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# CONTENT

## The Appendix

In honor of the first Doctor of Mathematical Sciences Acad. Blagoj Popov, a mathematician dedicated to differential equations, the idea of holding the "Day of Differential Equations" was born, prompted by Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski, and Prof. Ph.D. Lazo Dimov. Acad. Blagoj Popov presented his doctoral dissertation on 05.05.1952 in the field of differential equations. This is the main reason for holding the " Day of Differential Equations" at the beginning of May.

This year on May 10th, the "Day of Differential Equations" was held for the fifth time at the Faculty of Computer Sciences at "Goce Delcev" University in Stip under the auspices of Dean Prof. Ph.D. Cveta Martinovska - Bande, organized by Prof. Ph.D. Biljana Zlatanovska.

Acknowledgments to Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski and Prof. Ph.D. Lazo Dimov for the wonderful idea and the successful realization of the event this year and in previous years.

Acknowledgments to the Dean of the Faculty of Computer Sciences, Prof. Ph.D. Cveta Martinovska - Bande for her overall support of the organization and implementation of the "Day of Differential Equations".

The papers that emerged from the "Day of Differential Equations" are in the appendix to this issue of BJAMI.

UDC 517.912

# ON EXISTENCE AND CONSTRUCTION OF A POLYNOMIAL SOLUTION OF A CLASS OF MATRIX DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

**Boro M. Piperevski**

**Abstract.** A class of matrix differential equations is observed in this article. The conditions under which this class has a polynomial solution are obtained. The formula of that polynomial solution is also obtained, and some special cases of this type of equations related to orthogonal polynomial are observed.
Keywords: matrix differential equation, differential equation, orthogonal polynomial, polynomial solution.

## 1. Introduction

The relation between orthogonal polynomials and differential equations is well known.

Polynomials $\{\pi_n\}$ orthogonal on a semicircle with respect to the complex inner product

$$(f,g)=\int_0^{\pi} f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})d\theta,$$

have been introduced by Walter Gautschi, Henry J. Landau, and Gradimir V. Milovanovic [2].
In that paper, in the Gegenbauer case

$$w(z)=(1-z)^{\lambda-\frac{1}{2}}(1+z)^{\lambda-\frac{1}{2}},\ \lambda>-\frac{1}{2}, \tag{1.1}$$

a linear second-order differential equation for $\{\pi_n\}$ is obtained.

Polynomials $\{\pi_n^R\}$ orthogonal on a circular arc with respect to the complex inner product

$$(f,g)=\int_{\varphi}^{\pi-\varphi} f_1(\theta)g_1(\theta)w_1(\theta)d\theta,$$

where $\varphi\in\left(0,\frac{1}{2}\pi\right)$, and the function $f_1(\theta)$ in terms $f(z)$ is defined by

$$f_1(\theta)=f(-iR+e^{i\theta}\sqrt{R^2+1}),\ \ R=\tan\varphi,$$

have been introduced by de Bruin [1].

In the Jacobi case

$$w(z)=(1-z)^{\alpha}(1+z)^{\beta},\ \ \alpha,\beta>-1 \tag{1.2}$$

Gradimir V. Milovanovic and Predrag M. Rajkovic [3], obtained a linear second-order differential equation for $\{\pi_n^R\}$ .

In [4] a class of systems

$$ax_1' + bx_2' + Ax_1 = 0,$$
$$cx_1' + dx_2' + Bx_2 = 0,$$

or in matrix form

$$\boldsymbol{P}\boldsymbol{X}' + \boldsymbol{M}\boldsymbol{X} = \boldsymbol{0}, \tag{1.3}$$

where

$$\boldsymbol{P} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \boldsymbol{M} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \boldsymbol{X} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \boldsymbol{X}' = \begin{bmatrix} x'_1(t) \\ x'_2(t) \end{bmatrix},$$

$$a = a_1t + a_2, \qquad b = b_1t + b_2, \qquad c = c_1t + c_2, d = d_1t + d_2,$$
$$A, B, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in R, \quad x_1'(t) = \frac{dx_1}{dt}, x_2'(t) = \frac{dx_2}{dt},$$

is considered.

In that article, the necessary conditions

$$b' = 0, B + nd' = 0, k\,a' + A \neq 0, k < n, k \text{ a positive integer},$$

for a polynomial solution of degree $n$, $x_1(t) = P_{n-1}(t), x_2(t) = Q_n(t)$, or in matrix form

$$\boldsymbol{X_n} = \begin{bmatrix} P_{n-1}(t) \\ Q_n(t) \end{bmatrix},$$

where $P_{n-1}(t)$ is a polynomial of degree $n$-1and $Q_n(t)$, a polynomial of degree $n$ is obtained.

