
- ✓ The S&P 500 fell from a peak of **~1,565 in Oct 2007** to a low of **~676 in March 2009** — a **~57% drop**.

Figure 19 S&P 500 sector losses during 2008 Financial crisis (2007-2009)



Source: Author's calculations based on available historical data for S&P 500

On the next table we present the worsening effects of cheap talk on crisis .

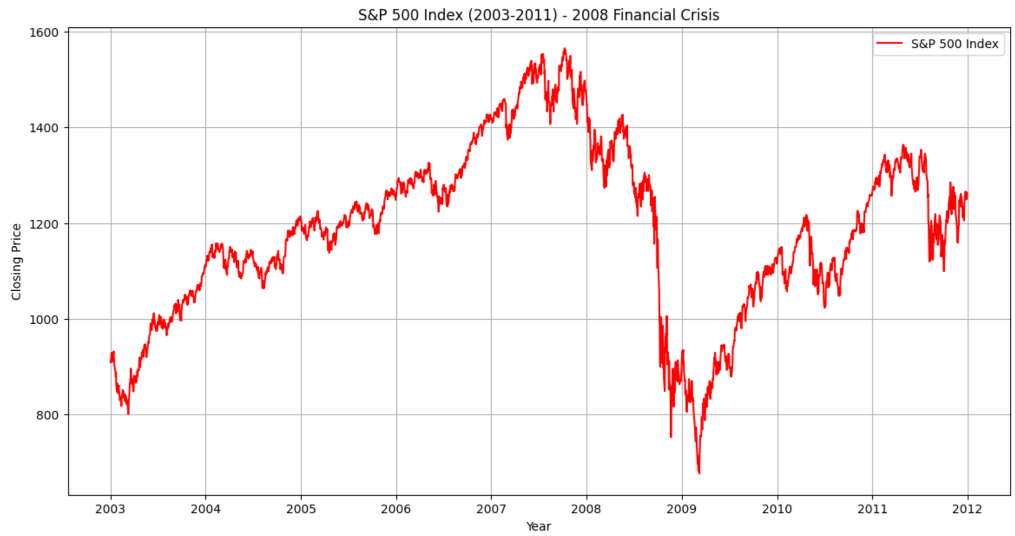
Table 5 How Cheap Talk Worsened the Crisis

Mechanism	Cheap Talk Element	Effect
Misleading ratings & advice	Unverifiable praise of toxic assets	Inflated demand
Regulatory communication	Reassurance without accountability	Delayed reaction
Borrower-lender communication	Misrepresentation of risk	Overleveraging
Market herd behavior	Belief-based imitation	Bubble amplification

Source: Author's own investigation through different relevant sources

Since there was irrational exuberance leading up to the 2008 financial crisis, primarily in the U.S. housing market and related financial markets. The crisis was fueled by subprime mortgage lending, securitization of risky assets, and excessive speculation in housing prices. The S&P 500 Index and Case-Shiller U.S. National Home Price Index are commonly used to analyze this bubble.

Figure 20 S&P index 2003 -2011



Source: Author's calculations based on available data at yfinance

Now we will explain some of the asset bubble indicators.

- ✓ The Case-Shiller Index is a widely used measure of U.S. residential real estate prices, tracking changes in the value of the housing market over time. It's considered one of the most accurate gauges of U.S. home price trends.
- ✓ Formally called the S&P CoreLogic Case-Shiller Home Price Index, it reflects the average change in home prices in a particular region or across the country, based on repeat sales of the same properties over time.

The index is based on repeat sales regression:

equation 92

$$\ln(P_{it}) - \ln(P_{is}) = \sum_{k=s+1}^{t} \beta_k D_{ik} + \varepsilon_i$$

- ✓ P_{it} is the price of house i at time t
- ✓ P_{is} is the price of the same house i at earlier time s
- ✓ β_k are coefficients representing the price change in period k
- ✓ D_{ik} Dummy variable =1 if transaction occurred in period k
- ✓ ε_i Error term

The log difference in prices is regressed on time dummies to estimate the price change over time. Each pair of repeat sales (same home sold at different times) is treated as one observation. The difference in log prices gives the percentage change in price. A time series regression is run across all repeat sales to extract a smooth index of housing prices over time. The final Case-Shiller index is normalized:

equation 93

$$CSindex_t = 100 \times \exp(\hat{\beta}_t)$$

More on Case-Shiller index see in Case, Karl E. and Shiller, Robert J. (1987) and at https://fred.stlouisfed.org/series/CSUSHPINSA . The model is based on a **hedonic regression** using **repeat sales** of the same home to control for quality and size. The regression takes the difference in the logarithm of prices of a house sold at two points in time and regresses it against time dummies. This allows for estimating price changes over time without relying on average prices, which can be misleading due to changes in the mix of homes sold[20]. The VIX stands for the Volatility Index and is published by the Chicago Board Options Exchange (CBOE). It measures the market's expectation of volatility over the next 30 days, based on the prices of options on the S&P 500 Index (SPX). The VIX doesn't measure past volatility, but rather the market's expectation of how volatile the S&P 500 will be. It's computed using a complex formula based on the Black-Scholes model, but in practice, it captures how much investors are paying for protection (options):

- ✓ High VIX (e.g., 40+): Markets are expected to be highly volatile → fear/panic.
- ✓ Low VIX (e.g., <15): Markets are expected to be calm → confidence/stability.

