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CONTENT

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DYNAMICAL ANALYSIS OF ONE TWO-DIMENSIONAL MAP

BILJANA ZLATANOVSKA

Abstract. In this paper we will present a dynamical analysis of a two- dimensional map via an example. The fixed points, the classification of their character (stable or unstable), the visualization of some orbits and plotting of the bifurcation diagrams will be the main aspects of research for the two-dimensional map. As a computer support, mathematical software Mathematica will be used.

1. Introduction

The dynamical analysis of the dynamical systems appeared as a result of the development of the science for the needs of the technique, a description of the natural processes and laws, the economics etc. The dynamical analysis of the dynamical systems can be seen in a lot of scientific papers (as examples [1]-[7]).

The dynamical analysis of the maps and their application in physics, economics, biologic process, engineering and other areas as discrete dynamical systems described via difference equations are presented in the mathematical literature ([1], [2], [3], [5], [6], [7]).

The simplest maps for a dynamical analysis are one-dimensional maps. They can be seen in [1], [2], [3], [5], [6], [7].

Any dynamical analysis of the dynamical systems (discrete or continuous) is unimaginable without using a mathematical software. For this goal, we will use a mathematical software Mathematica which contains tools and techniques of algebraic, numerical and graphical nature (as in [1], [2], [3], [4], [5]).

The dynamical analysis of the two-dimensional maps includes finding fixed and periodical points, classification of their character (stable or unstable), visualization of orbits, calculation, and visualization of Lyapunov functions, plotting of the bifurcation diagrams, calculations of Lyapunov numbers and visualizations of Lyapunov specter. Such a dynamical analysis can be seen in numerous mathematical literature (as in [5], [6], [7]).

Date: December 6, 2018.

Keywords. Discrete dynamical system, dynamical analysis, two-dimensional map, fixed point, orbit, bifurcation diagram.

The simplest two-dimensional maps are the liner maps $F(x,y) = (ax+by, cx+dy)$ where a,b,c,d are real parameters and the appropriate two-dimensional linear system of difference equations is

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned}$$

where n=1,2,.... This system has one fixed point $(0,0)$ for $(a-1)(d-1) - bc \neq 0$. Its dynamics are simple.

A dynamical analysis of the two-dimensional map will be given in this paper

$$F(x,y) = ((a-x-by)x, (a-cx-y)y) \tag{1.1}$$

where a,b,c are real parameters. This two-dimensional map is an unsolved exercise in the paper [5] as exercise 2.17 on page 197 in which they look for a dynamical analysis for the system of difference equations

$$\begin{aligned} x_{n+1} &= (a-x_n-by_n)x_n \\ y_{n+1} &= (a-cx_n-y_n)y_n \end{aligned} \tag{1.2}$$

where n=1,2,.... In the recommendation of the authors from the paper [5], this a system (1.2) is given in the paper from 1989, [S. Eubank, W. Miner, T. Tajima, J. Wiley "Interactive computer simulation and analysis of Newtonian dynamics", Am. J. Phys. 57].

The dynamical analysis of the two-dimensional map (1.1) will be presented with a finding fixed points, classification of their character (stable or unstable), visualization of orbits and plotting of the bifurcation diagrams.

2. **Theoretical basis**

The two-dimensional maps $F = (f,g)$ reviewed as discrete dynamical systems are reviewed with corresponding systems of difference equations

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{aligned} \tag{2.1}$$

where n=1,2,... and f,g are given functions.

The following definitions and theorems are given in [5]:

Definition 2.1. a) *A fixed point of the map $F = (f,g)$ as an equilibrium solution of the system (2.1) is a point $(\overline{x}, \overline{y})$ that satisfies*

$$\overline{x} = f(\overline{x}, \overline{y})$$
$$\overline{y} = g(\overline{x}, \overline{y}) \qquad (2.2)$$

b) *Let (x_0, y_0) be a given element of R^2. The pairs $(x_1, y_1), (x_2, y_2), \ldots$ defined inductively by (2.1) are called the iterates of (x_0, y_0), and the sequence $(x_n, y_n)_{n=0}^{\infty}$ is called the positive orbit of (x_0, y_0) and is denoted by $\gamma^+((x_0, y_0))$; that is $\gamma^+((x_0, y_0)) = \{(x_0, y_0), F(x_0, y_0), \ldots, F^k(x_0, y_0), \ldots\}$.*

c) *If the map F is invertible, we define the negative orbit of (x_0, y_0) to be $\gamma^-((x_0, y_0)) = \{(x_0, y_0), F^{-1}(x_0, y_0), \ldots, F^{-k}(x_0, y_0), \ldots\}$ where F^{-n} denotes the n - th iterate of F^{-1}.*

d) *When both the positive and negative orbits eist, the complete orbit $\gamma((x_0, y_0))$ is the union of the positive and negative orbits $\gamma((x_0, y_0)) = \gamma^+((x_0, y_0)) \bigcup \gamma^-((x_0, y_0))$.*

Definition 2.2. **a)** *Let $(\overline{x}, \overline{y})$ be a fixed point of a map $F = (f, q)$, where f and g are continuously differentiable function at $(\overline{x}, \overline{y})$. The Jacobian matrix of F at $(\overline{x}, \overline{y})$ is the matrix*

$$J_F(\overline{x}, \overline{y}) = \begin{bmatrix} \frac{\partial f}{\partial x}(\overline{x}, \overline{y}) & \frac{\partial f}{\partial y}(\overline{x}, \overline{y}) \\ \frac{\partial g}{\partial x}(\overline{x}, \overline{y}) & \frac{\partial g}{\partial y}(\overline{x}, \overline{y}) \end{bmatrix} \qquad (2.3)$$

