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SOME GENERALIZATIONS OF RECURSIVE DERIVATES OF k -ary OPERATIONSAleksandra Mileva¹, Vesna Dimitrova²

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Abstract. We present several results about recursive derivates of k -ary operations defined on finite set Q . They are generalizations of some binary cases given by Larionova-Cojocaru and Syrbu [7]. Also, we present several experimental results about recursive differentiability of ternary quasigroups of order 4. We also prove that the multiplication group of k -ary quasigroups obtained by recursive differentiability of a given k -ary quasigroup (Q, f) is a subgroup of the multiplication group of the (Q, f) .

Keywords: Recursively t -differentiable quasigroups, k -ary operations
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1 Introduction

Let Q be a nonempty set and let i and k be a positive integers, $i \leq k$. We will use (x_i^k) to denote the $(k - i + 1)$ -tuple $(x_i, \dots, x_k) \in Q^{(k-i+1)}$, and $\overset{k}{x}$ to denote the k -tuple $(x, \dots, x) \in Q^k$. A k -ary operation f on the set Q is a mapping $f : Q^k \rightarrow Q$ defined by $f : (x_1^k) \rightarrow x_{k+1}$, for which we write $f(x_1^k) = x_{k+1}$. A k -ary groupoid ($k \geq 1$) is an algebra (Q, f) on a nonempty set Q as its universe and with one k -ary operation f . A k -ary groupoid (Q, f) is called a k -ary quasigroup (of order $|Q| = q$) if any k of the elements $a_1, a_2, \dots, a_{k+1} \in Q$, satisfying the equality

$$f(a_1^k) = a_{k+1},$$

uniquely specifies the remaining one.

The k -ary operations $f_1, f_2, \dots, f_d, 1 \leq d \leq k$, defined on a set Q are **orthogonal** if the system $\{f_i(x_1^k) = a_i\}_{i=1}^d$ has exactly q^{k-d} solutions for any $a_1, \dots, a_d \in Q$, where $q = |Q|$ [4, 3]. There is one-to-one correspondence between the set of all k -tuples of orthogonal k -ary operations $\langle f_1, f_2, \dots, f_k \rangle$ defined on a set Q and the set of all permutations $\theta : Q^k \rightarrow Q^k$ ([4]), given by

$$\theta(x_1^k) \rightarrow (f_1(x_1^k), f_2(x_1^k), \dots, f_d(x_1^k)).$$

The k -ary operation I_j , $1 \leq j \leq k$, defined on Q with $I_j(x_1^k) = x_j$ is called the j -th selector or the j -th projection.

A system $\Sigma = \{f_1, f_2, \dots, f_s\}_{s \geq k}$ of k -ary operations is called **orthogonal**, if every k operations of Σ are orthogonal. A system $\Sigma = \{f_1, f_2, \dots, f_r\}$, $r \geq 1$ of distinct k -ary operations defined on a set Q is called **strong orthogonal** if the system $\{I_1, \dots, I_k, f_1, f_2, \dots, f_r\}$ is orthogonal, where each I_j , $1 \leq j \leq k$, is j -th selector. It follows that each operation of a strong orthogonal system, which is not a selector, is a k -ary quasigroup operation.

A code $C \subseteq Q^n$ is called a **complete k -recursive code** if there exists a function $f : Q^k \rightarrow Q$ ($1 \leq k \leq n$) such that every code word $(u_0, \dots, u_{n-1}) \in C$ satisfies the conditions $u_{i+k} = f(u_i^{i+k-1})$ for every $i = 0, 1, \dots, n-k-1$, where $u_0, \dots, u_{k-1} \in Q$. It is denoted by $C(n, f)$.

$C(n, f)$ can be represented by

$$C(n, f) = \{(x_1^k, f^{(0)}(x_1^k), \dots, f^{(n-k-1)}(x_1^k)) : (x_1^k) \in Q^k\}$$

where

$$\begin{aligned} f^{(0)} &= f^{(0)}(x_1^k) = f(x_1^k), \\ f^{(1)} &= f^{(1)}(x_1^k) = f(x_2^k, f^{(0)}) \end{aligned}$$

...

$$\begin{aligned} f^{(k-1)} &= f^{(k-1)}(x_1^k) = f(x_k, f^{(0)}, \dots, f^{(k-2)}) \\ f^{(i+k)} &= f^{(i+k)}(x_1^k) = f(f^{(i)}, \dots, f^{(i+k-1)}) \text{ for } i \geq 0 \end{aligned}$$

are **recursive derivatives** of f . The general form of the recursive derivatives for any k -ary operation f is given in [6], and $f^{(n)} = f\theta^n$, where $\theta : Q^k \rightarrow Q^k$, $\theta(x_1^k) = (x_2^k, f(x_1^k))$.