Table 6 Approximate VIX level crashes and its normal range

Event	Approx. VIX Level
Dot-com crash (2001)	~40–45
Financial crisis (2008)	**~80**
COVID crash (Mar 2020)	~82
Normal range	12–20

Source: CBOE (Chicago Board Options Exchange) They created and maintain the VIX index. Website: https://www.cboe.com/

Figure 21 VIX Levels During Crashes (With Sources)

Event	Peak VIX Level	Approx. Date	Source
Dot-com Bust	~40	2001–2002	CBOE
9/11 Attacks	~49.35	Sep 2001	CBOE
2008 Financial Crisis	**~80.86**	Oct 24, 2008	CBOE
COVID-19 Crash	**~82.69**	Mar 16, 2020	CBOE
2022 Inflation Panic	~36.45	Mar 2022	Yahoo

Source: CBOE,Yahoo

The CBOE VIX is calculated as:

[20] For S&P methodology see : S&P Methodology (Public PDF)

equation 94

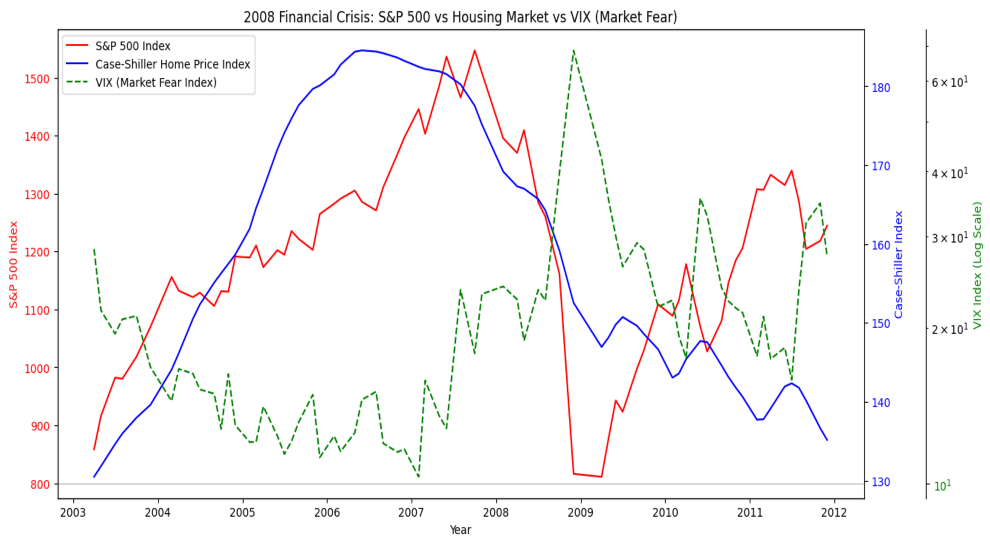
$$VIX = 100 \times \sqrt{\frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left(\frac{F}{K_0} - 1\right)^2}$$

VIX formula symbols meaning:

Symbol	Meaning
T	Time to expiration (in years)
K_i	Strike prices of out-of-the-money options
ΔK_i	Interval between strike prices
$Q(K_i)$	Average of bid-ask midpoints for each strike
R	Risk-free interest rate to expiration
F	Forward index level derived from option prices
K_0	First strike below the forward index level

Next, we will plot these indices.

Figure 22 Financial crisis, S&P vs Housing market and VIX market fear



Source: Standard & Poor's (S&P) / FactSet

In the Augmented Dickey–Fuller (ADF) test, the null hypothesis is that the time series has a unit root—that is, it is non-stationary. The "null statistic" you see in the output is the ADF test statistic computed from your data. This statistic is then compared against critical values (which are non-standard) to decide whether to reject the null hypothesis. For example, if the test statistic is less (more negative) than the critical value at a given significance level (e.g., 1%, 5%, or 10%), you reject the null hypothesis and conclude that the series is stationary. Otherwise, you fail to reject the null hypothesis, suggesting the presence of a unit root (non-stationarity) and potentially a bubble in the context of asset pricing. Table of ADF tests is presented below.

Table 7 ADF test for: S&P 500, Case-Shiller, VIX

	Test Statistic	p-value	Critical Value 1%	Critical Value 5%	Critical Value 10%
ADF Test for S&P 500:	-2.6038	0.0922	-3.5336	-2.9064	-2.5907
ADF Test for Case-Shiller:	-2.3634	0.1523	-3.5405	-2.9094	-2.5923
ADF Test for VIX:	-3.0851	0.0277	-3.5289	-2.9044	-2.5897

Source: Author's own calculations based on data available from Standard & Poor's (S&P) / FactSet

When a price series exhibits explosive non-stationarity, that can be a statistical symptom of a bubble.

