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CONTENT

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RESULT ON COLOMBEAU PRODUCT OF GENERALIZED FUNCTIONS CONTAINING DELTA DISTRIBUTION

MARIJA MITEVA AND LIMONKA KOCEVA LAZAROVA

Abstract. In this paper we consider the product of infinitely differentiable function and the $r-$th derivative of Dirak delta distribution. The product is obtained in Colombeau algebra of generalized functions, which is the most relevant algebraic construction for dealing with Schwartz distributions. Colombeau product of distributions generalizes classical products of distributions, but also allows us to obtain products of distributions that do not exist in the classical theory of distributions.

1. **Introduction**

The theory of distributions has its origins in the early 1950s and it was established as a result of the scientists' attempts to give mathematical meaning of many concepts in physics and engineering that were understood heuristically. Such concepts were Dirac delta function and its derivatives. As those concepts are very often used in science, the concept of distributions has large employment in mathematics and other natural sciences. The first sistematic theory of distributions was offered by the French mathematician Laurent Schwartz. But, although the scientists have found the theory of distributions a very useful branch of mathematics, they have realized that the theory of distributions comes across two main problems: multiplication of distributions (not any two distributions can always be multiplied) and differentiating the product of distributions (the product of distributions not always satisfies the Leibniz rule). The large application of distributions has imposed the need these two serious problems of this theory to be solved. Therefore, many attempts have been made by scientists to define the product of distributions [1, 2, 3], or rather to enlarge the number of existing products. Many attempts have also been made to include the distributions into differential algebras [4].

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Keywords. distribution, Colombeau algebra, Colombeau generalized function, Colombeau product of distributions.

According to the distribution theory [5, 6], we can distinguish two complementary points of view:

The first one is that distribution can be considered as a continuous linear functional f acting on a smooth function φ with compact support, i.e. we have a linear map $\varphi \to \langle f, \varphi \rangle$ where φ is called *test function*.

The second one is a sequential approach, which was introduced in order to enlarge the number of products of distributions that the first approach does not allow us to estimate: taking a sequence of smooth functions (φ_n) converging to the Dirac δ distribution, we obtain a family of regularization (f_n) by the convolution product

$$f_n(x) = (f * \varphi_n)(x) = \langle f(y), \varphi_n(x-y) \rangle \tag{1.1}$$

which converges weakly to the distribution f. We identify all the sequences that converge weakly to the same limit and consider them as an equivalence class. The elements of each equivalence class are called *representatives* of the appropriate distribution f. This way we obtain a sequential representation of distributions. Some authors use equivalence classes of nets of regularization, i.e. the δ - net $(\varphi_\varepsilon)_{\varepsilon>0}$ defined with $\varphi_\varepsilon = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$.

By the regularization process, the non-linear structure is lost in a way identifying sequences with their limit. Actually, all the operations then are done on the regularized functions (the sequences of smooth functions) and with the inverse process starting from the result, the function is returned from the regularization. So, we have to get a nonlinear theory of generalized functions that will work with regularization.

The sequential approach has partly solved the problem with multiplication of distributions, but the general solution was missing.

The optimal solution for overcoming the problems that Schwartz's theory of distributions is concerned with, was offered by J. F. Colombeau [7, 8]. He constructed a differential algebra of generalized functions $\mathcal{G}(\mathbf{R})$ which contains the space $\mathcal{D}'(\mathbf{R})$ of distributions as a subspace and the algebra of C^∞ - functions as subalgebra. This theory of generalized functions of Colombeau generalizes the theory of Schwartz distributions: these new Colombeau generalized functions can be differentiated in the same way as distributions, all products of distributions that exist in the clasical theory also exist in the Colombeau theory, but many products of distributions that are not defined in the classical theory, exist in the Colombeau theory. How Colombeau algebra $\mathcal{G}$ can be used for treating linear and nonlinear problems including singularities one can see in [9]. These new Colombeau generalized functions are very much related to the distributions in the sense that their definition may be considered as a natural evolution of Schwartz's definition of distribution.