A k -quasigroup (Q, f) is called **recursively t -differentiable** if all its recursive derivatives $f^{(0)}, \dots, f^{(t)}$ are k -ary quasigroup operations [5]. A k -quasigroup (Q, f) is called **t -stable** if the system of all recursive derivatives $f^{(0)}, \dots, f^{(t)}$ of f is an orthogonal system of k -ary quasigroup operations, i.e. $C(k+t+1, f)$ is an MDS code [5]. A k -ary quasigroup (Q, f) is called **strongly recursively t -differentiable** if it is recursively t -differentiable and $f^{(t+1)} = I_1$ (introduced for binary case in [1]). A k -ary quasigroup (Q, f) is strongly recursively 0-differentiable if $f^{(1)} = I_1$.

It is clear that if (Q, f) is a recursively t -differentiable k -ary quasigroup with recursive derivatives $f^{(0)}, \dots, f^{(t)}$, then its first recursive derivate $f^{(1)}$ is a recursively $(t-1)$ -differentiable k -ary quasigroup with recursive derivatives $f^{(1)}, \dots, f^{(t)}$ and $(t-1)$ -th recursive derivate $f^{(t-1)}$ is a recursively 1-differentiable k -ary quasigroup with recursive derivatives $f^{(t-1)}, f^{(t)}$.

2 Generalisation

The following results are generalisation of binary cases for recursive derivates from [7].

Lemma 1. *Let (Q, f) be a k -ary groupoid. For every $(x_1^k) \in Q^k$ and $n \in \mathbb{N}$ the following equalities hold:*

$$f^{(n)}(x_1^k) = f^{(n-1)}(x_2^k, f^{(0)}(x_1^k))$$

Proof. Let $f^{(i)} = f^{(i)}(x_1^k)$, for all $n \in N$. For $n = 1$, $f^{(1)}(x_1^k) = f^{(0)}(x_2^k, f^{(0)}(x_1^k))$.

Let us suppose that $f^{(n)}(x_1^k) = f^{(n-1)}(x_2^k, f^{(0)}(x_1^k))$ for $0 \leq n \leq s-1 < k-1$. Then for $n = s$, using this assumption, we get:

$$\begin{aligned} f^{(s-1)}(x_2^k, f^{(0)}(x_1^k)) &= f^{(s-2)}(x_3^k, f^{(0)}, f^{(0)}(x_2^k, f^{(0)})) = f^{(s-2)}(x_3^k, f^{(0)}, f^{(1)}) \\ &= f^{(s-3)}(x_4^k, f^{(0)}, f^{(1)}, f^{(0)}(x_3^k, f^{(0)}, f^{(1)})) = f^{(s-3)}(x_4^k, f^{(0)}, f^{(1)}, f^{(2)}) \\ &= \dots = \\ &= f^{(s-s)}(x_{s+1}^k, f^{(0)}, \dots, f^{(s-2)}, f^{(0)}(x_s^k, f^{(0)}, \dots, f^{(s-1)})) \\ &= f^{(0)}(x_{s+1}^k, f^{(0)}, \dots, f^{(s-2)}, f^{(s-1)}) \\ &= f^{(s)}(x_1^k) \end{aligned}$$

For $n = k$, we have

$$\begin{aligned} f^{(k-1)}(x_2^k, f^{(0)}(x_1^k)) &= f^{(k-2)}(x_3^k, f^{(0)}, f^{(0)}(x_2^k, f^{(0)})) = f^{(k-2)}(x_3^k, f^{(0)}, f^{(1)}) \\ &= \dots = \\ &= f^{(k-(k-1))}(x_k^k, f^{(0)}, \dots, f^{(k-3)}, f^{(0)}(x_{k-1}^k, f^{(0)}, \dots, f^{(k-3)})) \\ &= f^{(1)}(x_k, f^{(0)}, \dots, f^{(k-3)}, f^{(k-2)}) \\ &= f^{(0)}(f^{(0)}, \dots, f^{(k-2)}, f^{(0)}(x_k, f^{(0)}, \dots, f^{(k-3)}, f^{(k-2)})) \\ &= f^{(0)}(f^{(0)}, \dots, f^{(k-2)}, f^{(k-1)}) \\ &= f^{(k)}(x_1^k) \end{aligned}$$