- ✓ A random walk (unit root process) is a common non-stationary model of asset prices.
- ✓ A bubble is more extreme — it's a price process that grows faster than a random walk (called explosive behavior)

equation 95

$$P_t = \rho P_{t-1} + \epsilon_t, \rho > 0$$

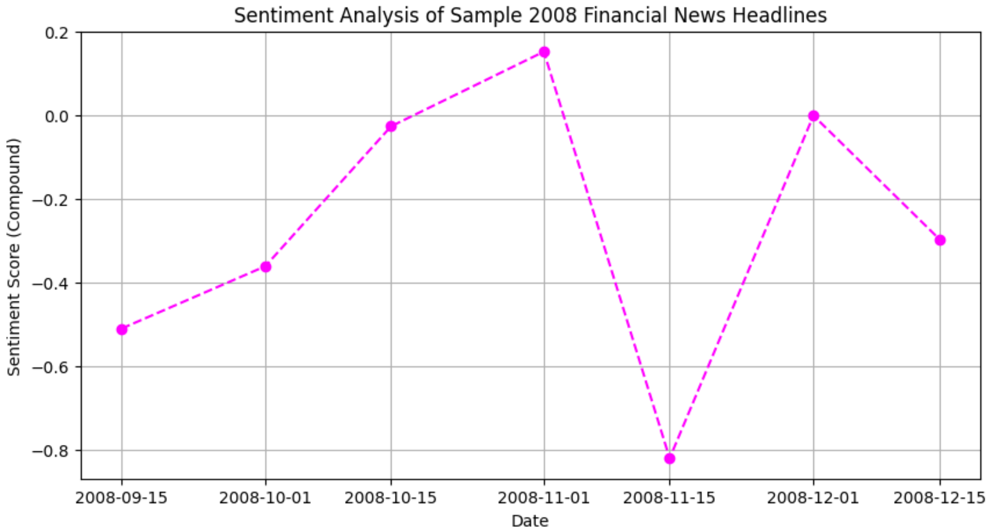
Table 8 Non-stationary, Bubble indicator, concept

Concept	**Non-stationary**	**Bubble Indicator?**	**Concept**
Random walk	Yes	Not necessarily a bubble	Random walk
Explosive process $(\rho > 1)$	Yes	**Yes, could indicate a bubble**	Explosive process $(\rho > 1)$
Structural breaks	Yes	Not necessarily, but may precede bubbles/crashes	Structural breaks

Source: Author's own calculation

VADER Sentiment Analysis. VADER (Valence Aware Dictionary and Sentiment Reasoner) is a lexicon and rule-based sentiment analysis tool that is specifically attuned to sentiments expressed in social media and works well on texts from other domains.

Figure 23 Sentiment analysis of sample 2008 Financial news headlines



Source: Daily aggregated compound sentiment scores, these are not real-time scraped headlines from a news API or archive. They are representative dummy headlines, manually input to demonstrate how VADER sentiment analysis works on financial news from 2008.

Table 9 News Headlines and their Sentiment Scores:

Date	Headline	Sentiment
2008-09-15	Lehman Brothers files for bankruptcy amid market panic	-0.5106
2008-10-01	Credit crisis deepens as banks tighten lending standards	-0.3612
2008-10-15	Government steps in with bailout plans to rescue troubled banks	-0.0258
2008-11-01	Stock markets plummeted as investors lose confidence	0.1531
2008-11-15	Economic recession fears intensify following major financial losses	-0.8214
2008-12-01	New regulations announced to stabilize the financial system	0.0000
2008-12-15	Slow recovery expected as market remains volatile and uncertain	-0.2960

Source: Daily aggregated compound sentiment scores, these are not real-time scraped headlines from a news API or archive. They are representative dummy headlines, manually input to demonstrate how VADER sentiment analysis works on financial news from 2008.

In sentiment analysis, VADER (Valence Aware Dictionary and Sentiment Reasoner) is a tool used to evaluate the sentiment of a given text. VADER provides a score that ranges from -1 to +1, with negative values indicating negative sentiment and positive values indicating positive sentiment. If the VADER score is negative, it means the text expresses a sentiment that is generally unfavorable or negative. The more negative the score, the stronger the negative sentiment detected in the text. A score close to -1 indicates a very strong negative sentiment.

10. Conclusion

The lower the bias equilibrium actions are closer to partition boundaries. In the Green,Stokey (2003) framework of two-person game of information transmission, principal's expected utility is much higher than the agent's. BNE equilibrium in previous game is in the intersection of principal's and agent's BR. If there are multiple sender's and receiver's BNE equilibrium is lower than Cournot equilibrium while

