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CONTENT

SOME FIXED POINT RESULTS FOR $\alpha - \psi -$ CONTRACTION SET - VALUED MAPPINGS IN CONE METRIC SPACES

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Abstract. In this paper are consider two $\alpha - \psi -$ contraction set valued mappings in cone metric spaces. For the composition of these two set - valued mappings is proved a fixed point theorem. With the idea of obtaining the uniqueness of the fixed point in the single - valued case is made an additional assumptions, too.

Key words: $\alpha -$ admissible set valued mapping; $\alpha - \psi -$ contraction set valued mappings; cone metric space; fixed point theorem; generalized Hausdorff distance.
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1. Introduction

Huang and Zhang [8] have examined cone metric spaces over solid Banach spaces. Cone metric spaces can be considered as a generalized version of the metric spaces. The above - mentioned researchers determined convergence and completeness in such spaces and provided a proof of some fixed point theorems for contractive single-valued mappings. Samet [15] has presented the notions of $\alpha -$ admissible and $\alpha - \psi -$ contractive type mappings and as a result established some fixed point theorems using these concepts. The reader can find more fixed point results via admissible operators and variations of $\alpha - \psi -$ contractive type mappings in [1],[2],[3],[4],[5],[6],[10],[12],[14],[15].

Let $(E, \|\cdot\|_E)$ be a real Banach space with zero element θ. Let P be a subset of E satisfying the following conditions:

(P1) P is non-empty closed and $P \neq \{\theta\}$;

(P2) $ax + by \in P$ for all $x, y \in P$ and $a, b \in R$, $a, b > 0$;

(P3) $P \bigcap (-P) = \{\theta\}$.

These three conditions mean that P is nonempty, closed, convex, pointed cone and P is not the trivial cone. Moreover, we will assume that the cone P has nonempty interior, i.e. P is solid.

Now we can define a partial order $\preceq$ on E with respect to the cone $P \subset E$. We will say that the element x precedes y and we will write $x \preceq y$, if and only if $y - x \in P$. We will say that x strictly precedes y and write $x \prec y$, if and only if $y - x \in P$, but $x \neq y$. Finally, we will write $x \ll y$ if and only if $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ stands for the interior of P.

The cone P is called normal if there is a number $K > 0$, such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \qquad \Rightarrow \qquad \|x\|_E \leq K \|y\|_E .$$

The least positive number K satisfying this inequality is called the normal constant of P. It is proved that $K \geq 1$.

Throughout the paper we will always suppose that E is a Banach space, P is a nonempty closed convex pointed cone in E with $\operatorname{int} P \neq \emptyset$. When the cone needs to be normal it will be mentioned in the statements in the theorems.

Definition 1.1. A cone metric space is an ordered pair (X, d), where X is any set and $d : X \times X \to E$ is a mapping, satisfying

(i) $d(x, y) \in P$, that is $\theta \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

The elements of cone metric space (X, d), are called points. Obviously, every metric space is a cone metric space over R. In [13] is shown that the theory of cone metric spaces over solid vector spaces is very close to the theory of the metric spaces.

Convergence in cone metric space is defined as follows.

