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CONTENT

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NOTION FOR CONNECTEDNESS AND PATH CONNECTEDNESS IN SOME TYPE OF TOPOLOGICAL SPACES

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Abstract: The notions of connectedness and path connectedness of topological spaces in the part of general topology are firmly related. In particular, path connectedness is a tougher condition of connectedness and reversal does not always apply. In a metric space, the notion of connectedness is more difficult to formulate precisely, while the path connectedness is a concept whose definition remains the same and easier to understand in these spaces. In this text, there are examples of connectedness, path-connectedness and simultaneously connected and path-connected topological spaces. Also, an additional condition is required for which the relation of the connected space is valid to be followed by a path connected.
Keywords: connectedness; path connectedness; metric space; locally Euclidean space; Euclidean space;

1. Introduction

In [5], there are topology space that are connected but not path connected. There are some examples in which is explained that some space when topology is connected than they cannot be a path connected. The main result in this paper is given in the Theorem 2.1. The result refers to the strong relation between path connected space and connected space. The opposite relation between these two notions is given in the paper. Specifically, [5] explained connected space in which cannot be find a path. In [1], [2], [3] and [4] are given the main properties of these notion and in all of references are given an implication that some topology space when is path connected then that space is connected. In [6], some special properties for topology of a metric space are represented, specifically Euclidean metric space. This will help us for examples in a part of results and discussion. So, in this paper we will looking for examples in which will be explained the opposite relation i.e. if the space is connected that we can find a path, so it would be a path connected.

2. Materials and methods

Let X be a topology space.

Definition 2.1. ([1]) Let $Y \subseteq X$. The set Y is called *connected* in X if it cannot be represented as a union of two separated subsets of X.

Definition 2.2. ([2]) A topological space X is said to be a *connected* if the only two subsets of X that are simultaneously open and closed are X itself and the empty set $\varnothing$.

Theorem 2.1. ([1]) Let X be a topological space and let $Y \subseteq Z \subseteq X$. A set Y is connected in topology space X if and only if is connected in topology space Z.

Proof. ([1], pp. 61)

Remark 2.1. A topological space X is connected if and only if cannot be a represent like a union of two open disjunct subsets of X.

Theorem 2.2. The intervals from a real line R are the only connected subsets.

Let (X, d) be a metric space.

Definition 2.3. ([6], Definition. 5.1.1) A metric space X is said to be connected if the only sets which are both open and closed in X are $\varnothing$ and the full space X, when X is a metric space.

Example 2.1. Intervals such as $[a,b]$ and (a,b) are connected subsets of the real line R. As an example of a subset of the real line that is not connected, let $X=[1,2]\cup(3,4)$. $[1,2]$ is relatively closed subset of X since $[1,2]$ is closed in R. At the same time $[1,2]$ is a relatively open subset of X, since $[1,2]=\left(\frac{1}{2},\frac{5}{2}\right)\cap X$. Finally, $[1,2]\neq\varnothing$ and $[1,2]\neq X$, so hence X is not connected. By the same token, the 'open interval' $(3,4)$ is also both relatively open and relatively closed in X.

Example 2.2. A topological space $X\subseteq R^2$, $X=\left\{\left(x,\frac{x}{n}\right)\middle| x\in[0,1], n\in N\right\}$ is also connected space. This space is pictured below, and it's called the deleted infinite broom.

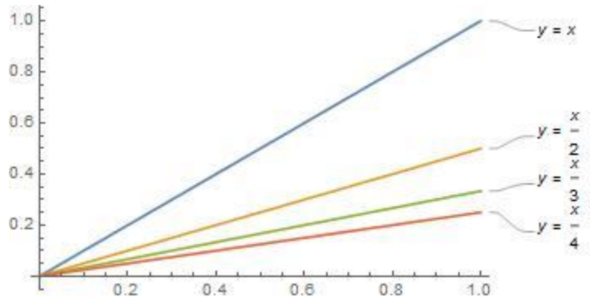

Figure 2.1. *Topological space – the deleted infinite broom*

Example 2.3. A topological space $Z=\left\{\left(x,\sin\frac{1}{x}\right)\middle| 0\leq x\leq 1\right\}\cup\{(0,0)\}$, $Z\subseteq R^2$ is connected space. This space is called topological sine curve and is pictured below.