Stackelberg equilibrium (leader-follower model) does not stand neither on principal neither on agent Stackelberg BR. In the Kripke model with partial separation and mixed strategy equilibrium, if high quality sender sends high quality message beliefs and payoffs are higher than if world (agent) is low quality and he pretends to be high quality and sends high quality message. In the $w_H m_L$(if agents pretend to be low quality), beliefs and payments are lower than if he is true to himself $w_L m_L$. In the Kripke frame with knowledge relations without and with cheap talk, agents can talk to each other and share information, but this communication does not change the structure of knowledge in the model. Asset price movements, dividend/price ratio and capital gain move differently in deterministic and bursting asset price bubble. On short-run cheap talk raises the asset price value much above fundamental value (asset bubble) but on a long-run cheap talk leads to asset price value much lower than the fundamental value. But in the canonical Crawford-Sobel (CS) model cheap talk leads to inflated asset prices over fundamental value in the long run as well as short run, while the expected value is in between. Crawford-Sobel cheap talk price is almost the same as Aumann's' common knowledge price. Nash equilibrium is much above CS, and Aumann's price which are identical with Kantian equilibrium price. Fundamental value is below all of them. Dot-com bubbe and financial crisis 2008 are a clear example of cheap talk and asset bubbles prove cheap talk as a boom-bust reason. ADF Test for :S&P 500, Case-Shiller, VIX proved that simple econometrics tools are not reliable in identifying asset bubbles in the making(Case-Shiller index was barely stationary at 10% probability, while VIX and . Sender's utility is higher in the partially informative equilibrium than in babbling equilibrium. This paper proved that cheap talk is not so cheap.

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УДК: 005.57 316.776

Appendix 1 Babbling equilibrium

In a babbling equilibrium: The receiver ignores the sender's message. The sender does not convey any information—message is uninformative. Actions are based on priority (not on the message). Setup here is as follows: A state space is $\Theta \subseteq \mathbb{R}$ with realizations $\theta \in \Theta$.A message space is M ,an action space is $A \subseteq R$.Sender's utility:$u_S(a, \theta)$, receiver's utility is:$u_R(a, \theta)$. Sender observes θ, sends message $m \in M$.Receiver observes m,chooses action a. Messages are costless and non-binding (cheap talk). Sender and Receiver have common knowledge of:

- ✓ The game structure
- ✓ Preferences u_S and u_R
- ✓ Beliefs are common knowledge and updated via Bayes' Rule (if possible)

In a babbling equilibrium:The receiver ignores the sender's message.The sender does not convey any information—message is uninformative.Actions are based on the prior (not on the message).Intuition here is that: The sender has an incentive to mislead due to misaligned preferences, so the receiver disregards the message altogether. Strategy profiles:

equation 96

$$\sigma: \Theta \to \Delta(M), m \in M \text{ , } random\ messaging, sender's\ strategy$$
$$\rho: M \to \Delta(A), \rho(m) = \alpha, \forall m, chooses\ same\ action\ a^* \ \forall m - receiver's\ strategy$$

Formal Babbling Equilibrium Conditions are: Receiver chooses a^* to maximize expected utility given the prior $F(\theta)$:

equation 97

$$a^* \in \arg\max_{a \in A} \int_{\Theta} u_R(a, \theta) dF(\theta)$$

Sender sends any message $m \in M$ regardless of θ, since:

equation 98

$$\forall \theta \in \Theta, m \in \arg\max_{m' \in M} u_S(a^*, \theta)$$

Since $a^*$is fixed, sender has no incentive to send different messages, hence any message is equally optimal. Common Knowledge & Equilibrium

- ✓ The fact that the receiver will ignore the message becomes common knowledge.
- ✓ Thus, the sender knows it is pointless to signal truthfully.
- ✓ The equilibrium becomes self-enforcing: no deviation is profitable.

Example: Crawford and Sobel (1982) Setup

Let:

- ✓ $\theta \in [0,1]$
- ✓ $A = [0,1]$
- ✓ $u_R(a, \theta) = -(a - \theta)^2$
- ✓ $u_S(a, \theta) = -(a - \theta - b)^2$, for bias $b > 0$

Example: Crawford and Sobel (1982) Setup

equation 99

$$a^* = \arg\max_{a\in[0,1]} \int_0^1 -(a-\theta)^2 d\theta = \mathbb{E}[\theta] = \frac{1}{2}$$

Sender has utility:

equation 100

$$u_S(a^*, \theta) = -\left(\frac{1}{2} - \theta - b\right)^2$$

Sender cannot improve by sending another message since a^* is fixed. Bias $b > 0$ means sender prefers a higher action than the receiver.Now we will compare babbling and partially Informative Equilibria. Receiver picks action a^* based on prior:

equation 101

$$a^* = \arg\max_{a\in[0,1]} \int_0^1 -(a-\theta)^2 d\theta = \arg\min_a \int_0^1 (1-\theta)^2 d\theta \Rightarrow a^* = \mathbb{E}[\theta] = \frac{1}{2}$$

Sender again has utility: $u_S(a^*, \theta) = -\left(\frac{1}{2} - \theta - b\right)^2$. Partition the state space into N intervals:

equation 102

$$[0, \theta_1], [\theta_1, \theta_2), \dots, [\theta_{N-1}, 1]$$

Sender reports which interval θ falls into.Receiver's best response in each interval:

equation 103

$$a_i = \mathbb{E}[\theta | \theta \in [\theta_{i-1}, \theta_i)] = \frac{\theta_{i-1} + \theta_i}{2}$$