The linear map $J_F(\overline{x}, \overline{y}) : R^2 \to R^2$ given by

$$\begin{bmatrix} \frac{\partial f}{\partial x}(\overline{x}, \overline{y})x + \frac{\partial f}{\partial y}(\overline{x}, \overline{y})y \\ \frac{\partial g}{\partial x}(\overline{x}, \overline{y})x + \frac{\partial g}{\partial y}(\overline{x}, \overline{y})y \end{bmatrix}$$

is called the linearization of the map F at the fixed point $(\overline{x}, \overline{y})$.
The equation

$$det(J_F(\overline{x}, \overline{y}) - \lambda E) = 0$$

i.e

$$\lambda^2 - trJ_F(\overline{x}, \overline{y})\lambda + detJ_F(\overline{x}, \overline{y}) = 0 \qquad (2.4)$$

is called the characteristic equation at the fixed point $(\overline{x}, \overline{y})$, where

$$E = [e_{i,j}] = \begin{cases} 1 & if\ i = j, \\ 0 & if\ i \neq j \end{cases} \qquad (2.5)$$

is an identity matrix from second order.
The solution λ of the characteristic equation (2.4) is called an eigenvalue of the Jacobian matrix (2.3).

b) *A fixed point $(\overline{x}, \overline{y})$ of the map F is said to be hyperbolic if the linearization of F is hyperbolic, that is, if the Jacobian matrix $J_F(\overline{x}, \overline{y})$ at $(\overline{x}, \overline{y})$ has no eigenvalues* *on the unit circle. If $J_F(\overline{x},\overline{y})$ has at least one eigenvalue on the unit cicle, the it is a nonhyperbolic fixed point.*

The classification of fixed points is given in the following theorem:

Theorem 2.1. *(Linearized Stability theorem) Let $F = (f, g)$ be a continuously differentiable function defined on an open set W in R^2 and let $(\overline{x},\overline{y})$ in W be a fixed point of F.*

a) *If all the eigenvalues of the Jacobian matrix $J_F(\overline{x},\overline{y})$ have modulus less than one, then the equilibrium point $(\overline{x},\overline{y})$ is asymptotically stable;*

b) *If at least one of the eigenvalues of the Jacobian matrix $J_F(\overline{x},\overline{y})$ has a modulus greater than one, then the equilibrium point $(\overline{x},\overline{y})$ is unstable.*

By analysis of the characteristic equation (2.4) of the Jacobian matrix (2.3) it can obtain explicit conditions for the local behaviour of a fixed point. For a two-dimensional map (1.1), an analysis will follow the general method Schur-Cohn criterion which is described in [5]:

Theorem 2.2. a) *An equilibrium point $(\overline{x},\overline{y})$ of (2.1) is locally asymptotically stable if and only if every solution of the characteristic equation (2.4) lies inside the unit circle that is if and only if $|trJ_F(\overline{x},\overline{y})| < 1 + detJ_F(\overline{x},\overline{y})) < 2$.*

b) *An equilibrium point $(\overline{x},\overline{y})$ of (2.1) is locally a repeller if and only if every solution of the characteristic equation (2.4) lies outside the unit circle, that is, if and only if $|trJ_F(\overline{x},\overline{y})| < |1 + detJ_F(\overline{x},\overline{y}))|$ and $|detJ_F(\overline{x},\overline{y})| > 1$.*

c) *An equilibrium point $(\overline{x},\overline{y})$ of (2.1) is locally a saddle point if and only if the characteristic equation (2.4) has one root that lies inside the unit circle and one root lies outside the unit circle, that is, if and only if $|trJ_F(\overline{x},\overline{y})| > |1+detJ_F(\overline{x},\overline{y}))|$ and $(trJ_F(\overline{x},\overline{y}))^2 - 4detJ_F(\overline{x},\overline{y})) > 0$.*

d) *An equilibrium point $(\overline{x},\overline{y})$ of (2.1) is non-hyperbolic if and only if the characteristic equation (2.4) has at least one root that lies on the unit circle, that is, if and only if $|trJ_F(\overline{x},\overline{y})| = |1 + detJ_F(\overline{x},\overline{y})|$ or $detJ_F(\overline{x},\overline{y}) = 1$ and $trJ_F(\overline{x},\overline{y}) \leq 2$.*

Monitoring changes in a map depending on the parameters resulting by a change of the parameter gives the dynamics of that map viewed as a discrete dynamical system. These qualitative changes are analyzed by drawing bifurcation diagrams for which the dynamical system is reviewed as a function depending on a parameter.

3. **Dynamical analysis of the two-dimensional map (1.1)**

The finding fixed points, classification of their character (stable or unstable), visualization of some orbits and plotting of the bifurcation diagrams for the two-dimensional map (1.1) will be shown in this part.

3.1. **Fixed points.** The fixed points of the two-dimensional map (1.1) are given with the following theorem:

Theorem 3.1. *The two-dimensional map (1.1) has four fixed points* $F_0(0,0)$, $F_1(0,a-1)$, $F_2(a-1,0)$ *and* $F_3(\frac{ab-a-b+1}{bc-1}, \frac{ac-a-c+1}{bc-1})$ $(bc-1 \neq 0)$.