Definition 1.2. ([8])Let (X,d) be a cone metric space, let $\{x_n\}$ be a sequence in X and $x \in X$. If for any $c \in P$ with $\theta \ll c$, there is $N \geq 1$ such that for all $n \geq N$, $d(x_n,x) \ll c$, then $\{x_n\}$ is said to be convergent. We will say $\{x_n\}$ converges to x and write $x_n \to x$, as $n \to \infty$.
Further in the paper we will use the following results.
Lemma 1.3. ([8]) *Let (X,d) be a cone metric space with cone P. Let $\{x_n\}$ be a sequence in X. Then*
(i) $\{x_n\}$ converges to x if $d(x_n,x) \to 0$, as $n \to \infty$;
(ii) if P is a normal cone then $\{x_n\}$ converges to x if and only if $d(x_n,x) \to 0$, as $n \to \infty$.
Lemma 1.4. ([8]) *Let (X,d) be a cone metric space, P is a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.*
Definition 1.5. ([8]) Let (X,d) be a cone metric space, $\{x_n\}$ be a sequence in X and let $x \in X$. If for any $c \in P$ with $\theta \ll c$, there is $N \geq 1$ such that for all $n,m \geq N$, $d(x_n,x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.
Definition 1.6. ([8]) Let (X,d) be a cone metric space. If every Cauchy sequence is convergent in X, then (X,d) is called a complete cone metric space.
Definition 1.7. ([8]) Let (X,d) be a cone metric space. We say that a subset A of X is closed if for any sequence $\{x_n\}$ in A convergent to x we have $x \in A$.
Lemma 1.8. ([9],[13]) *Let (X,d) be a cone metric space with cone P. Then the following properties hold:*
(i)If $u \preceq v$ and $v \ll w$, then $u \ll w$.
(ii) If $u \ll v$ and $v \preceq w$, then $u \ll w$.
(iii) If $u \ll v$ and $v \ll w$, then $u \ll w$.
(iv) If $\theta \preceq u \ll c$ for each $c \in intP$, then $u = \theta$.
(v) If $c \in intP$, and a_n is a sequence in E such that $\theta \preceq a_n$ for all $n \in \mathbb{N}$ and $a_n \to \theta$ as $n \to \infty$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $a_n \ll c$.
(vi) $\operatorname{int}P + P \subset \operatorname{int}P$ i.e., if $u \ll a$ and $v \preceq b$, then $u + v \ll a + b$.
Following the notion of $\alpha-$admissible mapping (see Samet [15]), Asl [5] introduced the multi-valued version of this concept α_*-admissible operator. Mohammadi [12] extended this last notion to $\alpha-$admissible operator as follows.
Definition 1.9. ([12]) Let X be a non-empty set, $F : X \rightrightarrows X$ and $\alpha : X \times X \to [0;\infty)$ be two given mappings. We say that F is $\alpha-$admissible whenever for each $x_1 \in X$ and $x_2 \in Fx_1$ with $\alpha(x_1,x_2) \geq 1$, we have $\alpha(x_2,x_3) \geq 1$, for all $x_3 \in Fx_2$.
From now on we will suppose that Ψ is a family of functions $\psi : P \to P$ that satisfy the following three assumptions
(Ψ 1) $\psi(\theta) = \theta$ and $\theta \prec \psi(t) \prec t$ for $t \in P - \{\theta\}$(consequently $\psi(t) \preceq t$ for all $t \in P$);
(Ψ 2) $\psi(t) \ll t$ for all $\theta \ll t$;
(Ψ 3) $\psi(a) \prec \psi(b)$ whenever $a \prec b$ (i.e. ψ is strictly order preserving).
For a cone metric space we denote

$$CB(X) := \{A : A \text{ is non-empty, closed and bounded subset of X}\},$$

$$s(p) := \{q \in E : p \preceq q\}, \quad \text{for } p \in E$$

and

$$s(a,B) := \bigcup_{b\in B} s\big(d(a,b)\big), \text{ for } a\in X \text{ and } B\in CB(X).$$

Cho and Bae [7] introduced the concept of the generalized Hausdorff distance operator in cone metric spaces and obtained a fixed point results for multi-valued operators using this concept. For $A,B\in CB(X)$ they considered the set

$$s(A,B) := \left(\bigcap_{a\in A} s(a,B)\right) \bigcap \left(\bigcap_{b\in B} s(A,b)\right)$$

and make the following remarks.

Remark 1.10. ([7]) Let (X,d) be a cone metric space. If $E=R$ and $P=[0,\infty)$, then (X,d) is a metric space. Moreover, for $A,B\in CB(X)$, $H(A,B)=\inf s(A,B)$ is the Hausforff distance induced by d.

Remark 1.11. ([7]) Let (X,d) be a cone metric space. Then, $s(\{a\},\{b\})=s(d(a,b))$ for $a,b\in X$.

Definition 1.12. Let (X,d) be a cone metric space, $\psi: P\to P$ and $\alpha: X\times X\to[0,\infty)$ are two given mappings. The multi-valued operator $F: X\to CB(X)$ is said to be an $\alpha-\psi-$ contraction if

$$\psi(d(x',x''))\in\alpha(x',x'')s(Fx',Fx''), \quad \text{for all } x',x''\in X.$$

Lemma 1.13. *Let (X,d) is a cone metric space in a real Banach space E.*

(i) Let $q\in P$ and $A,B\in CB(X)$. If $q\in s(A,B)$, then $q\in s(a,B)$, for all $a\in A$.