Now let suppose that $f^{(n)}(x_1^k) = f^{(n-1)}(x_2^k, f^{(0)}(x_1^k))$ for $k+1 \leq n \leq s-1$. Then for $n = s$, using this assumption, we get:

$$\begin{aligned} f^{(s-1)}(x_2^k, f^{(0)}(x_1^k)) &= f^{(s-2)}(x_3^k, f^{(0)}, f^{(0)}(x_2^k, f^{(0)})) = f^{(s-2)}(x_3^k, f^{(0)}, f^{(1)}) \\ &= f^{(s-3)}(x_4^k, f^{(0)}, f^{(1)}, f^{(0)}(x_3^k, f^{(0)}, f^{(1)})) = f^{(s-3)}(x_4^k, f^{(0)}, f^{(1)}, f^{(2)}) \\ &= \dots = \\ &= f^{(s-(k-1))}(x_k^k, f^{(0)}, \dots, f^{(k-3)}, f^{(0)}(x_{k-1}^k, f^{(0)}, \dots, f^{(k-3)})) \\ &= f^{(s-(k-1))}(x_k, f^{(0)}, \dots, f^{(k-3)}, f^{(k-2)}) \\ &= f^{(s-k)}(f^{(0)}, \dots, f^{(k-2)}, f^{(0)}(x_k, f^{(0)}, \dots, f^{(k-3)}, f^{(k-2)})) \\ &= f^{(s-k)}(f^{(0)}, \dots, f^{(k-2)}, f^{(k-1)}) \\ &= f^{(s-(k+1))}(f^{(1)}, \dots, f^{(k-1)}, f^{(0)}(f^{(0)}, \dots, f^{(k-2)}, f^{(k-1)})) \\ &= f^{(s-(k+1))}(f^{(1)}, \dots, f^{(k-1)}, f^{(k)}) \end{aligned}$$

$$\begin{aligned}
&= \dots = \\
&= f^{(s-(k+s-k))}(f^{(s-k)}, \dots, f^{(s-2)}, f^{(0)}(f^{(s-k-1)}, \dots, f^{(s-3)}, f^{(s-2)})) \\
&= f^{(0)}(f^{(s-k)}, \dots, f^{(s-2)}, f^{(s-1)}) \\
&= f^{(s)}(x_1^k)
\end{aligned}$$

\Rightarrow Lemma 1 is true for every $n \in N$.

Proposition 1. *Let (Q, f) be a k -ary groupoid. For every $(x_1^k) \in Q^k$ and for every $j = k - 1, \dots, n - 1$, where $n \geq k$, the following equalities hold:*

$$f^{(n)}(x_1^k) = f^{(n-j-1)}(f^{(j-k+1)}(x_1^k), \dots, f^{(j)}(x_1^k))$$

Proof. Let $f^{(i)} = f^{(i)}(x_1^k)$, for all $n \in N$. For $n = k$ and $j = k - 1$, we have

$$f^{(k)}(x_1^k) = f^{(0)}(f^{(0)}(x_1^k), \dots, f^{(k-1)}(x_1^k))$$

Suppose that $f^{(i)}(x_1^k) = f^{(i-j-1)}(f^{(j-k+1)}(x_1^k), \dots, f^{(j)}(x_1^k))$ for every $j = k - 1, \dots, i - 1$, $i = n$. For $i = n + 1$, we have:

$$\begin{aligned}
f^{(n+1)}(x_1^k) &= f^{(0)}(f^{(n-k+1)}, \dots, f^{(n)}) \\
&= f^{(0)}(f^{(n-k+1)-j-1}(f^{(j-k+1)}, \dots, f^{(j)}), \dots, f^{(n-j-1)}(f^{(j-k+1)}, \dots, f^{(j)})) \\
&= f^{(n+1)-j-1}(f^{(j-k+1)}(x_1^k), \dots, f^{(j)}(x_1^k))
\end{aligned}$$

\Rightarrow Proposition 1 is true for every $n \geq k$ and $j = k - 1, \dots, n - 1$.