Proof. By using (2.2) from Definition 2.1 the following system of algebraic equations is obtained:

$$(a - \overline{x} - b\overline{y})\overline{x} = \overline{x}$$
$$(a - c\overline{x} - \overline{y})\overline{y} = \overline{y}$$

By solving this system of algebraic equations (using of the mathematical software Mathematica) four fixed points are obtained:

i) the point $F_0(0,0)$ is a trivial fixed point;

ii) the points $F_1(0,a-1)$ and $F_2(a-1,0)$ are fixed points which exist $lla \in R$;

iii) the point $F_3(\frac{ab-a-b+1}{bc-1}, \frac{ac-a-c+1}{bc-1})$ is a fixed point which exists for $bc-1 \neq 0$. □

For the classification of fixed points for the two-dimensional map (1.1) Jacobian matrix is used (Theorem 2.1. and Theorem 2.2.). The Jacobian matrix (2.3) at the point $(\overline{x}, \overline{y})$ of the two-dimensional map (1.1) is

$$J_F(\overline{x}, \overline{y}) = \begin{bmatrix} a - 2\overline{x} - b\overline{y} & -b\overline{x} \\ -c\overline{y} & a - c\overline{x} - 2\overline{y} \end{bmatrix} \tag{3.1}$$

For the classification of fixed points, the map (1.1), $F = (f, g)$ must satisfy conditions of Definition 2.2 and Theorem 2.1. From
$f(x,y) = (a - x - by)x, g(x,y) = (a - cx - y)y$, we conclude that the map (1.1) satisfies conditions of Definition 2.2 and Theorem 2.1.

The fixed point $F_0(0,0)$: For the fixed point $F_0(0,0)$, we will give the following theorem,

Theorem 3.2. *For the two-dimensional map (1.1), the appropriate the characteristic equation of the Jacobian matrix (3.1) at the fixed point* $F_0(0,0)$ *is*

$$\lambda^2 - 2a\lambda + a^2 = 0$$

with eigenvalues $\lambda_1 = \lambda_2 = a$.

Proof. The Jacobian matrix (3.1) at the fixed point $F_0(0,0)$ has a form

$$J_F(F_0) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

From

$$det(J_F(F_0) - \lambda E) = \begin{vmatrix} a-\lambda & 0 \\ 0 & a-\lambda \end{vmatrix} = 0$$

the characteristic equation is obtained

$$\lambda^2 - 2a\lambda + a^2 = 0$$

where E is the identity matrix (2.5).
The eigenvalues of the Jacobian matrix $J_F(F_0)$ are $\lambda_1 = \lambda_2 = a$. □

For the classification of the fixed point $F_0(0,0)$ for the map (1.1) (by using Theorem 2.2. and the mathematical software Mathematica), we have the following conclusions:

i) When $-1 < a < 1$, the point $F_0(0,0)$ is an asymptotically stable point (locally asymptotically stable);

ii) When $a > 1$ or $a < -1$, the point $F_0(0,0)$ is a repeller (unstable point);

iii) When $a = 1$ or $a = -1$, the point $F_0(0,0)$ is a non-hyperbolic point.

We conclude that the behaviour of the map (1.1) in a neighbourhood of the fixed point $F_0(0,0)$ does not depend on the values of the parameters b and c. The point $F_0(0,0)$ cannot be the saddle point for any real value of parameters a, b, c .

The fixed point $F_1(0, a-1)$: For the fixed point $F_1(0, a-1)$, we will give the following theorem,

Theorem 3.3. *For the two-dimensional map (1.1), the appropriate characteristic equation of the Jacobian matrix (3.1) at the fixed point* $F_1(0, a-1)$ *is*

$$\lambda^2 - (2 + b - ab)\lambda + (2a - a^2 + 2b - 3ab + a^2 b) = 0$$

with eigenvalues $\lambda_1 = 2 - a, \lambda_2 = a + b - ab$.

Proof. The Jacobian matrix (3.1) at the fixed point $F_1(0, a-1)$ is

$$J_F(F_1) = \begin{bmatrix} a - b(a-1) & 0 \\ -b(a-1) & 2-a \end{bmatrix}$$

From

$$det(J_F(F_1) - \lambda E) = \begin{vmatrix} a - b(a-1) - \lambda & 0 \\ -b(a-1) & 2-a-\lambda \end{vmatrix} = 0$$

the characteristic equation is obtained

$$\lambda^2 - (2 + b - ab)\lambda + (2a - a^2 + 2b - 3ab + a^2 b) = 0$$

where E is the identity matrix (2.5).
The eigenvalues of the Jacobian matrix $J_F(F_1)$ are $\lambda_1 = 2 - a$, $\lambda_2 = a + b - ab$. □

For the classification of the fixed point $F_1(0, a-1)$ for the map (1.1) (by using Theorem 2.2. and the mathematical software Mathematica), we have the following conclusions:

i) When $1 < a < 3$ and $1 < b < \frac{a+1}{a-1}$, the point $F_1(0, a-1)$ is an asymptotically stable point (locally asymptotically stable);

ii) When $(a < 1$ and $(b > 1 \vee b < \frac{a+1}{a-1}))$ or $(a > 3$ and $(b < 1 \vee b > \frac{a+1}{a-1}))$, the point $F_1(0, a-1)$ is a repeller (unstable point);

iii) When $(a < 1$ and $\frac{a+1}{a-1} < b < 1)$ or $(1 < a < 3$ and $(b < 1 \vee b > \frac{a+1}{a-1}))$ or $(a > 3$ and $1 < b < \frac{a+1}{a-1})$ the point $F_1(0, a-1)$ is a saddle point;

iv) When $(a < 1$ and $(b = \frac{a+1}{a-1} \vee b = 1)$ or $(a = 1)$ or $(1 < a < 3$ and $(b = 1 \vee b = \frac{a+1}{a-1})$) or $(a = 3)$ or $(a > 3$ and $(b = \frac{a+1}{a-1} \vee b = 1))$, the point $F_1(0, a-1)$ is a non-hyperbolic point.

The behaviour of the map (1.1) in the neighbourhood of the fixed point $F_1(0, a-1)$ does not depend on the parameter c .