(ii) Let $q\in P$ and $\lambda\ge 0$, then $\lambda s(q)\subset s(\lambda q)$.

In [11], Kutbi and Situnavarat proved a contraction mapping principle for $\alpha-\psi-$ contractive type set valued mapping. In this paper, our aim is to prove a similar result for a composition of two set valued mappings.

2. Main result

Theorem 2.1. (Double contraction principle for admissible set - valued mappings)

Let (X,d) and (Y,D) are complete cone metric spaces with solid cone P in a real Banach space E and $\psi,\ \varphi\in\Psi$. Let $\alpha: X\times X\to[0,\infty)$ and $\beta: Y\times Y\to[0,\infty)$ are two given mappings. Let $T: X\to CB(Y)$ and $S: Y\to CB(X)$ are set valued mappings such that the mapping $S\circ T$ is α-admissible operator and $T\circ S$ is β-admissible operator. Suppose that the following conditions are satisfied:

(i) $\psi(d(x,x'))\in\alpha(x,x')s(Tx,Tx')$, for all $x,x'\in X$;

(ii) $\varphi(D(y,y'))\in\beta(y,y')s(Sy,Sy')$, for all $y,y'\in Y$;

(iii) $\lim_{n\to+\infty}\psi^n(t)=\theta$, for all $t\in P-\{\theta\}$;

(iv) there exist $y_0\in Y$, $x_0\in Sy_0$, $y_1\in Tx_0$ and $x_1\in Sy_1$, such that $\alpha(x_0,x_1)\ge 1$ and $\beta(y_0,y_1)\ge 1$.

(v) if the sequences $\{x_n\}\subset X$ and $\{y_n\}\subset Y$ are such that $\alpha(x_n,x_{n-1})\ge 1$, $\beta(y_n,y_{n-1})\ge 1$ for all $n\in N$ and $\{x_n\}\to x$, $\{y_n\}\to y$ as $n\to\infty$ then $\alpha(x_n,x)\ge 1$ and $\beta(y_n,y)\ge 1$ for all $n\in N$.

Then there exist $\overline{x}\in X$ and $\overline{y}\in Y$ such that $\overline{x}\in S\overline{y}$ and $\overline{y}\in T\overline{x}$.

Proof. Let $y_0\in Y$, $x_0\in Sy_0$, $y_1\in Tx_0$, and $x_1\in Sy_1$ such that $\alpha(x_0,x_1)\ge 1$ and $\beta(y_0,y_1)\ge 1$. Then, using (i), we have

$$\psi(d(x_0,x_1))\in\alpha(x_0,x_1)s(Tx_0,Tx_1)$$

and by Lemma 1.13 (i)

$$\psi(d(x_0,x_1)) \in \alpha(x_0,x_1)s(y_1,Tx_1).$$

By the definition of s there exists $y_2 \in Tx_1$ such that

$$\psi(d(x_0,x_1)) \in \alpha(x_0,x_1)s(D(y_1,y_2)).$$

Furthermore, using Lemma 1.13 (ii) we have

$$\psi(d(x_0,x_1)) \in s(\alpha(x_0,x_1)D(y_1,y_2)).$$

By (iv) we have $\alpha(x_0,x_1) \geq 1$ then $D(y_1,y_2) \preceq \alpha(x_0,x_1)D(y_1,y_2)$. According to the definition of s, this means that $\alpha(x_0,x_1)D(y_1,y_2) \preceq \psi(d(x_0,x_1))$. And we come to

$$D(y_1,y_2) \preceq \psi(d(x_0,x_1)). \tag{1}$$

Now using (ii), we have

$$\varphi(D(y_1,y_2)) \in \beta(y_1,y_2)s(Sy_1,Sy_2)$$

and since $x_1 \in Sy_1$ we come to

$$\varphi(D(y_1,y_2)) \in \beta(y_1,y_2)s(x_1,Sy_2).$$

According to the definition of s, there exists $x_2 \in Sy_2$ such that

$$\varphi(D(y_1,y_2)) \in \beta(y_1,y_2)s(d(x_1,x_2)).$$

Then, by Lemma 1.13 (ii),

$$\varphi(D(y_1,y_2)) \in s(\beta(y_1,y_2)d(x_1,x_2)).$$

Moreover by (iv) since $y_2 \in Tx_1 \subset T \circ Sy_1$ and $T \circ S$ is β- admissible we have $\beta(y_1,y_2) \geq 1$, consequently $d(x_1,x_2) \preceq \beta(y_1,y_2)d(x_1,x_2)$ and we come to $d(x_1,x_2) \preceq \varphi(D(y_1,y_2))$. Since $\varphi \in \Psi$, we have $\varphi(D(y_1,y_2)) \preceq D(y_1,y_2)$. By (1) we have