The notion 'association' in $\mathcal{G}$ is a faithful generalization of the equality of distributions and enables us to interpret the results we obtain in the Colombeau algebra of generalized functions, in terms of distributions again.

Due to all these properties, Colombeau theory has reached large application in different natural sciences and engineering, especially in fields where the products of distributions with coinciding singularities are considered. About the applications of Colombeau theory of generalized functions one can read papers [10, 11, 12, 13, 14, 15].

In this paper we consider the product of infinitely differentiable function with the derivatives of Dirac delta distribution, as embedded in Colombeau algebra. The results obtained in this way are a generalization of the results existing in the classical theory of distributions. We will mention here that the product of infinitely differentiable function with distribution is defined in the classical theory of distributions, but, we obtain the result in terms of associated distribution. The result obtained in the last part of this paper is associated with the term consisting of only delta distribution and its derivatives. Other products of distributions, evaluated in the same way, can be found in [16, 17, 18, 19, 20, 21, 22, 23].

2. Colombeau Algebra

In this section we will give notations and definitions from Colombeau theory that we have used while evaluating the main results.

$\mathbf{N_0}$ is the set of non-negative integers, i.e. $\mathbf{N_0} = \mathbf{N} \cup \{0\}$.

Let $\mathcal{D}(\mathbf{R})$ be the space of all smooth functions $\varphi : \mathbf{R} \to \mathbf{C}$ with compact support.

For $q \in \mathbf{N_0}$ we denote

$$A_q(\mathbf{R}) = \left\{ \varphi(x) \in \mathcal{D}(\mathbf{R}) \;\middle|\; \int_{\mathbf{R}} \varphi(x)\,dx = 1 \ and \ \int_{\mathbf{R}} x^j \varphi(x)\,dx = 0, j = 1, ..., q \right\} \tag{2.1}$$

The elements of the set $A_q(\mathbf{R})$ are called *test functions.*

It is obvious that $A_1 \supset A_2 \supset A_3 ...$. In his books Colombeau has proved that the sets A_k are non empty for all $k \in \mathbf{N}$.

For $\varphi \in A_q(\mathbf{R})$ and $\varepsilon > 0$ it is denoted $\varphi_\varepsilon = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ and $\overset{\vee}{\varphi}(x) = \varphi(-x)$.

Wanting to obtain an algebra containing the space of distributions, which elements could be multiplied and diferentiated as well as C^∞ functions, Colombeau started with $\mathcal{E}(\mathbf{R})$, the algebra of functions $f(\varphi, x) : A_0(\mathbf{R}) \times \mathbf{R} \to \mathbf{C}$ that are infinitely differentiable with respect to the second variable, x. The embedding of distributions into such an algebra should be done in a way that the embedding of C^∞ functions will be identity. Let f and g be C^∞ functions. Taking the sequence

$(f * \varphi_\varepsilon)_{\varepsilon>0}$, which converges to f in $\mathcal{D}'$, as a representative of f, we obtain an embedding of $\mathcal{D}'$ into $\mathcal{E}(\mathbf{R})$. So, if we consider f and g as distributions, we look at the sequences $(f * \varphi_\varepsilon)_{\varepsilon>0}$ and $(g * \varphi_\varepsilon)_{\varepsilon>0}$. The product of f and g as distributions embedded into this algebra not always coincide with their classical product considered as a distribution embedded in it, i.e.

$$(f * \varphi_\varepsilon)(g * \varphi_\varepsilon) \neq (fg) * \varphi_\varepsilon \tag{2.2}$$

The idea therefore is to find an ideal $\mathcal{I}[\mathbf{R}]$ such that this diffrence will vanish in the resulting quotient. In order to determine $\mathcal{I}[\mathbf{R}]$ it is obviously enough to find an ideal containing the differences $((f * \varphi_\varepsilon) - f)_{\varepsilon>0}$.