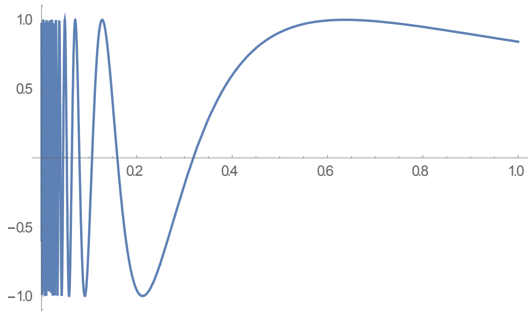

Figure 2.2. *Topological sine curve*

Let X be a topological space.

Definition 2.4. ([3]) A continuous function $f:[0,1]\to X$ is called a *path* in X. The path f is said to be *connected* or *join* the point $f(0)$ to the point $f(1)$. $f(0)$ is called the *initial point* and $f(1)$ is called the *terminal point* of the path f.

Definition 2.5. ([1]) *Path* in a topological space X is every continuous map $k:I\to X,\ I=[0,1]$. If $k(0)=x,\ k(1)=y$ then $k:I\to X,\ I=[0,1]$ is a path from point x to point y.

Definition 2.6. ([1]) A topological space X is called *line connected* if for every two points $x,y\in X$, then there exist a path in X from x to y.

Let (X,d) be a metric space.

Definition 2.7. ([6], Definition. 5.2.1) A *path* in X is a continuous map $\gamma:[0,1]\to X$ in X. If $\gamma(0)=x$ and $\gamma(1)=y$, then γ is said to be a path joining the points x to y or simply a path from x to y. (Figure 3)

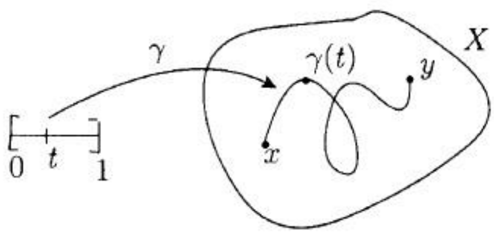

Figure 2.3. *Path between x and y*

Theorem 2.3. ([2]) Let X be a path connected topological space, then X is connected.

Proof.([2], 164)

Example 2.4. Any interval in R is path connected. This is true because for every two real numbers $a,b\in R$, the continuous map $f:[0,1]\to R$, is a path in R defined by $f(t)=a+(b-a)t$, for $t\in[0,1]$ and we have a connection $f(0)=a$ and $f(1)=b$.

Example 2.5. If X and Y are two topology space which is path connected, then the direct product $X\times Y$ is also path connected.

Example 2.6. Path connected spaces are the spheres, $S^n, n>0$.

Theorem 2.4. ([4], pg. 25-39) Let R^n be a Euclidean space. Let X be a connected open subset of $\mathbb{R}^n$. Then X is path connected.

Proof. ([4]) Let $a \in X$. Let $A \subseteq X$ be the subset of points in X which can be joined to a by a path in X. Let $B = X \setminus A$. Let $x \in A$. Then, $\exists \varepsilon > 0 : B_\varepsilon(x) \subseteq X$, where $B_\varepsilon(x)$ is the open ε - ball of x. Given any $y \in B_\varepsilon(x)$, there is a (straight line) path g in $B_\varepsilon(x) \subseteq X$ connecting x to y. But since $x \in A$, there is a path f in X joining a to x. From the Theorem for joining paths makes another path ([7]), traversing f and then g forms a path from a to y. It follows that $y \in A$ and therefore $B_\varepsilon(x) \subseteq A$. Thus, A is open.