Proposition 2. *If two k -ary groupoids (Q_1, f) and (Q_2, g) are isomorphic, then $(Q_1, f^{(n)}) \cong (Q_2, g^{(n)})$ for every $n \geq 1$.*

Proof. Let φ be an isomorphism from (Q_1, f) to (Q_2, g) . Then $\varphi(f(x_1^k)) = g(\varphi(x_1), \dots, \varphi(x_k))$ for every $(x_1^k) \in Q_1^k$. For $n = 1$, we have

$$\begin{aligned}
\varphi(f^{(1)}(x_1^k)) &= \varphi(f(x_2, f(x_1^k))) = g(\varphi(x_2), \dots, \varphi(x_k), \varphi(f(x_1^k))) = \\
&= g(\varphi(x_2), \dots, \varphi(x_k), g(\varphi(x_1), \dots, \varphi(x_k))) = g^{(1)}(\varphi(x_1), \dots, \varphi(x_k))
\end{aligned}$$

Suppose that $\varphi(f^{(i)}(x_1^k)) = g^{(i)}(\varphi(x_1), \dots, \varphi(x_k))$ for $2 \leq i \leq n - 1$. Because $f^{(0)} = f$, we have

$$\begin{aligned}
\varphi(f^{(n)}(x_1^k)) &= \varphi(f^{(n-1)}(x_2, f^{(0)}(x_1^k))) = g^{(n-1)}(\varphi(x_2), \dots, \varphi(x_k), \varphi(f^{(0)}(x_1^k))) \\
&= g^{(n-1)}(\varphi(x_2), \dots, \varphi(x_k), g^{(0)}(\varphi(x_1), \dots, \varphi(x_k))) = g^{(n)}(\varphi(x_1), \dots, \varphi(x_k))
\end{aligned}$$

So, we have $(Q_1, f^{(n)}) \cong (Q_2, g^{(n)})$ for every $n \geq 1$.

Proposition 3. *If (Q, f) is a k -ary groupoid, then the following holds:*

$$Aut(Q, f) \leq Aut(Q, f^{(n)}), \forall n \geq 1$$

Proof. If $\varphi \in \text{Aut}(Q, f)$, then $\varphi(f(x_1^k)) = f(\varphi(x_1), \dots, \varphi(x_k))$ for every $(x_1^k) \in Q^k$. For $n = 1$, we have

$$\begin{aligned} \varphi(f^{(1)}(x_1^k)) &= \varphi(f(x_2^k, f(x_1^k))) = f(\varphi(x_2), \dots, \varphi(x_k), \varphi(f(x_1^k))) = \\ &= f(\varphi(x_2), \dots, \varphi(x_k), f(\varphi(x_1), \dots, \varphi(x_k))) = f^{(1)}(\varphi(x_1), \dots, \varphi(x_k)) \end{aligned}$$

So, $\varphi \in \text{Aut}(Q, f^{(1)})$. Suppose that $\varphi \in \text{Aut}(Q, f^{(k)})$ for every $2 \leq k \leq n - 1$.

$$\begin{aligned} \varphi(f^{(n)}(x_1^k)) &= \varphi(f^{(n-1)}(x_2^k, f^{(0)}(x_1^k))) = f^{(n-1)}(\varphi(x_2), \dots, \varphi(x_k), \varphi(f^{(0)}(x_1^k))) \\ &= f^{(n-1)}(\varphi(x_2), \dots, \varphi(x_k), f^{(0)}(\varphi(x_1), \dots, \varphi(x_k))) = f^{(n)}(\varphi(x_1), \dots, \varphi(x_k)) \end{aligned}$$

So, $\varphi \in \text{Aut}(Q, f^{(n)})$.

The **center** of a k -ary quasigroup (Q, f) , denoted by $C(Q, f)$, consists of all those elements, c , such that

$$f(x_1^{i-1}, c, x_{i+1}^k) = f(x_1^{j-1}, c, x_{j+1}^k)$$

for all $(x_1^k) \in Q^k$ ([2]).