The fixed point $F_2(a-1, 0)$: For the fixed point $F_2(a-1, 0)$, we will give the following theorem,

Theorem 3.4. *For the two-dimensional map (1.1), the appropriate characteristic equation of the Jacobian matrix (3.1) at the fixed point $F_2(a-1, 0)$ is*

$$\lambda^2 - (2 + c - ac)\lambda + (2a - a^2 + 2c - 3ac + a^2c) = 0$$

with eigenvalues $\lambda_1 = 2 - a, \lambda_2 = a + c - ac.$

Proof. The Jacobian matrix (3.1) at the fixed point $F_2(a-1, 0)$ is

$$J_F(F_2) = \begin{bmatrix} 2-a & -b(a-1) \\ 0 & a - c(a-1) \end{bmatrix}$$

From

$$det(J_F(F_2) - \lambda E) = \begin{vmatrix} 2-a-\lambda & -b(a-1) \\ 0 & a - c(a-1) - \lambda \end{vmatrix} = 0$$

the characteristic equation is obtained

$$\lambda^2 - (2 + c - ac)\lambda + (2a - a^2 + 2c - 3ac + a^2c) = 0$$

where E is the identity matrix (2.5).
The eigenvalues of the Jacobian matrix $J_F(F_2)$ are $\lambda_1 = 2-a$, $\lambda_2 = a+c-ac$. $\square$

For the classification of the fixed point $F_2(a-1, 0)$ for the map (1.1) (by using Theorem 2.2. and the mathematical software Mathematica), we have the following conclusions:

i) When $1 < a < 3$ and $1 < c < \frac{a+1}{a-1}$, the point $F_2(a-1, 0)$ is an asymptotically stable point (locally asymptotically stable);
ii) When $(a < 1$ and $(c > 1 \vee c < \frac{a+1}{a-1})$) or $(a > 3$ and $(c < 1 \vee c > \frac{a+1}{a-1}))$, the point $F_2(a-1, 0)$ is a repeller (unstable point);
iii) When $(a < 1$ and $\frac{a+1}{a-1} < c < 1)$ or $(1 < a < 3$ and $(c < 1 \vee c > \frac{a+1}{a-1})$) or $(a > 3$ and $1 < c < \frac{a+1}{a-1})$, the point $F_2(a-1, 0)$ is a saddle point;
iv) When $(a < 1$ and $(c = \frac{a+1}{a-1} \vee c = 1))$ or $(a = 1)$ or $(1 < a < 3$ and $(c = 1 \vee c = \frac{a+1}{a-1})$) or $(a = 3)$ or $(a > 3$ and $(c = \frac{a+1}{a-1} \vee c = 1))$, the point $F_2(a-1, 0)$ is a non-hyperbolic point.

The behaviour of the map (1.1) in the neighbourhood of the fixed point $F_2(a-1, 0)$ does not depend on the parameter b .

The fixed point $F_3(\frac{ab-a-b+1}{bc-1}, \frac{ac-a-c+1}{bc-1})$: For the fixed point $F_3(\frac{ab-a-b+1}{bc-1}, \frac{ac-a-c+1}{bc-1})$, we will give the following theorem,

Theorem 3.5. *For the two-dimensional map (1.1), the appropriate characteristic equation of the Jacobian matrix (3.1) at the fixed point* $F_3(\frac{ab-a-b+1}{bc-1}, \frac{ac-a-c+1}{bc-1})$ *is*

$$\lambda^2 - \frac{2bc - ac + c - ab + b + 2a - 4}{bc-1}\lambda + \frac{(2-a)(abc - ac - ab + a + b + c - 2)}{bc-1} = 0$$

with eigenvalues $\lambda_1 = 2 - a$, $\lambda_2 = \frac{a+b+c-ab-ac+abc-2}{bc-1}$.

Proof. The Jacobian matrix (3.1) at the fixed point F_3 is

$$J_F(F_3) = \begin{bmatrix} \frac{a+b-ab+bc-2}{bc-1} & \frac{ab+b^2-ab^2-b}{bc-1} \\ \frac{ac+c^2-ac^2-c}{bc-1} & \frac{a+c-ac+bc-2}{bc-1} \end{bmatrix}$$

From

$$det(J_F(F_3) - \lambda E) = \begin{vmatrix} \frac{a+b-ab+bc-2}{bc-1} - \lambda & \frac{ab+b^2-ab^2-b}{bc-1} \\ \frac{ac+c^2-ac^2-c}{bc-1} & \frac{a+c-ac+bc-2}{bc-1} - \lambda \end{vmatrix} = 0$$

by using Mathematica (because of the complexity of the equation $det(J_F(F_3) - \lambda E) = 0$) the characteristic equation is obtained

$$\lambda^2 - \frac{2bc - ac + c - ab + b + 2a - 4}{bc-1}\lambda + \frac{(2-a)(abc - ac - ab + a + b + c - 2)}{bc-1} = 0$$

where E is the identity matrix (2.5).
The eigenvalues of the Jacobian matrix $J_F(F_3)$ are $\lambda_1 = 2-a$, $\lambda_2 = \frac{a+b+c-ab-ac+abc-2}{bc-1}$.
□

For the classification of the fixed point $F_3(\frac{ab-a-b+1}{bc-1}, \frac{ac-a-c+1}{bc-1})$ for the map (1.1) (by using Theorem 2.2. and the mathematical software Mathematica), we have the following conclusions:

i) The point $F_3(\frac{ab-a-b+1}{bc-1}, \frac{ac-a-c+1}{bc-1})$ is an asymptotically stable point (locally asymptotically stable) when

$$1 < a < 3 \quad and \quad [(b < \frac{a-1}{a+1} \quad and \quad \frac{ab-a-b+3}{ab-a+b+1} < c < 1)\vee$$

$$(b = \frac{a-1}{a+1} \quad and \quad c < 1) \vee (\frac{a-1}{a+1} < b < 1 \quad and \quad (c < 1 \vee c > \frac{ab-a-b+3}{ab-a+b+1}))\vee$$

