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CONTENT

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SOME GENERALIZATIONS OF RECURSIVE DERIVATES OF *k*-ary OPERATIONS

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Abstract. We present several results about recursive derivates of k-ary operations defined on finite set Q. They are generalizations of some binary cases given by Larionova-Cojocaru and Syrbu [7]. Also, we present several experimental results about recursive differentiability of ternary quasigroups of order 4. We also prove that the multiplication group of k-ary quasigroups obtained by recursive differentiability of a given k-ary quasigroup (Q, f) is a subgroup of the multiplication group of the (Q, f).

Keywords: Recursively t-differentiable quasigroups, k-ary operations
2010 Mathematics Subject Classification. 20N05, 20N15, 05B15, 94B60.

1 Introduction

Let Q be a nonempty set and let i and k be a positive integers, $i \leq k$. We will use (x_i^k) to denote the $(k-i+1)$-tuple $(x_i, \ldots, x_k) \in Q^{(k-i+1)}$, and $\overset{k}{x}$ to denote the k-tuple $(x, \ldots, x) \in Q^k$. A $k-$*ary operation* f on the set Q is a mapping $f : Q^k \to Q$ defined by $f : (x_1^k) \to x_{k+1}$, for which we write $f(x_1^k) = x_{k+1}$. A k-ary groupoid $(k \geq 1)$ is an algebra (Q, f) on a nonempty set Q as its universe and with one k-ary operation f. A k-ary groupoid (Q, f) is called a k-*ary quasigroup* (of order $|Q| = q$) if any k of the elements $a_1, a_2, \ldots, a_{k+1} \in Q$, satisfying the equality

$$f(a_1^k) = a_{k+1},$$

uniquely specifies the remaining one.

The $k-$ary operations $f_1, f_2, \ldots, f_d, 1 \leq d \leq k$, defined on a set Q are **orthogonal** if the system $\{f_i(x_1^k) = a_i\}_{i=1}^d$ has exactly q^{k-d} solutions for any $a_1, \ldots, a_d \in Q$, where $q = |Q|$ [4, 3]. There is one-to-one correspondence between the set of all k-tuples of orthogonal k-ary operations $< f_1, f_2, \ldots, f_k >$ defined on a set Q and the set of all permutations $\theta : Q^k \to Q^k$ ([4]), given by

$$\theta(x_1^k) \to (f_1(x_1^k), f_2(x_1^k), \ldots, f_d(x_1^k)).$$

The k-ary operation I_j, $1 \le j \le k$, defined on Q with $I_j(x_1^k) = x_j$ is called the *j-th selector* or the *j-th projection*.

A system $\Sigma = \{f_1, f_2, \dots, f_s\}_{s \ge k}$ of k-ary operations is called **orthogonal**, if every k operations of Σ are orthogonal. A system $\Sigma = \{f_1, f_2, \dots, f_r\}, r \ge 1$ of distinct $k-$ary operations defined on a set Q is called **strong orthogonal** if the system $\{I_1, \dots, I_k, f_1, f_2, \dots, f_r\}$ is orthogonal, where each $I_j, 1 \le j \le k$, is $j-$th selector. It follows that each operation of a strong orthogonal system, which is not a selector, is a k-ary quasigroup operation.

A code $C \subseteq Q^n$ is called a **complete k-recursive code** if there exists a function $f : Q^k \to Q$ $(1 \le k \le n)$ such that every code word $(u_0, \dots, u_{n-1}) \in C$ satisfies the conditions $u_{i+k} = f(u_i^{i+k-1})$ for every $i = 0, 1, \dots, n-k-1$, where $u_0, \dots, u_{k-1} \in Q$. It is denoted by $C(n, f)$.