Expanding the last term in a Taylor series and having in mind the properties of $\varphi(x)$ as an element of $A_q(\mathbf{R})$, we can see that it will vanish faster then any power of ε, uniformly on compact sets, in all derivatives. The set of these differences will not be an ideal in $\mathcal{E}(\mathbf{R})$ but in a set of sequences whose derivatives are bounded uniformly on compact sets by negative power of ε. These sequences are called 'moderate' sequences and the set containing them is denoted with $\mathcal{E}_M[\mathbf{R}]$. Let

$\mathcal{E}(\mathbf{R})$ be the algebra of functions $f(\varphi, x) : A_0(\mathbf{R}) \times \mathbf{R} \to \mathbf{C}$ that are infinitely differentiable for fixed 'parameter' φ. The generalized functions of Colombeau are elements of the quotient algebra

$$\mathcal{G} \equiv \mathcal{G}(\mathbf{R}) = \frac{\mathcal{E}_M[\mathbf{R}]}{\mathcal{I}[\mathbf{R}]} \tag{2.3}$$

where $\mathcal{E}_M[\mathbf{R}]$ is the subalgebra of 'moderate' functions such that for each compact subset K of $\mathbf{R}$ and any $p \in \mathbf{N_0}$ there is a $q \in \mathbf{N}$ such that, for each $\varphi \in A_q(\mathbf{R})$ there are $c > 0$, $\eta > 0$ and it holds:

$$\sup_{x \in K} |\partial^p f(\varphi_\varepsilon, x)| \leq c\varepsilon^{-q} \tag{2.4}$$

for $0 < \varepsilon < \eta$ and $\mathcal{I}[\mathbf{R}]$ is an ideal of $\mathcal{E}_M[\mathbf{R}]$ consisting of all functions $f(\varphi, x)$ such that for each compact subset K of $\mathbf{R}$ and any $p \in \mathbf{N_0}$ there is a $q \in \mathbf{N}$ such that for every $r \geq q$ and each $\varphi \in A_r(\mathbf{R})$ there are $c > 0$, $\eta > 0$ and it holds:

$$\sup_{x \in K} |\partial^p f(\varphi_\varepsilon, x)| \leq c\varepsilon^{r-q} \tag{2.5}$$

for $0 < \varepsilon < \eta$.

The distributions on $\mathbf{R}$ are embedded in the Colombeau algebra $\mathcal{G}(\mathbf{R})$ by the map:

$$i : \mathcal{D}'(\mathbf{R}) \to \mathcal{G}(\mathbf{R}) : u \to \widetilde{u} = \left\{ \widetilde{u}(\varphi, x) = \left(u * \overset{\vee}{\varphi} \right)(x) : \varphi \in A_q(\mathbf{R}) \right\} \tag{2.6}$$

where $*$ denotes the convolution product of two distributions and is given by:

$$(f * g)(x) = \int_{\mathbf{R}} f(y)\, g(x-y) dy. \tag{2.7}$$

We should notice that the sequential approach (regularization method) mentioned in the previous section is used here. Thus, an element $f \in \mathcal{G}$ (a generalized function of Colombeau) is actually an equivalence class $[f] = [f_\varepsilon + \mathcal{I}]$ of an element $f_\varepsilon \in \mathcal{E}_M$ which is called *representative* of f. Multiplication and differentiation of generalized functions are performed on arbitrary representatives of the respective generalized functions.

The meaning of the term 'association' in $\mathcal{G}(\mathbf{R})$ is given with the next two definitions.