Sender prefers interval i iff:$\theta \in [\theta_{i-1}, \theta_i) \Rightarrow a_i$.This leads to the following indifference conditions at cutoff θ_i:

equation 104

$$u_S(a_i, \theta_i) = u_S(a_{i+1}, \theta_i) \Rightarrow (a_i - \theta_i - b)^2 = (a_{i+1} - \theta_i - b)^2$$

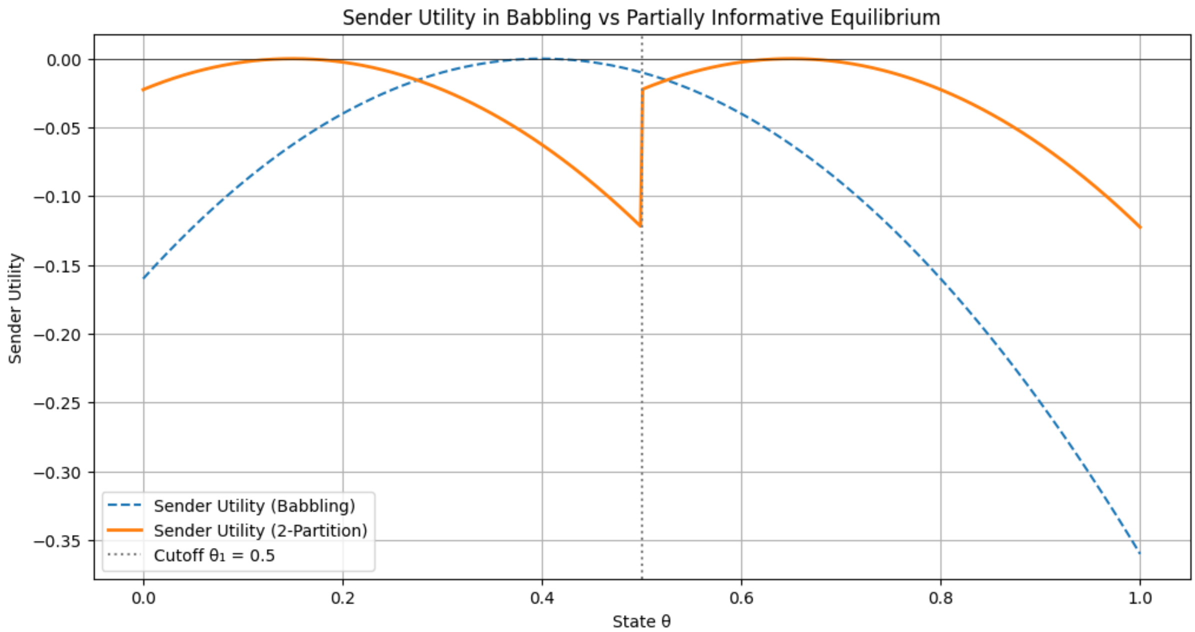
This gives the fixed point equations for cutoffs θ_i and actions a_i.Next, we'll: Simulate the sender's utility in both equilibria across the state space. Two plots below contain: Sender utility functions, strategies (actions based on messages), message cutoffs. In the plot dashed line: Utility in the babbling equilibrium, where the receiver ignores the message and always chooses action $a = 0.5$. Solid line: Utility in the 2-partition equilibrium, where the sender partially reveals their type using message intervals:

equation 105

$$m_1 \rightarrow \theta \in [0,0.5] \rightarrow receiver\ picks\ a = 0.25$$
$$m_2 \rightarrow \theta \in [0,0.5] \rightarrow receiver\ picks\ a = 0.75$$

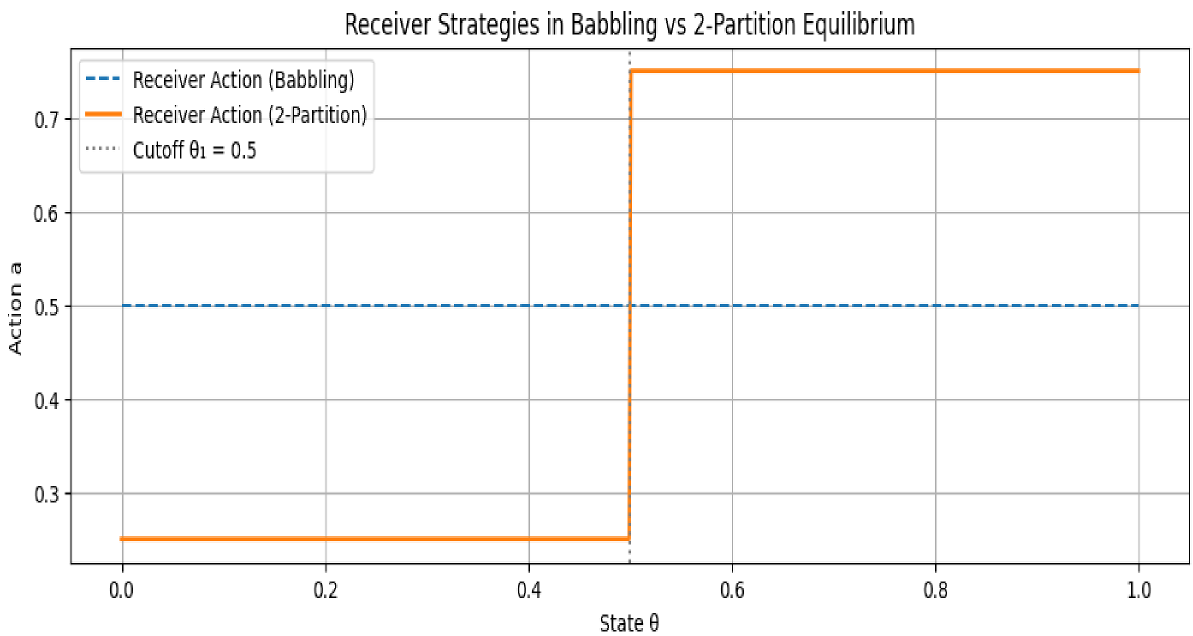
One can see the sender's utility is higher in the partially informative equilibrium than in babbling, since the receiver responds differently based on the message.