By a similar argument, B is also shown to be open: If $x \in B$, then $B_\varepsilon(x) \subseteq X$ for some $\varepsilon > 0$. If any point in $B_\varepsilon(x)$ can be joined to a by a path in X, then so could x.

It is clear that $A \cap B \neq \varnothing$ and $A \cup B = X$ by definition of set difference. As, trivially $a \in A$, we have $A \neq \varnothing$. Knowing that X is connected, if follows that $B \neq \varnothing$ and $A = X$.

Theorem 2.5. ([6]) Let $(x,y),(u,v) \in R^2$. Then the continuous function $f:[0,1] \to R^2$, defined by

$$f(t) = (u,v)t + (1-t)(x,y), \quad t \in [0,1]$$

is called a *path* in R^2.

3. Results and discussion

In a part 2, of this paper are given the general properties of connected and path connected topology spaces. The notion of connectedness and path connectedness are so strongly by each other and it's difficult to prove that in every space connectedness implies path connectedness. From general topology and topology on a metric space we saw that path connection is a tougher condition of connectedness and reversal does not always apply. In this part, we will see some examples that can show us when connected space can be and path connected.

In [7], by the theorem 12.10

Theorem 3.1. (Theorem 12.10, [7]) Any interval I in R is connected.

Proof. ([7]) Suppose that I is an interval in R and suppose for a contradiction that $\{A,B\}$ is a partition of I. Let $a \in A$, $b \in B$ and suppose without loss of generality that $a < b$. (Otherwise we may exchange the names of A and B). Since $a,b \in I$ and I is an interval, $[a,b] \subseteq I$.

Let $A' = A \cap [a,b]$ and $B' = B \cap [a,b]$. Since A and B are closed in I and $[a,b] \subseteq I$, we have that A' and B' are closed in $[a,b]$. Since also $[a,b]$ is closed in R, it follows that A' and B' are closed in R. Let $c = A'$. Then $c \in A'$ since A' is closed. Hence $c < b$ since $b \in B'$ and $A' \cap B' = \varnothing$. But A' is open in $[a,b]$ for some $\delta > 0$ we have $(c-\delta, c+\delta) \cap [a,b] \subseteq A'$. Since $c < b$ there exist points in $(c, c+\delta) \cap [a,b]$ greater then c and such points lie in A', contradicting the choice of c.

Example 3.1. Any interval in R is connected. Let we choose the interval $(0,1) \subseteq R$. By the previous theorem 12.10 ([7]), it is supposed that $(0,1) = A \bigcup B$ with A, B disjoint non – empty closed subsets. Choose $a \in A$ and $b \in B$ with $a < b$. Then let β be the least upper bound of the set $C = \{[a,b] \bigcap A\}$. This least upper bound exist by the standard properties of R. Since C is a closed subset it contains its limit points and so $\beta \in C$ and hence is in A. Since A is open β has an ε - neighborhood lying inside A and so unless $\beta = b$ it would both be an upper bound of C. But $\beta = b$ contradicts the fact that $b \in B = (0,1) \setminus A$.

Also, this interval $(0,1) \subseteq R$ is path connected. This follows by the definition of path in R. This means, if $f : [0,1] \to R$ is continuous map, then we define a path by $f(t) = a + (b-a)t$, for $t \in [0,1]$ and a connection $f(0) = a$ and $f(1) = b$. From the chosen interval, let $0 = a$ and $1 = b$. By definition of path, we have,

$$\begin{aligned} f(0) &= a + (b-a)0 = 0 + (1-0)0 = 0 \\ f(1) &= a + (b-a)1 = 0 + (1-0)1 = 1 \end{aligned}$$

So, we defined path between the two points on an interval i.e. path from 0 to 1.

Euclidean space $R \times R$ is connected and path connected. Connection follows by [8], Theorem 7.3, where the author says that every line segment in this space is always connected like a straight line who belong in connected $R \times R$ space made from vertical and horizontal slices. Path connection in this space clear follows by the definition on path in Euclidean space $R \times R$ (par 2, in this paper Theorem 2.5, taken by [6]). In [8], Theorem 7.3 says:

Theorem 3.2.([8]) Every line segment in R^2 is connected.