Definition 2.1. *Generalized functions $f, g \in \mathcal{G}(\mathbf{R})$ are said to be associated, denoted $f \approx g$, if for each representatives $f(\varphi_\varepsilon, x)$ and $g(\varphi_\varepsilon, x)$ and arbitrary $\psi(x) \in \mathcal{D}(\mathbf{R})$ there is a $q \in \mathbf{N_0}$ such that for any $\varphi(x) \in A_q(\mathbf{R})$*

$$\lim_{\varepsilon \to 0_+} \int_{\mathbf{R}} |f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x)| \psi(x)\, dx = 0 \tag{2.8}$$

Definition 2.2. *A generalized function $f \in \mathcal{G}$ is said to admit some $u \in \mathcal{D}'(\mathbf{R})$ as 'associated distribution', denoted $f \approx u$, if for each representative $f(\varphi_\varepsilon, x)$ of f and any $\psi(x) \in \mathcal{D}(\mathbf{R})$ there is a $q \in \mathbf{N_0}$ such that for any $\varphi(x) \in A_q(\mathbf{R})$*

$$\lim_{\varepsilon \to 0_+} \int_{\mathbf{R}} f(\varphi_\varepsilon, x) \psi(x)\, dx = \langle u, \psi \rangle \tag{2.9}$$

The representatives chosen in the above two definitions do not affect the result. The distribution associated, if it exists, is unique and the association is a faithful generalization of the equality of distributions.

If we multiply two distributions embedded in $\mathcal{G}$, as a result we always obtain a generalized function of Colombeau. But, it may not always be associated to a third distribution, so if the product of two distributions embedded in Colombeau algebra $\mathcal{G}$ admits an associated distribution, we say that *Colombeau product* of those two distributions exists. If the regularized model product of two distributions exists, then their Colombeau product also exists and it is the same as the first one.

3. Results and discussion

Theorem 3.1. *Let f be $C^{\infty}-$function on an open interval containing 0 and $\delta^{(r)}(x)$ is the $r-th$ derivative of Dirac Delta distribution. The Colombeau product of the generalized functions $\widetilde{f}(x)$ and $\widetilde{\delta^{(r)}}(x)$ exists (admits associated distribution) and it holds:*

$$\widetilde{f}(x)\cdot\widetilde{\delta^{(r)}}(x)\approx\sum_{i=0}^{r}\binom{r}{i}f^{(i)}(0)(-1)^{i+1}\delta^{(r-i)}(x) \tag{3.1}$$

Proof. We should embed first the function $f(x)$ and the distribution $\delta^{(r)}(x)$ in Colombeau algebra to obtain their representatives, and then multiply them as Colombeau generalized functions.
We will use the Taylor expansion for the function $f(x)$:

$$f(x)=\sum_{i=0}^{r}\frac{f^{(i)}(0)}{i!}\cdot x^{i}+O(\varepsilon) \tag{3.2}$$

According to the embedding rule, if we embed the function $f(x)$ in Colombeau algebra, we will obtain:

$$\begin{aligned}
&\widetilde{f}(\varphi_{\varepsilon},x)=\left(f*\overset{\vee}{\varphi}_{\varepsilon}\right)(x)\\
&=\int_{\mathbf{R}}f(y)\overset{\vee}{\varphi}_{\varepsilon}(x-y)\,dy=\frac{1}{\varepsilon}\int_{R}f(y)\varphi\left(\frac{y-x}{\varepsilon}\right)dy\\
&=\frac{1}{\varepsilon}\int_{\mathbf{R}}\left(\sum_{i=0}^{r}\frac{f^{(i)}(0)}{i!}y^{i}\right)\varphi\left(\frac{y-x}{\varepsilon}\right)dy+O(\varepsilon)\\
&=\frac{1}{\varepsilon}\sum_{i=0}^{r}\int_{\mathbf{R}}\frac{f^{(i)}(0)}{i!}y^{i}\varphi\left(\frac{y-x}{\varepsilon}\right)dy+O(\varepsilon)\\
&=\frac{1}{\varepsilon}\sum_{i=0}^{r}\frac{f^{(i)}(0)}{i!}\int_{\mathbf{R}}y^{i}\varphi\left(\frac{y-x}{\varepsilon}\right)dy+O(\varepsilon)
\end{aligned} \tag{3.3}$$