Figure 24 Sender Utility in babbling vs partially informative equilibrium



Source: Author's own calculation

Figure 25 Receiver strategies in babbling vs 2paprtition equilibrium



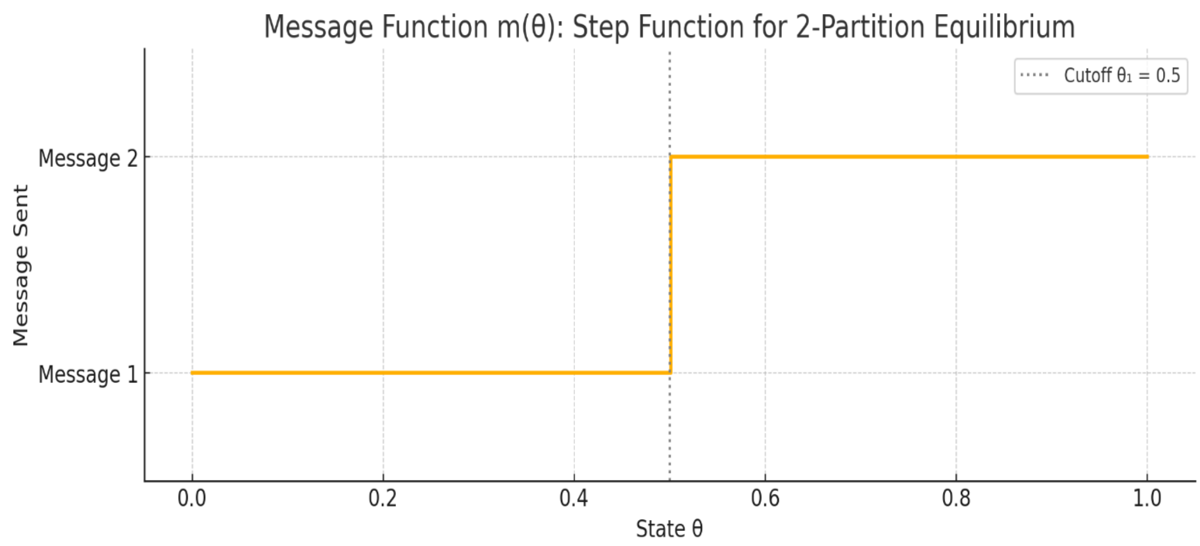
Source: Author's own calculation

Receiver Strategies

- ✓ In babbling, the action is constant: $a = 0.5$
- ✓ In 2-partition, the action depends on the sender's message:
- ✓ For $\theta \leq 0.5$, receiver chooses $a = 0.25$
- ✓ For $\theta > 0.5$, receiver chooses $a = 0.75$

The vertical line at $\theta = 0.5$ is the cutoff where the sender is indifferent between the two messages, derived from the indifference condition.

Figure 26 Message function $m(\theta)$ as a step function for 2-partition equilibrium



Source: Author’s own calculation

Here is the plot of the message function $m(\theta)$ as a step function for the 2-partition equilibrium:

- ✓ For $\theta \leq 0.5$, the sender sends message 1
- ✓ For $\theta > 0.5$, the sender sends message 2

This corresponds exactly to the cutoff $\theta_1 = 0.5$, derived from the indifference condition where the sender is just willing to switch messages.

Appendix 2 Cheap Talk with Noise or Non-Monotonic Strategies

1. Cheap Talk with Noise

The sender cannot perfectly control the message — with some probability ϵ, the message flips or is distorted.

2. Non-Monotonic Strategies

The sender's message does not increase with the type θ — maybe high types imitate low types or send random messages.

Let’s assume:

The sender wants to send message $m \in \{1,2\}$, but with probability w.p. $\epsilon \in (0,1)$,the message is flipped.