In this case, the second component of the matrix $\boldsymbol{X_n}$ is a polynomial solution of degree $n$ of the differential equation

$$(t^2 + Qt + R)(St + T)x_2'' + (\beta_2 t^2 + \beta_1 t + \beta_0)x_2' + (\gamma_1 t + \gamma_0)x_2 = 0, \tag{1.4}$$

and another polynomial solution of degree $k$, $k<n$ does not exist, if the conditions

$$\begin{aligned} &n^2 S + (\beta_2 - S)n + \gamma_1 = 0, \\ &S^2(\beta_0 + SR - QT) + T^2(S + \beta_2) - T\beta_1 S = 0, \\ &S^2(\beta_1\gamma_0 + \gamma_0^2 - \beta_0\gamma_1) + T(\gamma_1 + \beta_2)(T\gamma_1 - 2S\gamma_0) = 0, \end{aligned} \tag{1.5}$$

($n$ is the smaller one if both roots of the first condition are natural numbers), are satisfied.

Also the polynomial solution will be given by the formula

$$x_2(t) = Q_n(t) = e^{-F}\frac{d^{n-1}}{dt^{n-1}}[(t + K)(t^2 + Qt + R)^{n-1}e^F],$$

where

$$F = \int \frac{Mt + N}{t^2 + Qt + R}dt, \; M = \frac{\beta_2 - S}{S}, N = \frac{\beta_1 - T}{S} - \frac{T\beta_2}{S^2},$$
$$K = \frac{T\gamma_1 - n(S\beta_1 - 2T\beta_2 - T\gamma_1 + S\gamma_0)}{S\gamma_1}.$$

The complex polynomial orthogonal on the semicircle or on the circular arc [2,3] is the unique polynomial solution of the differential equation of type (1.4), satisfying the conditions (1.5) [5].

In particular [6], classes of complex polynomial $\pi_n^\lambda(z)$, in the case of the Gegenbauer weight function (1.1), are the unique polynomial solutions of the differential equation (1.4), where

$$Q=0, R=-1, S=2(2n+2\lambda-1)(n+\lambda-1)i\theta_{n-1}, T=-\frac{1}{(2n+2\lambda-1)^2}S,$$
$$\beta_2=2\lambda S, \beta_1=(2\lambda+1)T, \beta_0=S, \gamma_1=-n(n+2\lambda-1)S,$$
$$\gamma_0=(n+2\lambda-1)[n(2n+2\lambda-1)-(n-1)T],$$
$$\theta_k=\frac{k(k+2\lambda-1)}{4(k+\lambda)(k+\lambda-1)\theta_{k-1}}, k\in N,\ \theta_0=\frac{\Gamma(\lambda+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(\lambda+1)}.$$

Therefore, the system

$$\left(z-\frac{S}{n(2n+2\lambda-1)}\right)\frac{dG_{n-1}^\lambda(z)}{dz}-\frac{1}{n(n+2\lambda-1)}\frac{d\pi_{n-1}^\lambda(z)}{dz}+(n+2\lambda-1)G_{n-1}^\lambda(z)=0, \quad (1.6)$$
$$(Sz+T)\frac{dG_{n-1}^\lambda(z)}{dz}+\left(z-\frac{S}{(n+2\lambda-1)(2n+2\lambda-1)}\right)\frac{d\pi_{n-1}^\lambda(z)}{dz}-n\pi_n^\lambda(z)=0,$$

where $G_{n-1}^\lambda(z)$ is the Gegenbauer polynomial of degree $n$-1, solutions of the equation

$$(1-z^2)x_1''-(2\lambda+1)zx_1'+(n-1)(n-1+2\lambda)x_1=0,$$

are obtained.