We can suppose (without loss of generality) that supp$\varphi\subseteq[-l,l]$, so if $\frac{y-x}{\varepsilon}=-l$ then $y=x-\varepsilon l$ and if $\frac{y-x}{\varepsilon}=l$ then $y=x+\varepsilon l$ and we will have:

$$\widetilde{f}(\varphi_{\varepsilon},x)=\frac{1}{\varepsilon}\sum_{i=0}^{r}\frac{f^{(i)}(0)}{i!}\int_{x-\varepsilon l}^{x+\varepsilon l}y^{i}\varphi\left(\frac{y-x}{\varepsilon}\right)dy+O(\varepsilon) \tag{3.4}$$

Using substitution $\frac{y-x}{\varepsilon}=t$ we obtain the representatives of the function f in Colombeau algebra:

$$\widetilde{f}(\varphi_{\varepsilon},x)=\sum_{i=0}^{r}\frac{f^{(i)}(0)}{i!}\int_{-l}^{l}(x+\varepsilon t)^{i}\varphi(t)\,dt+O(\varepsilon) \tag{3.5}$$

In a similar way, using the embedding rule, we obtain the embedding of the distribution $\delta^{(r)}(x)$ in the Colombeau algebra:

$$\widetilde{\delta^{(r)}}(\varphi_\varepsilon, x) = \frac{(-1)^r}{\varepsilon^{r+1}}\varphi^{(r)}\left(-\frac{x}{\varepsilon}\right) \tag{3.6}$$

Let us now calculate the product of these two Colombeau generalized functions and check the existance of the Colombeau product of the considered distributions. For any $\psi(x) \in \mathcal{D}(\mathbf{R})$ we have:

$$\begin{aligned}&\left\langle \widetilde{f}(\varphi_\varepsilon, x)\cdot\widetilde{\delta^{(r)}}(\varphi_\varepsilon, x), \psi(x)\right\rangle = \\ &= \frac{(-1)^r}{\varepsilon^{r+1}}\sum_{i=0}^{r}\frac{f^{(i)}(0)}{i!}\int_{\mathbf{R}}\left(\int_{-l}^{l}(x+\varepsilon t)^i\varphi(t)\,dt\right)\varphi^{(r)}\left(-\tfrac{x}{\varepsilon}\right)\psi(x)\,dx + O(\varepsilon)\end{aligned} \tag{3.7}$$

We have supposed that supp$\varphi \in [-l, l]$, so if $-\frac{x}{\varepsilon} = -l$ then $x = \varepsilon l$ and if $-\frac{x}{\varepsilon} = l$ then $x = -\varepsilon l$, thus we have:

$$\begin{aligned}&\left\langle \widetilde{f}(\varphi_\varepsilon, x)\cdot\widetilde{\delta^{(r)}}(\varphi_\varepsilon, x), \psi(x)\right\rangle = \\ &= \frac{(-1)^{r+1}}{\varepsilon^{r+1}}\sum_{i=0}^{r}\frac{f^{(i)}(0)}{i!}\int_{-\varepsilon l}^{\varepsilon l}\left(\int_{-l}^{l}(x+\varepsilon t)^i\varphi(t)\,dt\right)\varphi^{(r)}\left(-\tfrac{x}{\varepsilon}\right)\psi(x)\,dx + O(\varepsilon)\end{aligned} \tag{3.8}$$