Table 10 Cheap talk with noise

True Type Range	Intended Msg	Actual Msg (with noise)
$\theta \leq \theta_1$	1	$1\, w.p. 1-\epsilon, 2 w.p. \epsilon$
$\theta > \theta_1$	2	$2\, w.p. 1-\varepsilon, 1\, w.p. \varepsilon$

Source: Author’s own calculation

Now, Bayes' rule updates the receiver's belief:

equation 106

$$\mathbb{E}[\theta|m = 1] = \frac{(1-\varepsilon)\mathbb{E}[\theta|\theta \leq \theta_1]P_1 + \varepsilon\mathbb{E}[\theta > \theta_1]P_2}{(1-\varepsilon)P_1 + P_2}$$

Where:

equation 107

$$P_1 = \mathbb{P}(\theta \leq \theta_1) = \theta_1$$
$$P_2 = 1 - \theta_1$$

This makes the receiver's action more conservative — less responsive to the message due to uncertainty. Non-Monotonic Strategies. Here, imagine:

- ✓ Types in the middle $\theta \in (0.4,0.6)$ send message 2
- ✓ Low and high types send message 1

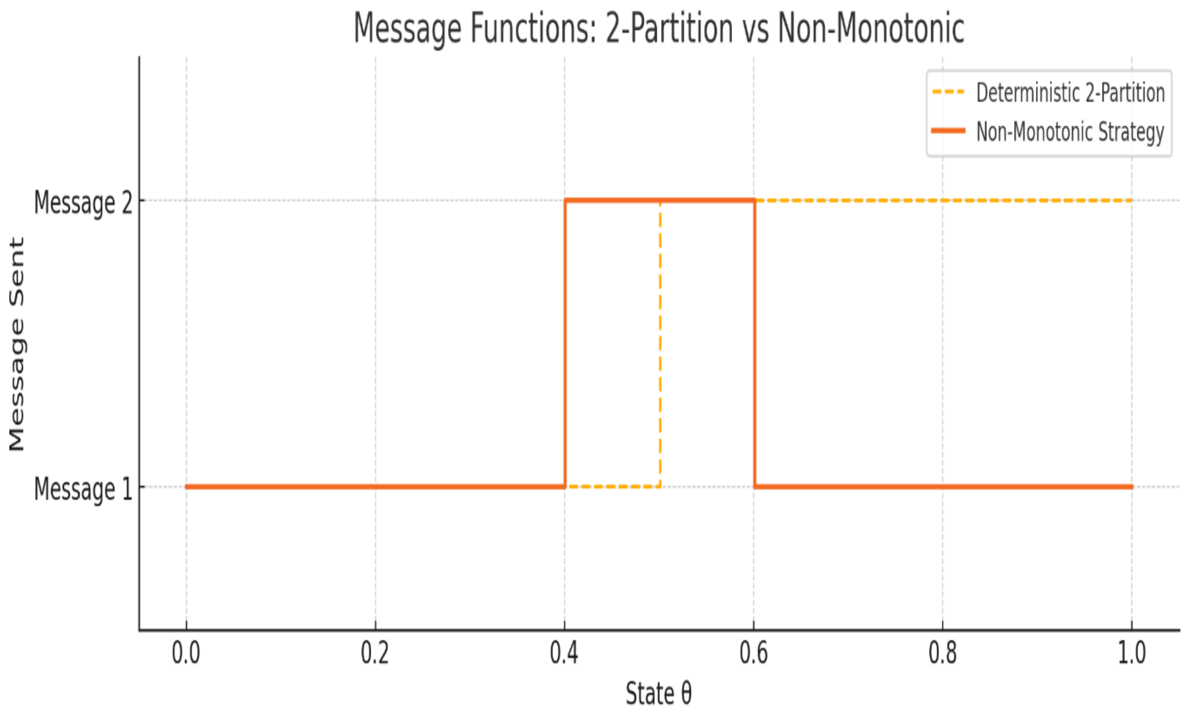
Such behavior arises in signaling games with pooling or discontinuous preferences. In formulas:

equation 108

$$m(\theta) = \begin{cases} 1 & \theta \in (0,0.4) \cup (0.6,1) \\ 2 & \theta \in (0.4,0.6) \end{cases}$$