$C(n, f)$ can be represented by

$$C(n, f) = \{(x_1^k, f^{(0)}(x_1^k), \dots, f^{(n-k-1)}(x_1^k)) : (x_1^k) \in Q^k\}$$

where
$f^{(0)} = f^{(0)}(x_1^k) = f(x_1^k)$,
$f^{(1)} = f^{(1)}(x_1^k) = f(x_2^k, f^{(0)})$
...
$f^{(k-1)} = f^{(k-1)}(x_1^k) = f(x_k, f^{(0)}, \dots, f^{(k-2)})$
$f^{(i+k)} = f^{(i+k)}(x_1^k) = f(f^{(i)}, \dots, f^{(i+k-1)})$ for $i \ge 0$
are **recursive derivatives** of f. The general form of the recursive derivatives for any k-ary operation f is given in [6], and $f^{(n)} = f\theta^n$, where $\theta : Q^k \to Q^k, \theta(x_1^k) = (x_2^k, f(x_1^k))$.

A k-quasigroup (Q, f) is called **recursively t-differentiable** if all its recursive derivatives $f^{(0)}, \dots, f^{(t)}$ are k-ary quasigroup operations [5]. A k-quasigroup (Q, f) is called **t-stable** if the system of all recursive derivatives $f^{(0)} \dots, f^{(t)}$ of f is an orthogonal system of k-ary quasigroup operations, i.e. $C(k+t+1, f)$ is an MDS code [5]. A k-ary quasigroup (Q, f) is called **strongly recursively t-differentiable** if it is recursively t-differentiable and $f^{(t+1)} = I_1$ (introduced for binary case in [1]). A k-ary quasigroup (Q, f) is strongly recursively 0-differentiable if $f^{(1)} = I_1$.

It is clear that if (Q, f) is a recursively t-differentiable k-ary quasigroup with recursive derivatives $f^{(0)} \dots, f^{(t)}$, then its first recursive derivate $f^{(1)}$ is a recursively $(t-1)$-differentiable k-ary quasigroup with recursive derivatives $f^{(1)} \dots, f^{(t)}$ and $(t-1)$-th recursive derivate $f^{(t-1)}$ is a recursively 1-differentiable k-ary quasigroup with recursive derivatives $f^{(t-1)}, f^{(t)}$.

2 Generalisation

The following results are generalisation of binary cases for recursive derivates from [7].

Lemma 1. *Let (Q, f) be a k-ary groupoid. For every $(x_1^k) \in Q^k$ and $n \in N$ the following equalities hold:*

$$f^{(n)}(x_1^k) = f^{(n-1)}(x_2^k, f^{(0)}(x_1^k))$$

Proof. Let $f^{(i)} = f^{(i)}(x_1^k)$, for all $n \in N$. For $n = 1$, $f^{(1)}(x_1^k) = f^{(0)}(x_2^k, f^{(0)}(x_1^k))$.

Let us suppose that $f^{(n)}(x_1^k) = f^{(n-1)}(x_2^k, f^{(0)}(x_1^k))$ for $0 \leq n \leq s-1 < k-1$. Then for $n = s$, using this assumption, we get:

$$f^{(s-1)}(x_2^k, f^{(0)}(x_1^k)) = f^{(s-2)}(x_3^k, f^{(0)}, f^{(0)}(x_2^k, f^{(0)})) = f^{(s-2)}(x_3^k, f^{(0)}, f^{(1)})$$
$$= f^{(s-3)}(x_4^k, f^{(0)}, f^{(1)}, f^{(0)}(x_3^k, f^{(0)}, f^{(1)})) = f^{(s-3)}(x_4^k, f^{(0)}, f^{(1)}, f^{(2)})$$
$$= \ldots =$$
$$= f^{(s-s)}(x_{s+1}^k, f^{(0)}, \ldots, f^{(s-2)}, f^{(0)}(x_s^k, f^{(0)}, \ldots, f^{(s-1)}))$$
$$= f^{(0)}(x_{s+1}^k, f^{(0)}, \ldots, f^{(s-2)}, f^{(s-1)})$$
$$= f^{(s)}(x_1^k)$$