Proposition 4. *If (Q, f) is a recursively 1-differentiable k -ary quasigroup, then the following holds:*

$$C(Q, f) \leq C(Q, f^{(1)})$$

Proof. Let $c \in C(Q, f)$, then $f^{(0)}(x_1^{i-1}, c, x_{i+1}^k) = f^{(0)}(x_1^{j-1}, c, x_{j+1}^k)$ for all $(x_1^k) \in Q^k$. We have

$$\begin{aligned} f^{(1)}(x_1^{i-1}, c, x_{i+1}^k) &= f^{(0)}(x_2^{i-1}, c, x_{i+1}^k, f^{(0)}(x_1^{i-1}, c, x_{i+1}^k)) = \\ &= f^{(0)}(x_2^{i-1}, c, x_{i+1}^k, f^{(0)}(x_1^{j-1}, c, x_{j+1}^k)) = f^{(0)}(x_2^{j-1}, c, x_{j+1}^k, f^{(0)}(x_1^{j-1}, c, x_{j+1}^k)) = \\ &= f^{(1)}(x_1^{j-1}, c, x_{j+1}^k) \end{aligned}$$

Corollary 1. *If (Q, f) is a recursively t -differentiable k -ary quasigroup, then the following holds:*

$$C(Q, f) \leq C(Q, f^{(n)}), 1 \leq n \leq t$$

Let (a_1^{k-1}) be an arbitrary element of Q^{k-1} . The mapping $L_{i, (a_1^{k-1})} : Q \rightarrow Q$ ($i = 1, \dots, k$) defined by

$$L_{i, (a_1^{k-1})}(x) = f(a_1^{i-1}, x, a_i^{k-1})$$

is called i -translation of the k -ary groupoid (Q, f) with respect to (a_1^{k-1}) . If (Q, f) is k -ary quasigroup, the group generated by the set of all i -translations of the (Q, f) is called the multiplication group of a quasigroup (Q, f) , and can be represented by:

$$M(Q, f) = \langle L_{i, (a_1^{k-1})} | (a_1^{k-1}) \in Q^{k-1}, i = 1, \dots, k \rangle.$$

Proposition 5. *If (Q, f) is a recursively 1-differentiable k -ary quasigroup, then the following holds:*

$$M(Q, f^{(1)}) \leq M(Q, f)$$

Proof. For $i = 1$, we have

$$\begin{aligned} L_{1, (a_1^{k-1})}^{(1)}(x) &= f^{(1)}(x, a_1^{k-1}) = f^{(0)}(a_1^{k-1}, f^{(0)}(x, a_1^{k-1})) \\ &= f^{(0)}(a_1^{k-1}, L_{1, (a_1^{k-1})}(x)) = L_{k, (a_1^{k-1})} \circ L_{1, (a_1^{k-1})}(x) \\ &\Rightarrow L_{1, (a_1^{k-1})}^{(1)} \in M(Q, f) \end{aligned}$$

For $2 \leq i \leq k$, we have

$$\begin{aligned} L_{i, (a_1^{k-1})}^{(1)}(x) &= f^{(1)}(a_1^{i-1}, x, a_i^{k-1}) = f^{(0)}(a_2^{i-1}, x, a_i^{k-1}, f^{(0)}(a_1^{i-1}, x, a_i^{k-1})) \\ &= f^{(0)}(a_2^{i-1}, x, a_i^{k-1}, L_{i, (a_1^{k-1})}(x)) \\ &= L_{i-1, (a_2^{k-1}, L_{i, (a_1^{k-1})}(x))}^{(1)}(x) \end{aligned}$$

Because $L_{i, (a_1^{k-1})}(x) \in Q \Rightarrow (a_2^{k-1}, L_{i, (a_1^{k-1})}(x)) \in Q^{k-1} \Rightarrow L_{i, (a_1^{k-1})}^{(1)} \in M(Q, f)$.
So, $M(Q, f^{(1)}) \leq M(Q, f)$.

Corollary 2. *If (Q, f) is a recursively t -differentiable k -ary quasigroup, then the following holds:*

$$M(Q, f^{(n)}) \leq M(Q, f), 1 \leq n \leq t$$

An element $e \in Q$ is called an i -th unit of the k -ary groupoid (Q, f) if the following equation holds:

$$f(\overset{i-1}{e}, x, \overset{k-i}{e}) = x$$

for any $x \in Q$.

Lemma 2. *If (Q, f) is a recursively 1-differentiable k -ary quasigroup with 1-th unit, then the mapping $x \rightarrow f(\overset{k}{x})$ is a bijection.*

Proof. If the k -ary quasigroup (Q, f) has the 1-th unit e , then

$$f^{(1)}(e, \overset{k-1}{x}) = f(\overset{k-1}{x}, f(e, \overset{k-1}{x})) = f(\overset{k-1}{x}, x) = f(\overset{k}{x})$$

for every $x \in Q$, so the mapping $x \rightarrow f(\overset{k}{x})$ is a bijection on Q .