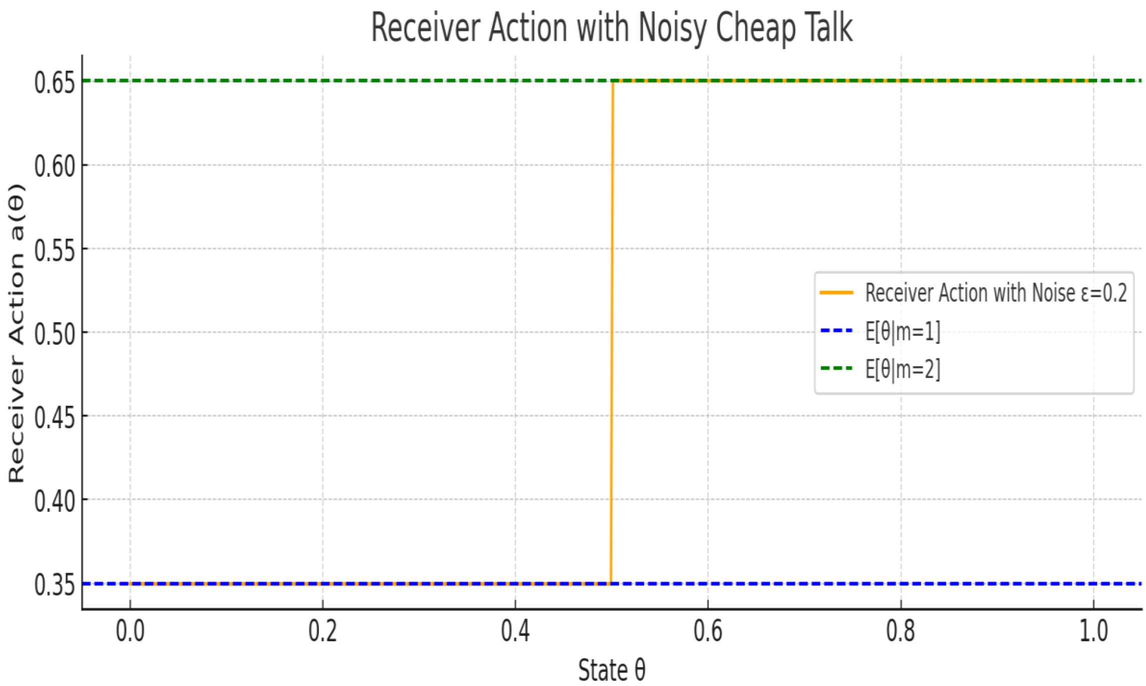
This causes non-monotonicity in $m(\theta)$ and non-monotonic belief updating by the receiver. This is plotted next.

Figure 27 message functions:2-partition vs non-monotonic



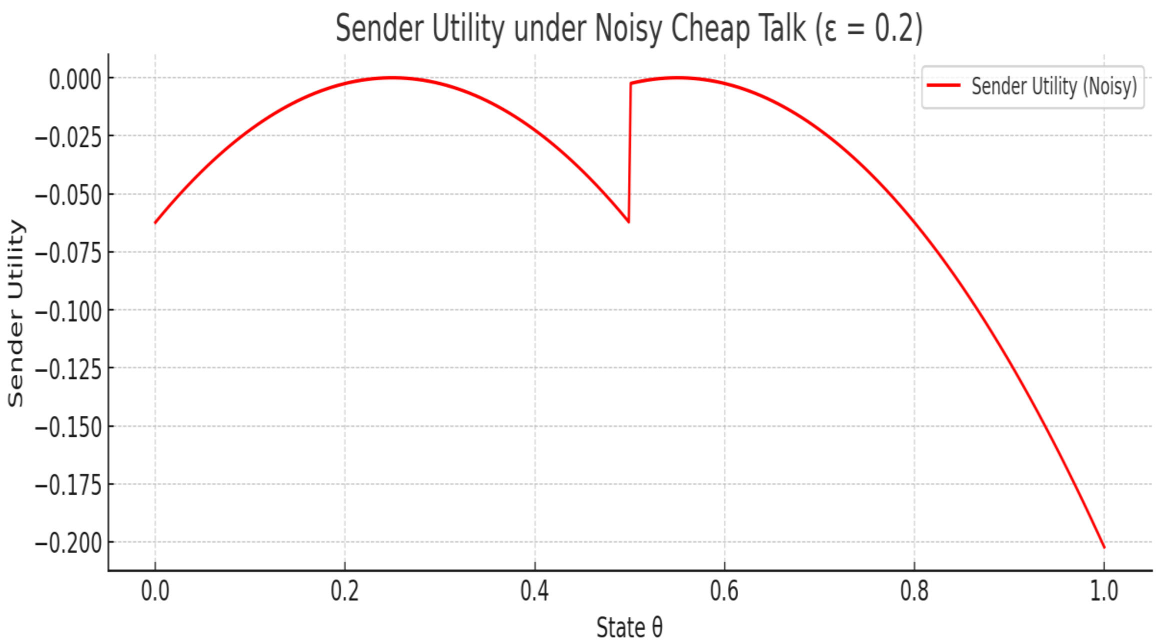
Source: Author's own calculation

Figure 28 Receiver action with Noisy cheap talk



Source: Author's own calculation

Figure 29 Sender utility under noisy cheap talk $\varepsilon = 0.2$



Source: Author's own calculation

- ✓ Message Functions Compared
- ✓ Dashed line: Standard 2-partition — monotonic: low types send Message 1, high types send Message 2.

- ✓ Solid line: Non-monotonic strategy — low and high types send Message 1, middle types send Message 2.
- ✓ This could arise when mid-types want to differentiate from others or due to discontinuous incentives.

With noise $\varepsilon = 0.2$, the message is less reliable. Receiver plays $a = \mathbb{E}[\theta|m]$ but because of noise:

- ✓ Beliefs are less extreme.
- ✓ For Message 1: action moves upward
- ✓ For Message 2: action moves downward
- ✓ This reduces the information content of the message — moving us closer to babbling.

Sender's utility

equation 109

$$u_S = -(a(\theta) - \theta - b)^2$$