Also, we obtain the formula

$$\pi_n^\lambda(z)=(n+2\lambda-1)^2(z^2-1)^{-\lambda+\frac{1}{2}}\frac{d^{n-1}}{dz^{n-1}}\left[\left(z-\frac{S}{(n+2\lambda-1)(2n+2\lambda-1)}\right)(z^2-1)^{n+\lambda-\frac{3}{2}}\right],$$

$$G_{n-1}^\lambda(z)=(z^2-1)^{-\lambda+\frac{1}{2}}\frac{d^{n-1}}{dz^{n-1}}\left[(z^2-1)^{n+\lambda-\frac{3}{2}}\right].$$

More generally [6], classes of complex polynomials, $\pi_n^R(z)$ in the case of the Jacobi weight function (1.2), are the unique polynomial solutions of the differential equation (1.4), where

$$Q=0, R=-1, S=(2n+\alpha+\beta)(2n+\alpha+\beta-1)i\theta_{n-1},$$
$$T=-\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)+(\beta^2-\alpha^2)S+S^2}{(2n+\alpha+\beta)^2},$$
$$\beta_2=(\alpha+\beta+1)S, \beta_1=-(\beta-\alpha)S+(\alpha+\beta+2)T, \beta_0=-(\beta-\alpha)T+S,$$
$$\gamma_1=-n(n+\alpha+\beta)S, \gamma_0=-n(n+\alpha+\beta+1)T+\frac{n(\beta-\alpha)S-S^2}{2n+\alpha+\beta},$$

$$\theta_{k-1} = \frac{1}{i}\frac{\rho_k(-iR)}{\rho_{k-1}(-iR)}, \; k \geq 1, \;\; \rho_k(z) = \int_{-1}^{1} \frac{P_k^{\alpha,\beta}(x)}{z-x}(1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

Here, $P_k^{\alpha,\beta}(z)$ is Jacobi polynomial of degree *k*.

.

Therefore, we obtain the system

$$\left(z + \frac{n(\beta-\alpha)-S}{n(2n+\alpha+\beta)}\right)\frac{dP_{n-1}^{\alpha,\beta}(z)}{dz} - \frac{1}{n(n+\alpha+\beta)}\frac{d\pi_n^R(z)}{dz} + (n+\alpha+\beta)P_{n-1}^{\alpha,\beta}(z) = 0, \quad (1.7)$$

$$(Sz+T)\frac{dP_{n-1}^{\alpha,\beta}(z)}{dz} + \left(z - \frac{(\beta-\alpha)(n+\alpha+\beta)+S}{(n+\alpha+\beta)(2n+\alpha+\beta)}\right)\frac{d\pi_n^R(z)}{dz} - n\pi_n^R(z) = 0,$$

where $P_{n-1}^{\alpha,\beta}(z)$ is the Jacobi polynomial of degree $n$-1, solutions of the equation

$$(z^2-1)x_1'' + [(\alpha+\beta+2)z - (\beta-\alpha)]x_1' - (n-1)(n+\alpha+\beta)x_1 = 0.$$

Also, we obtain the following formula

$$\pi_n^R(z) = \frac{n+\alpha+\beta}{2n+\alpha+\beta}(z-1)^{-\alpha}(1+z)^{-\beta}\frac{d^{n-1}}{dz^{n-1}}\{[(n+\alpha+\beta)(2n+\alpha+\beta)z -$$
$$-(\beta-\alpha)(n+\alpha+\beta) - S](z-1)^{n+\alpha-1}(z+1)^{n+\beta-1}\},$$

$$P_{n-1}^{\alpha,\beta}(z) = (z-1)^{-\alpha}(z+1)^{-\beta}\frac{d^{n-1}}{dz^{n-1}}[(z-1)^{n+\alpha-1}(z+1)^{n+\beta-1}].$$

**Remark 1.1:**
Of the corresponding systems (1.6) and (1.7) respectively, appropriate relations between the complex orthogonal polynomials of Jacobi or Gegenbauer can be obtained. It can also be shown that the corresponding differential equations whose solutions are complex orthogonal polynomials, do not have a polynomial as a second particular solution because the required condition (the degree of the second polynomial solution being the root of the characteristic equation) is not satisfied. Namely, the second root of the characteristic equation is $-(n+2\lambda-1) < 0 \; (\notin N)$ in the first case, and it is $-(n+\alpha+\beta) < 0 \; (\notin N)$ in the second case.

## 2. Main Result

In [6] we consider a class of matrix differential equations (1.3).

**Definition 2.1**: We say that (1.3) has a polynomial solution of degree *n* if polynomials $P_n(t), Q_n(t)$ of degree *n* exist such that

$$\boldsymbol{X}_n = \begin{bmatrix} P_n(t) \\ Q_n(t) \end{bmatrix}$$

is the solution (1.3).