For $n = k$, we have

$$f^{(k-1)}(x_2^k, f^{(0)}(x_1^k)) = f^{(k-2)}(x_3^k, f^{(0)}, f^{(0)}(x_2^k, f^{(0)})) = f^{(k-2)}(x_3^k, f^{(0)}, f^{(1)})$$
$$= \ldots =$$
$$= f^{(k-(k-1))}(x_k^k, f^{(0)}, \ldots, f^{(k-3)}, f^{(0)}(x_{k-1}^k, f^{(0)}, \ldots, f^{(k-3)}))$$
$$= f^{(1)}(x_k, f^{(0)}, \ldots, f^{(k-3)}, f^{(k-2)})$$
$$= f^{(0)}(f^{(0)}, \ldots, f^{(k-2)}, f^{(0)}(x_k, f^{(0)}, \ldots, f^{(k-3)}, f^{(k-2)}))$$
$$= f^{(0)}(f^{(0)}, \ldots, f^{(k-2)}, f^{(k-1)})$$
$$= f^{(k)}(x_1^k)$$

Now let suppose that $f^{(n)}(x_1^k) = f^{(n-1)}(x_2^k, f^{(0)}(x_1^k))$ for $k+1 \leq n \leq s-1$. Then for $n = s$, using this assumption, we get:

$$f^{(s-1)}(x_2^k, f^{(0)}(x_1^k)) = f^{(s-2)}(x_3^k, f^{(0)}, f^{(0)}(x_2^k, f^{(0)})) = f^{(s-2)}(x_3^k, f^{(0)}, f^{(1)})$$
$$= f^{(s-3)}(x_4^k, f^{(0)}, f^{(1)}, f^{(0)}(x_3^k, f^{(0)}, f^{(1)})) = f^{(s-3)}(x_4^k, f^{(0)}, f^{(1)}, f^{(2)})$$
$$= \ldots =$$
$$= f^{(s-(k-1))}(x_k^k, f^{(0)}, \ldots, f^{(k-3)}, f^{(0)}(x_{k-1}^k, f^{(0)}, \ldots, f^{(k-3)}))$$
$$= f^{(s-(k-1))}(x_k, f^{(0)}, \ldots, f^{(k-3)}, f^{(k-2)})$$
$$= f^{(s-k)}(f^{(0)}, \ldots, f^{(k-2)}, f^{(0)}(x_k, f^{(0)}, \ldots, f^{(k-3)}, f^{(k-2)}))$$
$$= f^{(s-k)}(f^{(0)}, \ldots, f^{(k-2)}, f^{(k-1)})$$
$$= f^{(s-(k+1))}(f^{(1)}, \ldots, f^{(k-1)}, f^{(0)}(f^{(0)}, \ldots, f^{(k-2)}, f^{(k-1)}))$$
$$= f^{(s-(k+1)}(f^{(1)}, \ldots, f^{(k-1)}, f^{(k)})$$

$$= \ldots =$$
$$= f^{(s-(k+s-k))}(f^{(s-k)}, \ldots, f^{(s-2)}, f^{(0)}(f^{(s-k-1)}, \ldots, f^{(s-3)}, f^{(s-2)}))$$
$$= f^{(0)}(f^{(s-k)}, \ldots, f^{(s-2)}, f^{(s-1)})$$
$$= f^{(s)}(x_1^k)$$

$\Rightarrow$ Lemma 1 is true for every $n \in N$.

Proposition 1. *Let (Q, f) be a k-ary groupoid. For every $(x_1^k) \in Q^k$ and for every $j = k-1, \ldots, n-1$, where $n \geq k$, the following equalities hold:*

$$f^{(n)}(x_1^k) = f^{(n-j-1)}(f^{(j-k+1)}(x_1^k), \ldots, f^{(j)}(x_1^k))$$

Proof. Let $f^{(i)} = f^{(i)}(x_1^k)$, for all $n \in N$. For $n = k$ and $j = k-1$, we have

$$f^{(k)}(x_1^k) = f^{(0)}(f^{(0)}(x_1^k), \ldots, f^{(k-1)}(x_1^k))$$

Suppose that $f^{(i)}(x_1^k) = f^{(i-j-1)}(f^{(j-k+1)}(x_1^k), \ldots, f^{(j)}(x_1^k))$ for every $j = k - 1, \ldots, i-1$, $i = n$. For $i = n+1$, we have:

$$f^{(n+1)}(x_1^k) = f^{(0)}(f^{(n-k+1)}, \ldots, f^{(n)})$$
$$= f^{(0)}(f^{(n-k+1)-j-1}(f^{(j-k+1)}, \ldots, f^{(j)}), \ldots, f^{(n-j-1)}(f^{(j-k+1)}, \ldots, f^{(j)}))$$
$$f^{((n+1)-j-1)}(f^{(j-k+1)}(x_1^k), \ldots, f^{(j)}(x_1^k))$$