$$d(x_1,x_2) \preceq \psi(d(x_0,x_1)). \tag{2}$$

Now we can use (i) again and we will have

$$\psi(d(x_1,x_2)) \in \alpha(x_1,x_2)s(Tx_1,Tx_2).$$

Since $y_2 \in Tx_1$, we have

$$\psi(d(x_1,x_2)) \in \alpha(x_1,x_2)s(y_2,Tx_2).$$

By the definition of s, there exists $y_3 \in Tx_2$ such that

$$\psi(d(x_1,x_2)) \in \alpha(x_1,x_2)s(D(y_2,y_3)).$$

Using Lemma 1 (ii), we come to

$$\psi(d(x_1,x_2)) \in s(\alpha(x_1,x_2)D(y_2,y_3)).$$

But $S \circ T$ is α -admissible operator and $x_2 \in Sy_2 \subset S \circ Tx_1$, then using (iv), we have $\alpha(x_1,x_2) \geq 1$. Consequently $D(y_2,y_3) \preceq \alpha(x_1,x_2)D(y_2,y_3)$ and by the definition of s we have

$$D(y_2,y_3) \preceq \alpha(x_1,x_2)D(y_2,y_3) \preceq \psi(d(x_1,x_2)). \tag{3}$$

Then using (2), since ψ is order preserving we have

$$D(y_2,y_3) \preceq \psi^2(d(x_0,x_1)).$$

Now using again (ii) we have

$$\varphi(D(y_2,y_3)) \in \beta(y_2,y_3)s(Sy_2,Sy_3).$$

By Lemma 1.13(i) since $x_2 \in Sy_2$ we come to

$$\varphi(D(y_2,y_3)) \in \beta(y_2,y_3)s(x_2,Sy_3).$$

By the definition of s there exist $x_3 \in Sy_3$ such that

$$\varphi(D(y_2,y_3)) \in \beta(y_2,y_3)s(d(x_2,x_3)).$$

Then by Lemma 1.13 (ii)

$$\varphi(D(y_2,y_3)) \in s(\beta(y_2,y_3)d(x_2,x_3)).$$

But $T \circ S$ is β-admissible and $y_3 \in Tx_2 \subset T \circ Sy_2$ then we have $\beta(y_2,y_3) \geq 1$ and consequently $d(x_2,x_3) \preceq \beta(y_2,y_3)d(x_2,x_3)$. By the definition of s since $\varphi,\psi \in \Psi$ using ($\Psi1$), ($\Psi3$), (3) and (2) we come to

$$d(x_2,x_3) \preceq \varphi(D(y_2,y_3)) \preceq D(y_2,y_3) \preceq \psi(d(x_1,x_2)) \preceq \psi^2(d(x_0,x_1)).$$

In this way, proceeding by induction, we can construct a sequences $\{x_n\}$ in X and $\{y_n\}$ in Y such that

$$x_n \in Sy_n, \qquad y_{n+1} \in Tx_n, \tag{4}$$

$$d(x_n,x_{n+1}) \preceq \psi(d(x_{n-1},x_n)), \qquad D(y_n,y_{n+1}) \preceq \psi(d(x_{n-1},x_n)), \tag{5}$$

$$\alpha(x_n,x_{n-1}) \geq 1, \qquad \beta(y_n,y_{n-1}) \geq 1. \tag{6}$$

for all $n \in N$. Using (5) we obtain

$$d(x_n,x_{n+1}) \preceq \psi^n(d(x_0,x_1)), \qquad D(y_n,y_{n+1}) \preceq \psi^n(d(x_0,x_1)), \tag{7}$$

for all $n \in N$.