Now using substitution $u = -\frac{x}{\varepsilon}$, for the product of the above two Colombeau generalized functions we obtain:

$$\begin{aligned}&\left\langle \widetilde{f}(\varphi_\varepsilon, x)\cdot\widetilde{\delta^{(r)}}(\varphi_\varepsilon, x), \psi(x)\right\rangle = \\ &= \frac{(-1)^r}{\varepsilon^{r+1}}\sum_{i=0}^{r}\frac{f^{(i)}(0)}{i!}\int_{-l}^{l}\left(\int_{-l}^{l}(\varepsilon t-\varepsilon u)^i\varphi(t)\,dt\right)\varphi^{(r)}(u)\psi(-\varepsilon u)(-\varepsilon du) + O(\varepsilon) \\ &= \frac{(-1)^{r+1}}{\varepsilon^{r}}\sum_{i=0}^{r}\frac{f^{(i)}(0)}{i!}\int_{-l}^{l}\varphi^{(r)}(u)\psi(-\varepsilon u)\left(\int_{-l}^{l}(\varepsilon t-\varepsilon u)^i\varphi(t)\,dt\right)du + O(\varepsilon)\end{aligned} \tag{3.9}$$

Using Taylor's Theorem for the function ψ we have:

$$\psi(-\varepsilon u) = \sum_{j=0}^{r}\frac{\psi^{(j)}(0)}{j!}(-\varepsilon u)^j + \frac{\psi^{(r+1)}(-\varepsilon\eta u)}{(r+1)!}(-\varepsilon u)^{r+1} \tag{3.10}$$

for $0 < \eta < 1$. Substituting (3.10) in (3.9) we obtain:

$$\begin{aligned}
&\left\langle \widetilde{f}\left(\varphi_\varepsilon, x\right) \cdot \widetilde{\delta^{(r)}}\left(\varphi_\varepsilon, x\right), \psi\left(x\right)\right\rangle = \\
&= \frac{(-1)^{r+1}}{\varepsilon^r} \sum_{i=0}^{r} \frac{f^{(i)}(0)}{i!} \int_{-l}^{l} \varphi^{(r)}\left(u\right) \sum_{j=0}^{r} \frac{\psi^{(j)}(0)}{j!} (-\varepsilon u)^j \left(\int_{-l}^{l} \left(\varepsilon t - \varepsilon u\right)^i \varphi\left(t\right) dt \right) du + O\left(\varepsilon\right) \\
&= \frac{(-1)^{r+1}}{\varepsilon^r} \sum_{i=0}^{r} \frac{f^{(i)}(0)}{i!} \sum_{j=0}^{r} \frac{\psi^{(j)}(0)}{j!} (-\varepsilon)^j \int_{-l}^{l} \varphi^{(r)}\left(u\right) u^j \left(\int_{-l}^{l} \left(\varepsilon t - \varepsilon u\right)^i \varphi\left(t\right) dt \right) du + O\left(\varepsilon\right) \\
&= \sum_{i=0}^{r} \frac{f^{(i)}(0)}{i!} \sum_{j=0}^{r} \frac{(-1)^{r+j+1}\psi^{(j)}(0)}{\varepsilon^{r-j} j!} \int_{-l}^{l} \varphi\left(t\right) dt \int_{-l}^{l} \varphi^{(r)}\left(u\right) \left(\varepsilon t - \varepsilon u\right)^i u^j du + O\left(\varepsilon\right) \\
&= \sum_{i=0}^{r} \frac{f^{(i)}(0)}{i!} \sum_{j=0}^{r} \frac{(-1)^{r+j+1}\psi^{(j)}(0)}{\varepsilon^{r-j} j!} \cdot J_{i,j} + O\left(\varepsilon\right)
\end{aligned} \tag{3.11}$$

where

$$J_{i,j} = \int_{-l}^{l} \varphi\left(t\right) dt \int_{-l}^{l} \varphi^{(r)}\left(u\right) \left(\varepsilon t - \varepsilon u\right)^i u^j du \tag{3.12}$$