$\Rightarrow$ Proposition 1 is true for every $n \geq k$ and $j = k-1, \ldots, n-1$.

Proposition 2. *If two k-ary groupoids (Q_1, f) and (Q_2, g) are isomorphic, then $(Q_1, f^{(n)}) \cong (Q_2, g^{(n)})$ for every $n \geq 1$.*

Proof. Let φ be an isomorphism from (Q_1, f) to (Q_2, g). Then $\varphi(f(x_1^k)) = g(\varphi(x_1), \ldots, \varphi(x_k))$ for every $(x_1^k) \in Q_1^k$. For $n = 1$, we have

$$\varphi(f^{(1)}(x_1^k)) = \varphi(f(x_2^k, f(x_1^k)) = g(\varphi(x_2), \ldots, \varphi(x_k), \varphi(f(x_1^k))) =$$

$$g(\varphi(x_2), \ldots, \varphi(x_k), g(\varphi(x_1), \ldots, \varphi(x_k))) = g^{(1)}(\varphi(x_1), \ldots, \varphi(x_k)))$$

Suppose that $\varphi(f^{(i)}(x_1^k)) = g^{(i)}(\varphi(x_1), \ldots, \varphi(x_k))$ for $2 \leq i \leq n-1$. Because $f^{(0)} = f$, we have

$$\varphi(f^{(n)}(x_1^k)) = \varphi(f^{(n-1)}(x_2^k, f^{(0)}(x_1^k))) = g^{(n-1)}(\varphi(x_2), \ldots, \varphi(x_k), \varphi(f^{(0)}(x_1^k)))$$

$$= g^{(n-1)}(\varphi(x_2), \ldots, \varphi(x_k), g^{(0)}(\varphi(x_1), \ldots, \varphi(x_k))) = g^{(n)}(\varphi(x_1), \ldots, \varphi(x_k))$$

So, we have $(Q_1, f^{(n)}) \cong (Q_2, g^{(n)})$ for every $n \geq 1$.

Proposition 3. *If (Q, f) is a k-ary groupoid, then the following holds:*

$$Aut(Q, f) \leq Aut(Q, f^{(n)}), \forall n \geq 1$$

Proof. If $\varphi \in Aut(Q,f)$, then $\varphi(f(x_1^k)) = f(\varphi(x_1),\dots,\varphi(x_k))$ for every $(x_1^k) \in Q^k$. For $n=1$, we have

$$\varphi(f^{(1)}(x_1^k)) = \varphi(f(x_2^k, f(x_1^k)) = f(\varphi(x_2),\dots,\varphi(x_k),\varphi(f(x_1^k))) =$$

$$f(\varphi(x_2),\dots,\varphi(x_k),f(\varphi(x_1),\dots,\varphi(x_k))) = f^{(1)}(\varphi(x_1),\dots,\varphi(x_k)))$$

So, $\varphi \in Aut(Q, f^{(1)})$. Suppose that $\varphi \in Aut(Q, f^{(k)})$ for every $2 \le k \le n-1$.

$$\varphi(f^{(n)}(x_1^k)) = \varphi(f^{(n-1)}(x_2^k, f^{(0)}(x_1^k))) = f^{(n-1)}(\varphi(x_2),\dots,\varphi(x_k),\varphi(f^{(0)}(x_1^k)))$$

$$= f^{(n-1)}(\varphi(x_2),\dots,\varphi(x_k),f^{(0)}(\varphi(x_1),\dots,\varphi(x_k))) = f^{(n)}(\varphi(x_1),\dots,\varphi(x_k))$$

So, $\varphi \in Aut(Q, f^{(n)})$.