Proof. ([8])

Example 3.2. An open connected subset of Euclidean space $R \times R$ is path connected. From previous part, from Theorem 7.3 in [8], we can say that:

Recall that R is connected and notice that it is homeomorphic to vertical and horizontal slices of the form $\{a\} \times R$ and $R \times \{b\}$ so that these slices are also connected. Now fix the base point $(0,0) \in R^2$ and for any $a \in R$, consider the family of cross – shaped spaces of the form $X_a = (\{a\} \times R) \bigcup (R \times \{0\})$. By taking the union of all such cross – shaped space over all $a \in R$, we obtain the entire plane. Hence, since R^2 is the union of a collection of connected spaces that have a base point $(0,0)$ in common, we conclude that R^2 must also be connected, as desired.

To see that this space is also path connected, we choose two points. Let the first point is $(1,2)$ and second point is $(4,5)$. This means $(1,2),(4,5) \in R^2$. Definition on path in Euclidean space R^2 says that for every points

$(x,y),(u,v)\in R^2$ there is exist a continuous map $f:I\to R^2$, $I=[0,1]$ which is a path defined by $f(t)=(u,v)t+(1-t)(x,y)$, $t\in[0,1]$. By our example, we have:

$$f(0)=(4,5)0+(1-0)(1,2)=(1,2)$$
$$f(1)=(4,5)1+(1-0)(1,2)=(4,5)$$

Since this, we defined path between two points in R^2, where path starts in $(1,2)$ and ended in $(4,5)$. So, this can be and a straight line who connects this chosen two points.

The space in we are worked in Examples 3.1, 3.2 are metric space and it is easy to show that concept of connectedness is in relation with connectedness.

Author Conrad K, in paper [5], with Theorem 3.7 says that:

Theorem 3.3. ([5]) The topologist's sine curve is connected but not path connected.

The prove of this theorem is given by author in [5], but for that we can define a new space represent in a next example 3.3.

Example 3.3. The topological sine curve represents by the space

$$S=\left\{(x,y)\in R^2 \middle| y=\sin\frac{1}{x}, x>0\right\}\cup\left\{(x,y)\in R^2 \middle| x=0, y\in[-1,1]\right\},\quad S\subseteq R^2$$

is connected but not path connected. (Figure 1)

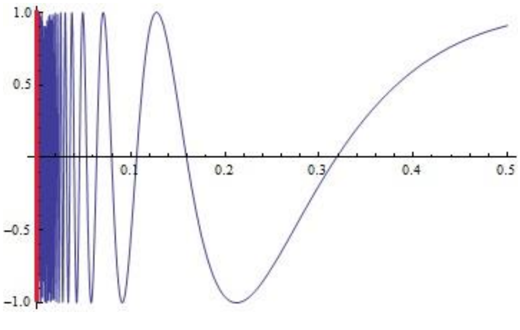

Figure 3.1. *Topological sine curve S*

This is a metric space by the definition in R^2. Let $S=A\cup B\subseteq R^2$. By [6] (ex. 5.2.23, pg. 119-121), the explanation on this example will go with:

Let $f:[0,1]\to S$ be a path joining $(0,0)$ to $(1,0)$. Then, we write $f(t)=(f_1(t),f_2(t))$. Since B is closed in S, the inverse image $f^{-1}(B)$ is closed, $0\in f^{-1}(B)$. Let t_0 be the least upper bound of this closed and bounded set. Obviously, $t_0 \in f^{-1}(B)$. Note that $0 < t_0 < 1$ and we claim that f_2 is not continuous at t_0. For any $\delta > 0$, with $t_0 + \delta \le 1$ we must have $f_1(t_0 + \delta) > 0$. Hence there exist $n \in N$ such that $f_1(t_0) < \frac{2}{4n+1} < f_1(t_0 + \delta)$. By the intermediate value theorem applied to the continuous function f_1, we can find t such that $t_0 < t < t_0 + \delta$ and such that $f_1(t) < \frac{2}{4n+1}$. Hence $f_2(t) = 1$ and $|f_2(t) - f_2(t_0)| \ge 1$. We therefore conclude that f_2 is not continuous at t_0.