$$(b > 1 \quad and \quad \frac{ab-a-b+3}{ab-a+b+1} < c < 1)];$$

ii) The point $F_3(\frac{ab-a-b+1}{bc-1}, \frac{ac-a-c+1}{bc-1})$ is a repeller (unstable point) when

$$(1) \quad a < -1 \quad and \quad [(b < 0 \quad and \quad (c < \frac{1}{b} \vee \frac{1}{b} < c < 1 \vee c > \frac{ab-a-b+3}{ab-a+b+1}))\vee$$

$$(b = 0 \quad and \quad (c < 1 \vee c > \frac{a-3}{a-1})) \vee (0 < b < 1 \quad and \quad (c < 1 \vee c > \frac{1}{b}$$

$$\vee \frac{ab-a-b+3}{ab-a+b+1} < c < \frac{1}{b})) \vee (1 < b < \frac{a-1}{a+1} \quad and \quad (\frac{ab-a-b+3}{ab-a+b+1} < c < \frac{1}{b}\vee$$

$$\frac{1}{b} < c < 1)) \vee (b = \frac{a-1}{a+1} \quad and \quad (c < \frac{1}{b} \vee \frac{1}{b} < c < 1)) \vee (b > \frac{a-1}{a+1} \quad and$$

$$(c < \frac{1}{b} \vee \frac{1}{b} < c < 1 \vee c > \frac{ab-a-b+3}{ab-a+b+1}))] \quad or$$

$$(2) \quad a = -1 \quad and \quad [(b < 0 \quad and \quad (c < \frac{1}{b} \vee \frac{1}{b} < c < 1 \vee c > \frac{1}{2}(4-2b)))\vee$$

$$(b = 0 \quad and \quad (c < 1 \vee c > 2))\vee(0 < b < 1 \quad and \quad (c < 1\vee\frac{1}{2}(4-2b) < c < \frac{1}{b}\vee c > \frac{1}{b}))$$

$$\vee(b > 1 \quad and \quad (\frac{1}{2}(4-2b) < c < \frac{1}{b} \vee \frac{1}{b} < c < 1))] \quad or$$

$$(3) \quad -1 < a < 1 \quad and \quad [(b < \frac{a-1}{a+1} \quad and \quad (\frac{ab-a-b+3}{ab-a+b+1} < c < \frac{1}{b}\vee\frac{1}{b} < c < 1))$$

$$\vee(b = \frac{a-1}{a+1} \quad and \quad (c < \frac{1}{b}\vee\frac{1}{b} < c < 1))\vee(\frac{a-1}{a+1} < b < 0 \quad and \quad (c < \frac{1}{b}\vee\frac{1}{b} < c < 1$$

$$\vee c > \frac{ab-a-b+3}{ab-a+b+1})) \vee (b = 0 \quad and \quad (c < 1 \vee c > \frac{a-3}{a-1})) \vee (0 < b < 1 \quad and$$

$$(c > 1\vee\frac{ab-a-b+3}{ab-a+b+1} < c < \frac{1}{b}\vee c > \frac{1}{b}))\vee(b > 1 \quad and \quad (\frac{ab-a-b+3}{ab-a+b+1} < c < \frac{1}{b}\vee$$

$$\frac{1}{b}<c<1))] \quad or$$

$$(4) \quad a>3 \quad and \quad [(b<0 \quad and \quad (c<\frac{1}{b}\vee\frac{1}{b}<c<\frac{ab-a-b+3}{ab-a+b+1}\vee c>1))$$
$$\vee(b=0 \quad and \quad (c<\frac{a-3}{a-1}\vee c>1))\vee(0<b<\frac{a-1}{a+1} \quad and \quad (c<\frac{ab-a-b+3}{ab-a+b+1}$$
$$\vee 1<c<\frac{1}{b}\vee c>\frac{1}{b}))\vee(b=\frac{a-1}{a+1} \quad and \quad (1<c<\frac{1}{b}\vee c>\frac{1}{b}))\vee(\frac{a-1}{a+1}<b<1 \quad and$$
$$(1<c<\frac{1}{b}\vee\frac{1}{b}<c<\frac{ab-a-b+3}{ab-a+b+1}))\vee(b>1 \quad and \quad (c<\frac{1}{b}\vee\frac{1}{b}<c<\frac{ab-a-b+3}{ab-a+b+1}$$
$$\vee c>1))];$$

iii) The point $F_3(\frac{ab-a-b+1}{bc-1},\frac{ac-a-c+1}{bc-1})$ is a saddle point when

$$(1) \quad a<-1 \quad and \quad [(b<1 \quad and \quad (1<c<\frac{ab-a-b+3}{ab-a+b+1}))\vee(1<b<\frac{a-1}{a+1}$$
$$and \quad (c<\frac{ab-a-b+3}{ab-a+b+1}\vee c>1))\vee(b=\frac{a-1}{a+1} \quad and \quad c>1)\vee(b>\frac{a-1}{a+1}$$
$$and \quad 1<c<\frac{ab-a-b+3}{ab-a+b+1})] \quad or$$

$$(2) \quad a=-1 \quad and \quad [(b<1 \quad and \quad (1<c<\frac{1}{2}(4-2b)))\vee(b>1 \quad and$$
$$(c<\frac{1}{2}(4-2b)\vee c>1))] \quad or$$

$$(3) \quad -1<a<1 \quad and \quad [(b<\frac{a-1}{a+1} \quad and \quad (c<\frac{ab-a-b+3}{ab-a+b+1}\vee c>1))$$
$$\vee(b=\frac{a-1}{a+1} \quad and \quad c>1)\vee(\frac{a-1}{a+1}<b<1 \quad and \quad 1<c<\frac{ab-a-b+3}{ab-a+b+1})$$
$$\vee(b>1 \quad and \quad (c<\frac{ab-a-b+3}{ab-a+b+1}\vee c>1))]$$