We will calculate now the last integral. Using a binomial expansion for the term $\left(\varepsilon t - \varepsilon u\right)^i$, we have:

$$\begin{aligned}
J_{i,j} &= \int_{-l}^{l} \varphi\left(t\right) dt \int_{-l}^{l} \left(\varepsilon t - \varepsilon u\right)^i u^j \varphi^{(r)}\left(u\right) du \\
&= \int_{-l}^{l} \varphi\left(t\right) dt \int_{-l}^{l} \left[\sum_{k=0}^{i} \binom{i}{k} (\varepsilon t)^k (-\varepsilon u)^{i-k} \right] u^j \varphi^{(r)}\left(u\right) du \\
&= \int_{-l}^{l} \varphi\left(t\right) dt \int_{-l}^{l} \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} \varepsilon^i t^k u^{i+j-k} \varphi^{(r)}\left(u\right) du \\
&= \varepsilon^i \int_{-l}^{l} t^k \varphi\left(t\right) dt \int_{-l}^{l} \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} u^{i+j-k} \varphi^{(r)}\left(u\right) du
\end{aligned} \tag{3.13}$$

Due to the properties of the function $\varphi \in \mathbf{A}_0$, the last integral will be non zero only for $k = 0$ and $i + j - k = r$, i.e for $i + j = r \Leftrightarrow j = r - i$. In this case we will obtain:

$$\begin{aligned}
J_{i,j} &= \varepsilon^i \int_{-l}^{l} \binom{i}{0} (-1)^i u^r \varphi^{(r)}\left(u\right) du \\
&= (-1)^i \varepsilon^i \int_{-l}^{l} u^r \varphi^{(r)}\left(u\right) du \\
&= (-1)^i \varepsilon^i \cdot (-1)^r r! \\
&= (-1)^{i+r} \varepsilon^i r!
\end{aligned} \tag{3.14}$$

Applying this result in (3.11), and having in mind that we have obtained it for $j = r - i$, about the product of Colombeau generalized functions we are considering, we have:

$$\begin{aligned}
&\left\langle \widetilde{f}\left(\varphi_{\varepsilon}, x\right) \cdot \widetilde{\delta^{(r)}}\left(\varphi_{\varepsilon}, x\right), \psi\left(x\right)\right\rangle = \\
&= \sum_{i=0}^{r} \frac{f^{(i)}(0)}{i!} \cdot \frac{(-1)^{r+j+1}\psi^{(j)}(0)}{\varepsilon^{r-j} j!} \cdot (-1)^{i+r} \varepsilon^{i} r! + O\left(\varepsilon\right) \\
&= \sum_{i=0}^{r} \frac{f^{(i)}(0)}{i!} \cdot \frac{(-1)^{2r-i+1}\psi^{(r-i)}(0)}{\varepsilon^{i}(r-i)!} \cdot (-1)^{i+r} \varepsilon^{i} r! + O\left(\varepsilon\right) \\
&= \sum_{i=0}^{r} \frac{r!}{i!(r-i)!} \cdot (-1)^{1-i} \psi^{(r-i)}\left(0\right) f^{(i)}\left(0\right) (-1)^{i+r} + O\left(\varepsilon\right) \\
&= \sum_{i=0}^{r} \binom{r}{i} f^{(i)}\left(0\right) (-1)^{r+1} \psi^{(r-i)}\left(0\right) + O\left(\varepsilon\right) \\
&= \sum_{i=0}^{r} \binom{r}{i} f^{(i)}\left(0\right) (-1)^{i+1} \left\langle \delta^{(r-i)}\left(x\right), \psi\left(x\right)\right\rangle + O\left(\varepsilon\right)
\end{aligned} \tag{3.15}$$

Finally, passing to the limit when $\varepsilon \to 0$, we obtain the relation:

$$\widetilde{f}\left(\varphi_{\varepsilon}, x\right) \cdot \widetilde{\delta^{(r)}}\left(\varphi_{\varepsilon}, x\right) \approx \sum_{i=0}^{r} \binom{r}{i} f^{(i)}\left(0\right) (-1)^{i+1} \delta^{(r-i)}\left(x\right) \tag{3.16}$$

which proves the theorem. □

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