The **center** of a k-ary quasigroup (Q,f), denoted by $C(Q,f)$, consists of all those elements, c, such that

$$f(x_1^{i-1}, c, x_{i+1}^k) = f(x_1^{j-1}, c, x_{j+1}^k)$$

for all $(x_1^k) \in Q^k$ ([2]).

Proposition 4. *If (Q,f) is a recursively 1-differentiable k-ary quasigroup, then the following holds:*

$$C(Q,f) \le C(Q,f^{(1)})$$

Proof. Let $c \in C(Q,f)$, then $f^{(0)}(x_1^{i-1}, c, x_{i+1}^k) = f^{(0)}(x_1^{j-1}, c, x_{j+1}^k)$ for all $(x_1^k) \in Q^k$. We have

$$f^{(1)}(x_1^{i-1}, c, x_{i+1}^k) = f^{(0)}(x_2^{i-1}, c, x_{i+1}^k, f^{(0)}(x_1^{i-1}, c, x_{i+1}^k)) =$$

$$= f^{(0)}(x_2^{i-1}, c, x_{i+1}^k, f^{(0)}(x_1^{j-1}, c, x_{j+1}^k)) = f^{(0)}(x_2^{j-1}, c, x_{j+1}^k, f^{(0)}(x_1^{j-1}, c, x_{j+1}^k)) =$$

$$= f^{(1)}(x_1^{j-1}, c, x_{j+1}^k))$$

Corollary 1. *If (Q,f) is a recursively t-differentiable k-ary quasigroup, then the following holds:*

$$C(Q,f) \le C(Q,f^{(n)}), 1 \le n \le t$$

Let (a_1^{k-1}) be an arbitraty element of Q^{k-1}. The mapping $L_{i,(a_1^{k-1})} : Q \to Q$ $(i = 1,\dots,k)$ defined by

$$L_{i,(a_1^{k-1})}(x) = f(a_1^{i-1}, x, a_i^{k-1})$$

is called i-translation of the k-ary groupoid (Q,f) with respect to (a_1^{k-1}). If (Q,f) is k-ary quasigroup, the group generated by the set of all i-translations of the (Q,f) is called the multiplication group of a quasigroup (Q,f), and can be represented by:

$$M(Q,f) = \langle L_{i,(a_1^{k-1})} | (a_1^{k-1}) \in Q^{k-1}, i = 1,\dots,k\rangle.$$

Proposition 5. *If (Q, f) is a recursively 1-differentiable k-ary quasigroup, then the following holds:*

$$M(Q, f^{(1)}) \leq M(Q, f)$$

Proof. For $i = 1$, we have

$$L^{(1)}_{1,(a_1^{k-1})}(x) = f^{(1)}(x, a_1^{k-1}) = f^{(0)}(a_1^{k-1}, f^{(0)}(x, a_1^{k-1}))$$

$$= f^{(0)}(a_1^{k-1}, L_{1,(a_1^{k-1})}(x)) = L_{k,(a_1^{k-1})} \circ L_{1,(a_1^{k-1})}(x)$$

$$\Rightarrow L^{(1)}_{1,(a_1^{k-1})} \in M(Q, f)$$

For $2 \leq i \leq k$, we have

$$L^{(1)}_{i,(a_1^{k-1})}(x) = f^{(1)}(a_1^{i-1}, x, a_i^{k-1}) = f^{(0)}(a_2^{i-1}, x, a_i^{k-1}, f^{(0)}(a_1^{i-1}, x, a_i^{k-1}))$$

$$= f^{(0)}(a_2^{i-1}, x, a_i^{k-1}, L_{i,(a_1^{k-1})}(x))$$

$$= L_{i-1,(a_2^{k-1}, L_{i,(a_1^{k-1})}(x))}(x)$$

Because $L_{i,(a_1^{k-1})}(x) \in Q \Rightarrow (a_2^{k-1}, L_{i,(a_1^{k-1})}(x)) \in Q^{k-1} \Rightarrow L^{(1)}_{i,(a_1^{k-1})} \in M(Q, f)$. So, $M(Q, f^{(1)}) \leq M(Q, f)$.