By assumption (iii) we have $\lim_{n\to+\infty} \psi^n(t) = \theta$. Now we fix $c \in E$ such that $\theta \ll c$. By ($\Psi1$), ($\Psi2$) we have $\theta \prec \psi(c) \ll c$ and consequently $\theta \ll c - \psi(c)$. Now using Lemma 1.8 since $\lim_{n\to+\infty} \psi^n(t) = \theta$ there exists $N \in \mathbb{N}$ such that $\psi^n(d(x_0,x_1)) \ll c$ and $\psi^n(d(x_0,x_1)) \ll c - \psi(c)$, for all $n \geq N$. Consequently by (7) we have $d(x_n,x_{n+1}) \ll c - \psi(c)$ and $d(x_n,x_{n+1}) \ll c$. For fixed $m > n \geq N$ using (3) since ψ is order preserving we have

$$d(x_{n+1},x_{n+2}) \preceq \psi(d(x_n,x_{n+1})) \preceq \psi(c),$$

$$d(x_{n+1},x_{n+3}) \preceq \psi(d(x_n,x_{n+2})) \preceq \psi(d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})) \preceq \psi(c),$$

....

$$d(x_{n+1},x_{n+k}) \preceq \psi(c), \quad \text{for all } k>1.$$

Then using Lemma 8 (vi) we have

$$d(x_n,x_m) \preceq d(x_n,x_{n+1}) + d(x_{n+1},x_m) \ll c - \psi(c) + \psi(c) = c.$$

This means that $\{x_n\}$ is a Cauchy sequence. Since X is complete cone metric space we have $\{x_n\} \to x$ as $n \to \infty$. Similarly, it is easy to see that

$$D(y_{n+1},y_{m+k}) \preceq \psi(c), \quad \text{for all } k>1$$

And the sequence $\{y_n\}$ is a Cauchy sequence. Since Y is complete cone metric space we have $\{y_n\} \to y$ as $n \to \infty$. Now using (4) and (v) we obtain $\alpha(x_n,x) \geq 1$ and $\beta(y_n,y) \geq 1$, for all $n \in N$. Using (i) again we have

$$\psi(d(x_n,x)) \in \alpha(x_n,x)s(Tx_n,Tx)$$

for all $n \in N$. By (4) $y_{n+1} \in Tx_n$ and using Lemma 1.13 (i) we obtain

$$\psi(d(x_n,x)) \in \alpha(x_n,x)s(y_{n+1},Tx)$$

for all $n \in N$. Then there exists a sequence $\overline{y}_n \in Tx$ such that

$$\psi(d(x_n,x)) \in \alpha(x_n,x)s(D(y_{n+1},\overline{y}_n)) \subseteq s(\alpha(x_n,x)D(y_{n+1},\overline{y}_n))$$

$n \in N$ and consequently

$$D(y_{n+1}, \overline{y}_n) \preceq \alpha(x_n, x) D(y_{n+1}, \overline{y}_n) \preceq \psi(d(x_n, x)) \preceq d(x_n, x)$$

for all $n \in N$. Since $\{x_n\} \to x$ as $n \to \infty$ it follows that for a given $c \in E$ and $\theta \ll c$ there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have $d(x_n, x) \ll \frac{c}{2}$. Then we have $D(y_{n+1}, \overline{y}_n) \ll \frac{c}{2}$ for all $n \geq N_1$. Since $\{y_n\} \to y$ as $n \to \infty$ we have $D(y_{n+1}, y) \ll \frac{c}{2}$ for all $n \geq N_1$. It follows that for all $n \geq N_1$.

$$D(\overline{y}_n, y) \preceq D(y_{n+1}, \overline{y}_n) + D(y_{n+1}, y) \ll \frac{c}{2} + \frac{c}{2} = c.$$

This means that the sequence $\{\overline{y}_n\} \in Tx$ converges to y. But Tx is closed then $y \in Tx$.
Furthermore using (ii) again we have

$$\varphi(D(y_n, y)) \in \beta(y_n, y) s(Sy_n, Sy)$$

for all $n \in N$. By (4) $x_n \in Sy_n$ and using Lemma 1.13 (i) we obtain

$$\varphi(D(y_n, y)) \in \beta(y_n, y) s(x_n, Sy)$$

for all $n \in N$. Then there exists a sequence $\overline{x}_n \in Sy$ such that

$$\varphi(D(y_n, y)) \in \beta(y_n, y) s(d(x_n, \overline{x}_n)) \subseteq s(\beta(y_n, y) d(x_n, \overline{x}_n))$$