Each of A and B are connected. Also, the point $(0,0)$ is a limit point of the set A and hence $A_1 = A \cup \{(0,0)\} \subseteq \overline{A}$ is connected. Since B and A_1 have a point in common their union S is connected.

In this space, represent in this example we cannot find a path between any two points from S.

If this space is represented like the follow figure (figure 3.2), then we can find a path between two points so, the space S will be a path connected. The space pictured below is called topological sine circle.

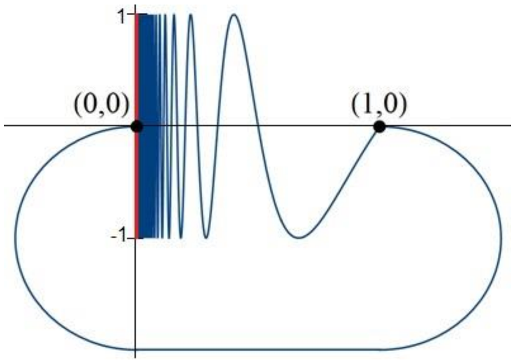

Figure 3.2. *Topologist's sine circle*

The topological sine circle is path connected. This is true because we have a continuous path joining $(0,0)$ and $(1,0)$ namely the arc added to the topologist's sine curve. Also, there is a continuous path joining $(0,0)$ with any point of the segment $\{0\} \times [-1,1]$ and we just have a straight line. Here, there is a continuous path joining $(1,0)$ with any point $\left(x, \sin\frac{1}{x}\right), x > 0$. We can simply take part of the graph of continuous function $x \mapsto \sin\frac{1}{x}$. Now, it is simple to combine the above to get a path joining any two points or we can argue that the above implies that any two points lie in the same path connected components.

More than all, the topologies sine circle, pictured in figure 3.2 is line connected. So, if it line connected this space must be a connected and path connected.

In all of this example, we saw that the notations of connectedness and path connectedness are in relation i.e. if some space is connected then we can find a path connection in that if the space is locally Euclidean[1]. In a metric space, the concept of connectedness it is difficult to estimate definite but understanding for it is to take just like a 'hole part'. Path connectedness is almost the same thing in metric space. So, depends of the dimension of the Euclidean space we can find relation between connectedness and path connectedness.

Example 3.4. If (X,d) be a connected metric space. Assume that each point of X has an open set U such that $x \in U$ and U is path connected. Than X is path connected.

Example 3.5. If A be a connected subset in R^n and $\varepsilon > 0$. Then it is clear that for ε - neighborhood of A defined by $U_\varepsilon(A) := \{x \in R^n \,|\, d_A(x) < \varepsilon\}$ is path connected.

4. Concluding remarks

In particular, spaces that are connected cannot be always path connected too. These notions are in the relation if the topology space has some properties. In this paper are represented some examples in which it can be easily seen that there are some connected topology spaces which can be path connected also. The conclusion from all of these is: if we are working with a metric space or we have a locally Euclidean space we can find a connected space in which also we can find a path between some pair of points. Metric spaces are interesting for work, because they are not so abstract and can be represented geometrically, so the concept of connectedness this space take like a 'hole part' and concept of path connectedness is easier to see there. Implication of connectedness to path connectedness can be shown if the space for working is metric (Euclidean). Examples 3.1, 3.2, 3.3, 3.4 and 3.5 are given in metric space and for that this implication can be true.

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[1] A topological space X is called locally Euclidean if there is a non negative integer n such that every point in X has a neighbourhood which is homeomorphic to the Euclidean space with specific dimension.