Corollary 2. *If (Q, f) is a recursively t-differentiable k-ary quasigroup, then the following holds:*

$$M(Q, f^{(n)}) \leq M(Q, f), 1 \leq n \leq t$$

An element $e \in Q$ is called an **i-th unit** of the k-ary groupoid (Q, f) if the following equation holds:

$$f(\overset{i-1}{e}, x, \overset{k-i}{e}) = x$$

for any $x \in Q$.

Lemma 2. *If (Q, f) is a recursively 1-differentiable k-ary quasigroup with 1-th unit, then the mapping $x \to f(\overset{k}{x})$ is a bijection.*

Proof. If the k-ary quasigroup (Q, f) has the 1-th unit e, then

$$f^{(1)}(e, \overset{k-1}{x}) = f(\overset{k-1}{x}, f(e, \overset{k-1}{x})) = f(\overset{k-1}{x}, x) = f(\overset{k}{x})$$

for every $x \in Q$, so the mapping $x \to f(\overset{k}{x})$ is a bijection on Q.

In general, the two converse statements are not always true. First, if (Q, f) is a k-ary quasigroup with 1-th unit, and the mapping $x \to f(\overset{k}{x})$ is a bijection on Q, than (Q, f) is not always a recursively 1-differentiable k-ary quasigroup. For example, the quasigroup $(Z_5, \cdot)$, where $x \cdot y = x + 3y + 3z(mod\ 5)$, is a ternary quasigroup with 1-th unit 0 and $x \to f(x, x, x)$ is a bijection on Q, but $(Z_5, \cdot)$ is not a recursively 1-differentiable ternary quasigroup. Second, if (Q, f) is a recursively 1-differentiable k-ary quasigroup, and the mapping $x \to f(\overset{k}{x})$ is a bijection on Q, than (Q, f) does not have always a 1-th unit. For example, the quasigroup $(Z_5, \cdot)$, where $x \cdot y = 2x + 2y + 2z(mod\ 5)$, is a recursively 1-differentiable ternary quasigroup and $x \to f(x, x, x)$ is a bijection on Q, but $(Z_5, \cdot)$ does not have a 1-th unit.

Corollary 3. *If (Q, f) is a recursively t-differentiable k-ary quasigroup* $(1 \leq t \leq k)$ *with the same* 1*-th to* t*-th unit* e*, then the mapping* $x \to f(\overset{k}{x})$ *is a bijection.*

Corollary 4. *If (Q, f) is a recursively t-differentiable k-ary loop* $(1 \leq t \leq k)$, *then the mapping* $x \to f(\overset{k}{x})$ *is a bijection.*

Corollary 5. *If (Q, f) is a recursively t-differentiable k-ary group* $(1 \leq t \leq k)$, *then the mapping* $x \to f(\overset{k}{x})$ *is a bijection.*

3 Some results for ternary quasigroups

By experiments, we obtained the following results:

- there are 96 recursively 1-differentiable ternary quasigroups of order 4, and all are 1-stable
- there are no recursively t-differentiable ternary quasigroups of order 4, for $t \geq 2$,
- there are 64 strongly recursively 0-differentiable ternary quasigroups of order 4,
- there are 8 strongly recursively 1-differentiable ternary quasigroups of order 4.

Bellow is example of strongly recursively 1-differentiable and 1-stable ternary quasigroups of order 4.

{{{1, 2, 3, 4}, {3, 4, 1, 2}, {4, 3, 2, 1}, {2, 1, 4, 3}}, {{2, 1, 4, 3}, {4, 3, 2, 1}, {3, 4, 1, 2}, {1, 2, 3, 4}},
{{3, 4, 1, 2}, {1, 2, 3, 4}, {2, 1, 4, 3}, {4, 3, 2, 1}}, {{4, 3, 2, 1}, {2, 1, 4, 3}, {1, 2, 3, 4}, {3, 4, 1, 2}}}

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