$n \in N$ and consequently

$$d(x_n, \overline{x}_n) \preceq \beta(y_n, y) d(x_n, \overline{x}_n) \preceq \varphi(D(y_n, y)) \preceq D(y_n, y) \ll \frac{c}{2}$$

for all $n \geq N_1$. Then we have

$$d(\overline{x}_n, x) \preceq d(x_n, \overline{x}_n) + d(x_n, x) \ll \frac{c}{2} + \frac{c}{2} = c$$

for all $n \geq N_1$. This means that the sequence $\{\overline{x}_n\} \in Sy$ converges to x. But Sy is closed then $x \in Sy$ and this completes the proof. □

We will notice that Theorem 2.1 can be similarly proved if we change assumptions (iii) and (iv) with:

(iii') $\lim\limits_{n \to +\infty} \varphi^n(t) = \theta$, for all $t \in P - \{\theta\}$;

(iv') there exist $x_0 \in X$, $y_0 \in Tx_0$, $x_1 \in Sy_0$ and $y_1 \in Tx_1$, such that $\alpha(x_0, x_1) \geq 1$ and $\beta(y_0, y_1) \geq 1$.

In this case we have to similarly construct sequences $\{x_n\}$ in X and $\{y_n\}$ in Y such that

$$x_{n+1} \in Sy_n, \qquad y_n \in Tx_n,$$

$$d(x_n, x_{n+1}) \preceq \varphi(d(x_{n-1}, x_n)), \qquad D(y_n, y_{n+1}) \preceq \varphi(d(x_{n-1}, x_n)),$$

$$\alpha(x_n, x_{n-1}) \geq 1, \qquad \beta(y_n, y_{n-1}) \geq 1.$$

$$d(x_n, x_{n+1}) \preceq \varphi^n(d(x_0, x_1)), \qquad D(y_n, y_{n+1}) \preceq \varphi^n(d(x_0, x_1)),$$

for all $n \in \mathbb{N}$. Hence we can make the same conclusion as in the proof of the theorem using the fact that $\varphi^n \to \theta$, instead of $\psi^n \to \theta$, as $n \to \infty$.

Example 2.2. Let

$E = C_R^1[0,1] := \{f : [0,1] \to R, f$ is first order continuously differentiable on $[0,1]\}$ and let

E be a real Banach space with norm $\|\cdot\|_E$ defined by $\|f\|_E = \sup_{t\in[0,1]}|f(t)| + \sup_{t\in[0,1]}|f'(t)|$ for all $f \in E$. The zero element $\theta := f_0 \in E$ is defined by $f_0(t) = 0$, for all $t\in[0,1]$. The cone P in E is defined by $P := \{g \in E : g(t) \geq 0 \text{ forall } t\in[0,1]\}$. This cone is solid and not normal (see [16]).
Let $X = [0,10]$, $Y = [0,1]$. We define cone metric $d : X\times X \to E$, for each $x_1, x_2 \in X$ and $D : Y\times Y \to E$ for each $y_1, y_2 \in Y$ as follows:

$$d(x_1,x_2)(t) = |x_1 - x_2| e^t, \qquad D(y_1,y_2)(t) = |y_1 - y_2| e^t,$$

for all $t\in[0,1]$. Define the mappings $\alpha : X\times X \to [0,\infty)$ and $\beta : Y\times Y \to [0,\infty)$ by

$$\alpha(x_1,x_2) = \begin{cases} x_1^2 + x_2^2 + 1, & x_1, x_2 \in [0,1] \\ 0, & \text{otherwise} \end{cases} \qquad \beta(y_1,y_2) = \begin{cases} y_1^2 + y_2^2 + 1, & y_1, y_2 \in \left[0, \frac{1}{10}\right] \\ 0, & \text{otherwise} \end{cases}$$

Now we define the mapping $T : X \to CB(Y)$ as

$$Tx = \frac{1}{10}x, \text{ for all } x \in X$$

and the set valued mapping $S : Y \to CB(X)$ as follows

$$Sy = \begin{cases} \dfrac{5y}{3}, & y \in \left[0, \dfrac{1}{10}\right] \\ \left[\dfrac{100y - 10}{9}, 10y\right], & y \in \left(\dfrac{1}{10}, 1\right] \end{cases}$$