$$(4) \quad 1<a<3 \quad and \quad [(b<0 \quad and \quad (c<\frac{1}{b}\vee\frac{1}{b}<c<\frac{ab-a-b+3}{ab-a+b+1}\vee c>1))$$
$$\vee(b=0 \quad and \quad (c<\frac{a-3}{a-1}\vee c>1))\vee(0<b<\frac{a-1}{a+1} \quad and$$
$$(c<\frac{ab-a-b+3}{ab-a+b+1}\vee 1<c<\frac{1}{b}\vee c>\frac{1}{b}))\vee(b=\frac{a-1}{a+1} \quad and \quad (1<c<\frac{1}{b}\vee c>\frac{1}{b}))\vee$$
$$(\frac{a-1}{a+1}<b<1 \quad and \quad (1<c<\frac{1}{b}\vee\frac{1}{b}<c<\frac{ab-a-b+3}{ab-a+b+1}))\vee(b>1 \quad and$$

$$c<\frac{1}{b}\vee\frac{1}{b}<c<\frac{ab-a-b+3}{ab-a+b+1}\vee c>1))] \quad or$$

$$(5)\quad a>3 \quad and \quad [(b<\frac{a-1}{a+1} \quad and \quad \frac{ab-a-b+3}{ab-a+b+1}<c<1)\vee(b=\frac{a-1}{a+1}$$
$$and \quad c<1)\vee(\frac{a-1}{a+1}<b<1 \quad and \quad (c>\frac{ab-a-b+3}{ab-a+b+1}\vee c<1))\vee(b>1 \quad and$$
$$\frac{ab-a-b+3}{ab-a+b+1}<c<1)];$$

iv) The point $F_3(\frac{ab-a-b+1}{bc-1},\frac{ac-a-c+1}{bc-1})$ is a non-hyperbolic point when

$$(1)\quad a<-1 \quad and \quad [(b<1 \quad and \quad (c=\frac{ab-a-b+3}{ab-a+b+1}\vee c=1))\vee(b=1 \quad and$$
$$(c<1\vee c>1))\vee(1<b<\frac{a-1}{a+1} \quad and \quad (c=\frac{ab-a-b+3}{ab-a+b+1}\vee c=1))\vee(b=\frac{a-1}{a+1}$$
$$and \quad c=1)\vee(b>\frac{a-1}{a+1} \quad and \quad (c=1\vee c=\frac{ab-a-b+3}{ab-a+b+1}))] \quad or$$

$$(2)\quad a=-1 \quad and \quad [(b<1 \quad and \quad (c=\frac{1}{2}(4-2b)\vee c=1))\vee(b=1 \quad and$$
$$(c<1\vee c>1)\vee(b>1 \quad and \quad (c=\frac{1}{2}(4-2b)\vee c=1))] \quad or$$

$$(3)\quad -1<a<1 \quad and \quad [(b<\frac{a-1}{a+1} \quad and \quad (c=\frac{ab-a-b+3}{ab-a+b+1}\vee c=1))\vee$$
$$(b=\frac{a-1}{a+1} \quad and \quad c=1)\vee(\frac{a-1}{a+1}<b<1 \quad and \quad (c=\frac{ab-a-b+3}{ab-a+b+1}\vee c=1))$$
$$\vee(b=1 \quad and \quad (c<1\vee c>1))\vee(b>1 \quad and \quad (c=\frac{ab-a-b+3}{ab-a+b+1}\vee c=1))] \quad or$$

$$(4)\quad a=1 \quad and \quad [(b<0 \quad and \quad (c<\frac{1}{b}\vee c>\frac{1}{b}))\vee(b=0)\vee(b>0 \quad and$$
$$(c<\frac{1}{b}\vee c>\frac{1}{b}))] \quad or$$

$$(5)\quad 1<a<3 \quad and \quad [(b<\frac{a-1}{a+1} \quad and \quad (c=\frac{ab-a-b+3}{ab-a+b+1}\vee c=1))\vee(b=\frac{a-1}{a+1}$$
$$and \quad c=1)\vee(\frac{a-1}{a+1}<b<1 \quad and \quad (c=\frac{ab-a-b+3}{ab-a+b+1}\vee c=1))\vee(b=1$$
$$and \quad (c<1\vee c>1))\vee(b>1 \quad and \quad (c=\frac{ab-a-b+3}{ab-a+b+1}\vee c=1))] \quad or$$

$$(6)\quad a=3 \quad and \quad [(b<0 \quad and \quad (c<\frac{1}{b}\vee c>\frac{1}{b}))\vee(b=0)\vee(b>0 \quad and \quad (c<\frac{1}{b}\vee c>\frac{1}{b}))] \quad or$$

$$(7)\quad a>3 \quad and \quad [(b<\frac{a-1}{a+1} \quad and \quad (c=\frac{ab-a-b+3}{ab-a+b+1}\vee c=1))\vee(b=\frac{a-1}{a+1} \quad and \quad c=1)\vee(\frac{a-1}{a+1}<b<1 \quad and \quad (c=\frac{ab-a-b+3}{ab-a+b+1}\vee c=1))\vee(b=1 \quad and \quad (c<1\vee c>1))\vee(b>1 \quad and \quad (c=1\vee c=\frac{ab-a-b+3}{ab-a+b+1}))].$$

For the classification of the fixed points for the map (1.1), we conclude that the most complex one is the point F_3 vs. the other fixed points F_0, F_1, F_2.

3.2. **Visualization of orbits for the map (1.1).** In this part a visualization of two orbits for the map (1.1) will be given.