In general, the two converse statements are not always true. First, if (Q, f) is a k -ary quasigroup with 1-th unit, and the mapping $x \rightarrow f(\overset{k}{x})$ is a bijection on Q , than (Q, f) is not always a recursively 1-differentiable k -ary quasigroup. For example, the quasigroup (Z_5, \cdot) , where $x \cdot y = x + 3y + 3z \pmod{5}$, is a ternary quasigroup with 1-th unit 0 and $x \rightarrow f(x, x, x)$ is a bijection on Q , but (Z_5, \cdot) is not a recursively 1-differentiable ternary quasigroup. Second, if (Q, f) is a recursively 1-differentiable k -ary quasigroup, and the mapping $x \rightarrow f(\overset{k}{x})$ is a bijection on Q , than (Q, f) does not have always a 1-th unit. For example, the quasigroup (Z_5, \cdot) , where $x \cdot y = 2x + 2y + 2z \pmod{5}$, is a recursively 1-differentiable ternary quasigroup and $x \rightarrow f(x, x, x)$ is a bijection on Q , but (Z_5, \cdot) does not have a 1-th unit.

Corollary 3. *If (Q, f) is a recursively t -differentiable k -ary quasigroup ($1 \leq t \leq k$) with the same 1-th to t -th unit e , then the mapping $x \rightarrow f(\overset{k}{x})$ is a bijection.*

Corollary 4. *If (Q, f) is a recursively t -differentiable k -ary loop ($1 \leq t \leq k$), then the mapping $x \rightarrow f(\overset{k}{x})$ is a bijection.*

Corollary 5. *If (Q, f) is a recursively t -differentiable k -ary group ($1 \leq t \leq k$), then the mapping $x \rightarrow f(\overset{k}{x})$ is a bijection.*

3 Some results for ternary quasigroups

By experiments, we obtained the following results:

- there are 96 recursively 1-differentiable ternary quasigroups of order 4, and all are 1-stable
- there are no recursively t -differentiable ternary quasigroups of order 4, for $t \geq 2$,
- there are 64 strongly recursively 0-differentiable ternary quasigroups of order 4,
- there are 8 strongly recursively 1-differentiable ternary quasigroups of order 4.

Bellow is example of strongly recursively 1-differentiable and 1-stable ternary quasigroups of order 4.

$$\begin{aligned} & \{\{1, 2, 3, 4\}, \{3, 4, 1, 2\}, \{4, 3, 2, 1\}, \{2, 1, 4, 3\}\}, \{\{2, 1, 4, 3\}, \{4, 3, 2, 1\}, \{3, 4, 1, 2\}, \{1, 2, 3, 4\}\}, \\ & \{\{3, 4, 1, 2\}, \{1, 2, 3, 4\}, \{2, 1, 4, 3\}, \{4, 3, 2, 1\}\}, \{\{4, 3, 2, 1\}, \{2, 1, 4, 3\}, \{1, 2, 3, 4\}, \{3, 4, 1, 2\}\} \end{aligned}$$

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References

1. Belyavskaya, G.: Recursively r -differentiable Quasigroups within S-systems and MDS-codes. Quasigroups and Related Systems 20, 157 – 168 (2012).

2. Belousov, V.D., Sandik, M.D.: n -ary Quasi-groups and Loops. Siberian Math. J+7 (1), 24–42 (1966)
3. Belousov, V.D., Yakubov, T. On orthogonal n -ary operations. (Russian), Voprosy Kibernetiki 16, 3–17 (1975)
4. Bektenov, A.S., Yakubov, T.: Systems of orthogonal n -ary operations. (Russian), Izv. AN Moldavskoi SSR, Ser. fiz.-teh. i mat. nauk, 3, 7–14 (1974)
5. Couselo, E., Gonsales, S., Markov, V., Nechaev, A.: Recursive MDS- codes and recursively differentiable quasigroup. Discrete Math. 10(2), 3–29, (1998)
6. Izbash, V. I., Syrbu, P.: Recursively differentiable quasigroups and complete recursive codes. Comment. Math. Univ. Carolinae 45, 257–263 (2004).
7. Larionova-Cojocaru, I., V. I., Syrbu, P.: On Recursive Differentiability of Binary Quasigroups. Studia Universitatis Moldavia 2(82), 53–60 (2015).