So we have the composition $S\circ T : X \to CB(X)$:

$$S\circ Tx = \begin{cases} \dfrac{x}{6}, & x \in [0,1] \\ \left[\dfrac{100x - 10}{9}, x\right], & \text{otherwise} \end{cases}$$

It is easy to see that $S\circ T$ is $\alpha-\psi-$ contraction mapping with $\psi : P \to P$ defined by $\psi(f) = \frac{1}{2}f$, for all $f \in P$. The mapping $S\circ T$ is $\alpha-$ admissible operator and there exist $x_0 = 1$ and $x_1 = \frac{1}{6} \in S\circ Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Also the condition (v) of Theorem 2.1 holds. (For more details see Example 3.2 [11]).
The same statements hold if we consider $T\circ S : Y \to CB(Y)$:

$$T\circ Sy = \begin{cases} \dfrac{y}{6}, & y \in \left[0, \dfrac{1}{10}\right] \\ \left[\dfrac{100y - 1}{9}, y\right], & y \in \left(\dfrac{1}{10}, 1\right] \end{cases}$$

We will show that this mapping is $\beta-\phi-$ contraction mapping with $\phi : P \to P$ defined by $\psi(f) = 0{,}17f$, for all $f \in P$. Let $y_1, y_2 \in [0, \frac{1}{10}]$,

$$\beta(y_1,y_2)D\left(\frac{y_1}{6},\frac{y_2}{6}\right)=(y_1^2+y_2^2+1)\left|\frac{y_1-y_2}{6}\right|$$
$$\preceq\left(\frac{1}{100}+\frac{1}{100}+1\right)\left|\frac{y_1-y_2}{6}\right|e^t=0.17\,|\,y_1-y_2\,|\,e^t$$
$$=\phi(D(y_1,y_2))(t)$$

Hence $\phi(D(y_1,y_2))\in\beta(y_1,y_2)s(T\circ Sy_1,T\circ Sy_2)$. It is obvious that the same relation holds when $y_1,y_2\in(\frac{1}{10},1]$. $T\circ S$ is $\beta-$ admissible operator and there exist $y_0=\frac{1}{10}$ and $y_1=\frac{1}{60}\in T\circ Sy_0$ such that $\beta(y_0,y_1)\geq 1$. Also the condition (v) of Theorem 2.1 holds. This shows that Theorem 2.1 can be used for this case and the mappings $T\circ S, S\circ T$ have fixed points.

Of course Theorem 2.1 holds when the mappings T and S are single valued but even in this case we can not guarantee the uniqueness of the fixed point. To investigate the uniqueness of a fixed point we shall introduce the following condition.

(A) For each $x,y\in Fix(S\circ T)$, we have $\alpha(x,y)\geq 1$ and $\beta(Tx,Ty)\geq 1$ where $Fix(S\circ T)$ is the set of all fixed points of $S\circ T$.

Theorem 2.3. (Double contraction principle for admissible single valued mappings)
Let (X,d) and (Y,D) are cone metric spaces with solid cone P in a real Banach space E and $\psi,\ \varphi\in\Psi$. Let $\alpha:X\times X\to[0,\infty)$ and $\beta:Y\times Y\to[0,\infty)$ are two given mappings. Let $T:X\to Y$ and $S:Y\to X$ are single valued mappings such that the mapping $S\circ T$ is α-admissible operator and $T\circ S$ is β-admissible operator. Suppose that the following conditions are satisfied:

(i) $\alpha(x,x')D(Tx,Tx')\preceq\psi(d(x,x'))$, for all $x,x'\in X$;

(ii) $\beta(y,y')d(Sy,Sy')\preceq\varphi(D(y,y'))$, for all $y,y'\in Y$;

(iii) $\lim\limits_{n\to+\infty}\psi^n(t)=\theta$ or $\lim\limits_{n\to+\infty}\varphi^n(t)=\theta$, for all $t\in P-\{\theta\}$;

(iv) there exist $y_0\in Y$, such that $\alpha(Sy_0,S\circ T\circ Sy_0)\geq 1$ and $\beta(y_0,T\circ Sy_0)\geq 1$;

(v) if the sequences $\{x_n\}\subset X$ and $\{Tx_n\}\subset Y$ are such that $\alpha(x_n,x_{n+1})\geq 1$, $\beta(Tx_n,Tx_{n-1})\geq 1$ for all $n\in\mathbb{N}$ and $\{x_n\}\to x$, $\{Tx_n\}\to Tx$ as $n\to\infty$ then $\alpha(x_n,x)\geq 1$ and $\beta(Tx_n,Tx)\geq 1$ for all $n\in\mathbb{N}$. Then there exist $\overline{x}\in X$ and $\overline{y}\in Y$ such that $\overline{x}=S\overline{y}$ and $\overline{y}=T\overline{x}$.