Let $a=-1.5, b=2.22431, c=2.5$, the positive orbit of $(x_0, y_0)=(0.1,0)$ (Definition 2.1) is

$$\gamma^{+}((0.1,0))=\{(0.1,0),(-0.16,0),(0.2144,0),(-0.367,0),(0.416,0),(-0.7976,0),(0.56,0),(-1.15,0),(0.399,0),(-0.757,0),(0.56,0)...\} \qquad (3.2)$$

In fig.1 the graphical presentation of the orbit (3.2) in the x-y plane is shown:

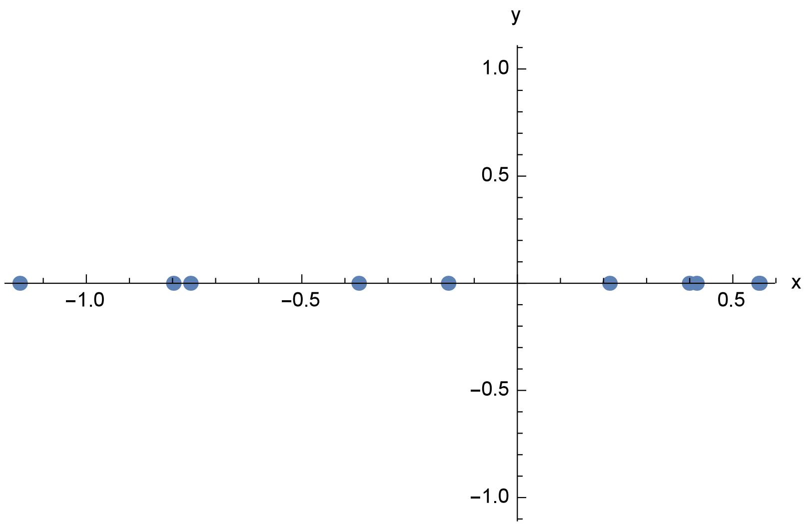

FIGURE 1. *Graphical visualization of orbit (3.2) for the map (1.1)*

For the parameters $a = -1.5, b = 2.22431, c = 2.5$,

1. the point $F_0(0, 0)$ has eigenvalues $\lambda_1 = \lambda_2 = a = -1.5$ and $|\lambda_1| = |\lambda_2| > 1$;

2. the point $F_1(0, -2.5)$ has eigenvalues $\lambda_1 = 2 - a = 3.5, \lambda_2 = a + b - ab \approx 4.1$ and $|\lambda_1| > 1, |\lambda_2| > 1$;

3. the point $F_2(-2.5, 0)$ has eigenvalues $\lambda_1 = 2 - a = 3.5, \lambda_2 = a + c - ac = 4.75$ and $|\lambda_1| > 1, |\lambda_2| > 1$;

i.e. F_0, F_1, F_2 are repellers (unstable points).

4. But the point $F_3(-0.6711, -0.8222)$ has eigenvalues $\lambda_1 = 2 - a = 3.5, \lambda_2 = \frac{a+b+c-ab-ac+abc-2}{bc-1} \approx -0.00666$ and $|\lambda_1| > 1, |\lambda_2| < 1$ i.e. F_3 is a saddle point.

By a small change of the parameter $b = 2.32622$, for the same values of the parameters $a = -1.5, c = 2.5$, the positive orbit of $(x_0, y_0) = (0.1, 10^{-30})$ is

$$\begin{aligned}
\gamma^+((0.1, 10^{-30})) = \{&(0.1, 10^{-30}), (-0.16, -1.749 \cdot 10^{-30}),\\
&(0.2144, 1.9249 \cdot 10^{-30}), (-0.367, -3.919 \cdot 10^{-30}), (0.416, 2.27 \cdot 10^{-30}),\\
&(-0.7976, -5.786 \cdot 10^{-30}), (0.56, -2.85 \cdot 10^{-30}), (-1.15, 8.29 \cdot 10^{-30}),\\
&(0.399, 1.15 \cdot 10^{-30}), (-0.757, -2.8 \cdot 10^{-30}), (0.56, -1.133 \cdot 10^{-30})...\}
\end{aligned} \tag{3.3}$$

In fig.2 the graphical presentation of the orbit (3.3) in the x-y plane is shown:

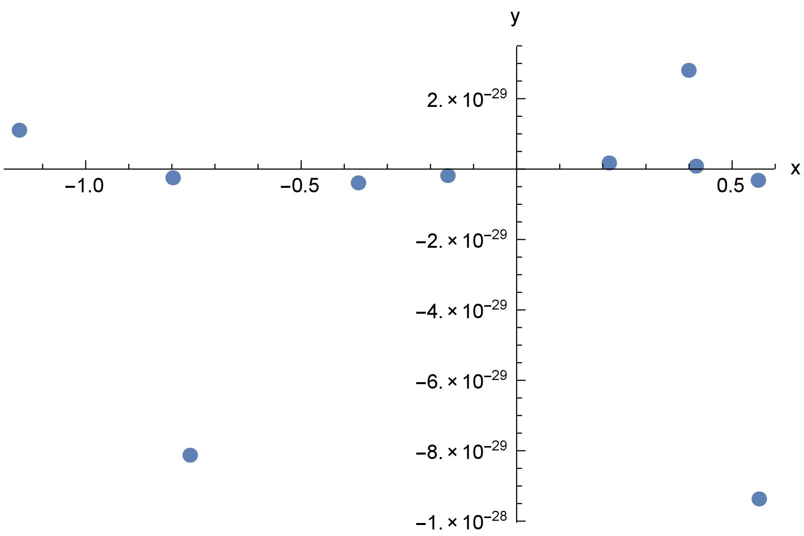

FIGURE 2. *Graphical visualization of orbit (3.3) for the map (1.1)*

For the parameters $a = -1.5, b = 2.32622, c = 2.5$,

1.the point $F_0(0,0)$ has eigenvalues $\lambda_1 = \lambda_2 = a = -1.5$ and $|\lambda_1| = |\lambda_2| > 1$;

2.the point $F_1(0,-2.5)$ has eigenvalues $\lambda_1 = 2 - a = 3.5, \lambda_2 = a + b - ab \approx 4.1$ and $|\lambda_1| > 1, |\lambda_2| > 1$;

3. the point $F_2(-2.5,0)$ has eigenvalues $\lambda_1 = 2 - a = 3.5, \lambda_2 = a + c - ac = 4.75$ and $|\lambda_1| > 1, |\lambda_2| > 1$;

i.e. F_0, F_1, F_2 are repellers (unstable points).

4. But the point $F_3(-0.6885,-0.7787)$ has eigenvalues $\lambda_1 = 2 - a = 3.5, \lambda_2 = \frac{a+b+c-ab-ac+abc-2}{bc-1} \approx -0.033$ and $|\lambda_1| > 1, |\lambda_2| < 1$ i.e. F_3 is a saddle point.

3.3. **Bifurcation diagrams of map (1.1).** From the experimental results which are made in the mathematical software Mathematica the parameter a appears as a bifurcation parameter.

In Figure 3 the bifurcation diagram of the map (1.1) is shown where the parameter a is changing in the interval $a \in [-2,-1]$ by step 0.005. The values of the other parameters are $b = 2.22431$ and $c = 2.5$. The initial values are $(0.1, 0)$.

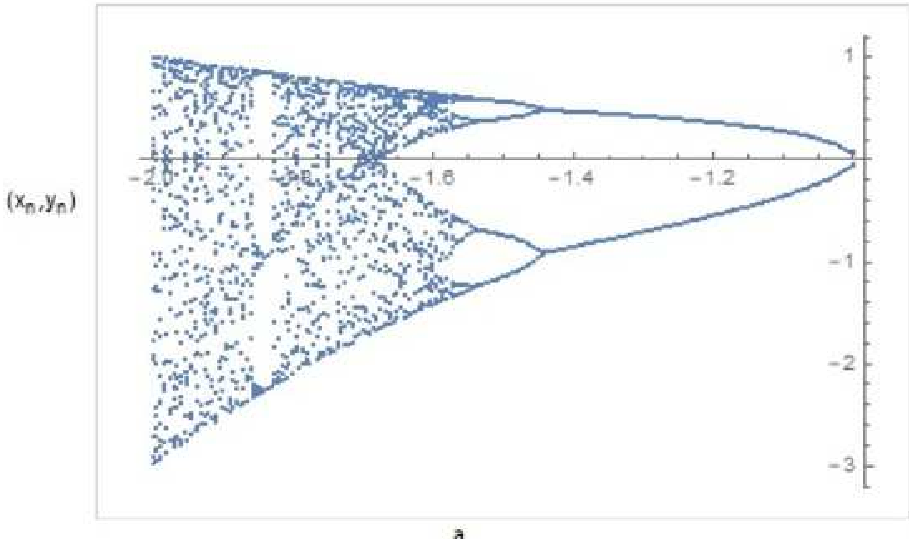

FIGURE 3. *Bifurcation diagram of the map (1.1) for* $a \in [-2,-1]$, *b=2.22431, c=2.5 and the initial values* $(0.1, 0)$

In Figure 4 the bifurcation diagram of the map (1.1) is shown where the parameter a is changing in the interval $a \in [-2,-1]$ by step 0.005. The values of the other parameters are $b = 2.32622$ and $c = 2.5$. The initial values are $(0.1, 10^{-30})$.

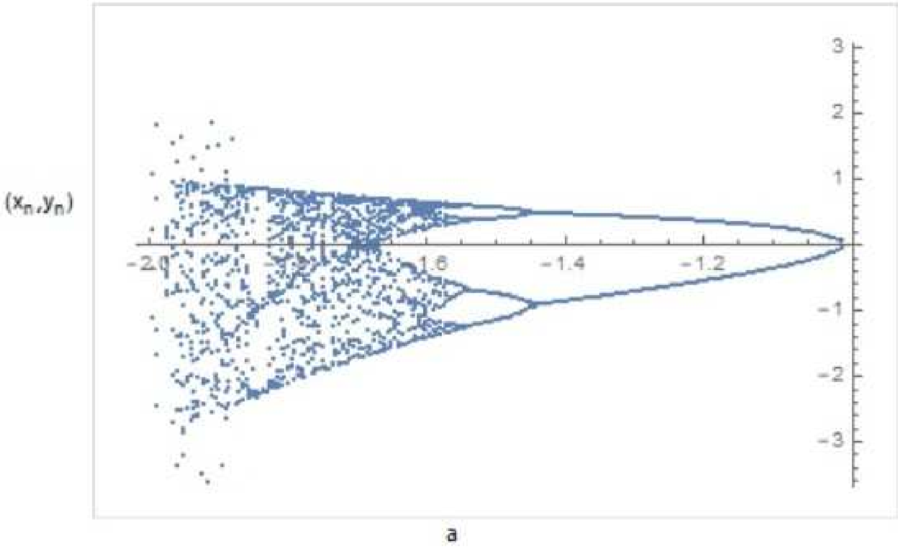

FIGURE 4. *Bifurcation diagram of the map (1.1) for* $a \in [-2, -1]$*, b=2.32622, c=2.5 and the initial values* $(0.1, 10^{-30})$

4. CONCLUSION

The dynamical analysis of the two-dimensional nonlinear maps is a complex process that requires a lot of research and a good computer. The two-dimensional map (1.1) is a nonlinear map which depends on three parameters a, b, c. Therefore, its dynamics is interesting for analysis. From this dynamical analysis of the map (1.1), we conclude that the two-dimensional map (1.1) has a complex structure. For a better picture of the behaviour of the map (1.1) we should make finding and classification of the character of periodical points, calculation and visualization of Lyapunov functions, calculations of Lyapunov numbers and visualizations of Lyapunov spectre, which is left for further research.

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