*Let in addition assumption **(A)** holds. Then the fixed point of $S\circ T$ is unique.*

Proof. Suppose to the contrary and take two distinct points $x^*=S\circ Tx^*$ and $x=S\circ Tx$. Then $\theta\prec d(x^*,x)$. Using **(A),** (i), (ii) and the properties of $\varphi,\psi\in\Psi$ we have

$$\begin{aligned}d(x^*,x)&=d(S\circ Tx^*,S\circ Tx)\\&\preceq\beta(Tx^*,Tx)d(S\circ Tx^*,S\circ Tx)\preceq\varphi(D(Tx^*,Tx))\\&\prec D(Tx^*,Tx)\preceq\alpha(x^*,x)D(Tx^*,Tx)\preceq\psi(d(x^*,x))\\&\prec d(x^*,x).\end{aligned}$$

And this is a contradiction which shows that the fixed point is unique. □

Corollary 2.4. *Let (X,d) is a cone metric space, P is normal cone with normal constant K. For $c\in E$ with $\theta\ll c$ and $\overline{x}\in X$, set $B(\overline{x},c)=\{x\in X\,|\,d(\overline{x},x)\preceq c\}$. Let $T:X\to Y$ and $S:Y\to X$ are single valued mappings such that*

(a) $d(\overline{x},ST\overline{x})\preceq c-\varphi\circ\psi(c)$;

(b) $D(Tx,Tx')\preceq\psi(d(x,x'))$, for all $x,x'\in B(\overline{x},c)$;

(c) $d(Sy,Sy')\preceq\varphi(D(y,y'))$, for all $y,y'\in B(T\overline{x},\psi(c))$;

(d) $\lim_{n\to+\infty} \psi^n(t)=\theta$ or $\lim_{n\to+\infty} \varphi^n(t)=\theta$, for all $t\in P-\{\theta\}$.

Then $S\circ T$ *has a unique fixed point in* $B(\bar{x},c)$.

Proof. The existence of the unique fixed point follows from Theorem 2.3, taking $\alpha=1$ and $\beta=1$. Since the cone P is normal then the metric d is continuous and consequently $B(\bar{x},c)$ is complete. More precisely if we take a Cauchy sequence x_n in $B(\bar{x},c)$ then x_n is in X which is complete and consequently x_n converges to some $x\in X$. Moreover by Lemma 1.1.3 we have $d(x_n,x)\to 0$ $(n\to\infty)$, since P is normal cone. Then we have

$$d(x,\bar{x})\preceq d(x,x_n)+d(x_n,\bar{x})\preceq d(x,x_n)+c.$$

Consequently

$$\theta\preceq -d(x,\bar{x})+d(x,x_n)+c.$$

Since P is closed and $d(x_n,x)\to 0$ $(n\to\infty)$, we get $d(x,\bar{x})\preceq c$. Hence $x\in B(\bar{x},c)$ and therefore $B(\bar{x},c)$ is complete. Now we will show that for every $x\in B(\bar{x},c)$, STx is in $B(\bar{x},c)$. Let $x\in B(\bar{x},c)$,

$$\begin{aligned} &d(\bar{x},STx)\\ &\preceq d(\bar{x},ST\bar{x})+d(ST\bar{x},STx)\\ &\preceq c-\varphi\circ\psi(c)+\varphi(D(T\bar{x},Tx))\\ &\preceq c-\varphi\circ\psi(c)+\varphi\circ\psi(d(\bar{x},x))\\ &\preceq c.\end{aligned}$$

Then $STx\in B(\bar{x},c)$. □

We will notice that if we take $\varphi\circ\psi=kt$, where $k\in[0,1]$ and $S\circ T=F:X\to X$, then the above corollary is equivalent to Corollary 1 in [8].

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