**Lemma 2.1**: Using the substitution

$$\boldsymbol{X} = \boldsymbol{T}\cdot\boldsymbol{Y}, \quad \boldsymbol{T} = \begin{bmatrix} f(t) & 0 \\ 0 & f(t) \end{bmatrix}, f(t) = e^{-\int \frac{aB+Ad}{ad-bc}dt},$$

in (1.3), we get $\boldsymbol{P_1 Y' + M_1 Y = 0}$, where

$$\boldsymbol{P}_1 = \begin{bmatrix} -Ad & bB \\ Ac & -aB \end{bmatrix}, \boldsymbol{M}_1 = \begin{bmatrix} AB & 0 \\ 0 & AB \end{bmatrix}.$$

**Theorem 2.1**: The equation (1.3) with condition $a \cdot b \cdot c \cdot d \cdot (ad - bc) \cdot A \cdot B \neq 0$ has a polynomial solution of degree *n*, and another polynomial solution of degree *k*, *k*<*n*, does not exist, if only there exists a positive integer n such that the conditions

$$r(\boldsymbol{M} + n\boldsymbol{P}') = 1, r(\boldsymbol{M} + k\boldsymbol{P}') = 2\,, k < n, k \text{ a positive integer,}$$
$$b' \neq 0, c' \neq 0, A + na' \neq 0, B + nd' \neq 0, \tag{2.1}$$

are satisfied ($r\ the\ rank\ of\ a\ matrix$).

The polynomial solution will be given by the formula

$$\boldsymbol{X}_n = \boldsymbol{T} \cdot [\boldsymbol{T}_1 \cdot \boldsymbol{U}_1]^{(n-1)}, \tag{2.2}$$

where

$$\boldsymbol{T} = \begin{bmatrix} f(t) & 0 \\ 0 & f(t) \end{bmatrix}, \boldsymbol{T}_1 = \begin{bmatrix} B(kb+a) & 0 \\ 0 & A(kd+c) \end{bmatrix}, \boldsymbol{U}_1 = \frac{(ad-bc)^{n-1}}{f(t)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$f(t) = e^{-\int \frac{aB+Ad}{ad-bc} dt}, \; k = -\frac{nc'}{B+nd'} = -\frac{A+na'}{nb'}.$$

**Proof**: Suppose that the conditions of the theorem are satisfied and consider the algebraic matrix equation (homogeneous linear algebraic system) $(\boldsymbol{M} + n\boldsymbol{P}') \cdot \boldsymbol{W} = \boldsymbol{0}$. Using the conditions, we write the solution of this equation in the form

$$\boldsymbol{W} = \boldsymbol{K} \neq \boldsymbol{0}, \; \boldsymbol{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, k_1 \neq 0, k_2 \neq 0.$$

If $\boldsymbol{K} = \boldsymbol{X}_n^{(n)}$, $\boldsymbol{X}_n^{(n+1)} = \boldsymbol{0}$, then $\boldsymbol{P} \cdot \boldsymbol{X}_n^{(n+1)} + (\boldsymbol{M} + n\boldsymbol{P}') \cdot \boldsymbol{X}_n^{(n)} = \boldsymbol{0}$, and $\boldsymbol{X} = \boldsymbol{X}_n$ is a polynomial solution of (1.3).

Suppose that equation (1.3) has a polynomial solution $\boldsymbol{X}_n$ of degree *n* and another polynomial solution of degree *k*, *k*<*n*, does not exist. That

$$\boldsymbol{X}_n^{(n)} = \boldsymbol{K} \neq \boldsymbol{0}, \; \boldsymbol{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, k_1 \neq 0, k_2 \neq 0, \boldsymbol{X}_n^{(n+1)} = \boldsymbol{0}.$$

Differentiating equation (1.3) *n* times, and $\boldsymbol{X} = \boldsymbol{X}_n$ we obtain the algebraic matrix equation (homogeneous linear algebraic system) $(\boldsymbol{M} + n\boldsymbol{P}')\boldsymbol{K} = \boldsymbol{0}$ where $\boldsymbol{X}_n^{(n)} = \boldsymbol{K} \neq \boldsymbol{0}$. Since

$$r(\boldsymbol{M} + n\boldsymbol{P}') = 1, r(\boldsymbol{M} + k\boldsymbol{P}') = 2\,, k < n, k \text{ a positive integer,}$$
$$b' \neq 0, c' \neq 0, A + na' \neq 0, B + nd' \neq 0,$$

Now, let us consider the matrix differential equation

$$\boldsymbol{P}^* \cdot \boldsymbol{Z}' + \boldsymbol{M}^* \cdot \boldsymbol{Z} = \boldsymbol{0}, \tag{2.3}$$

where

$$\boldsymbol{Z}=\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}\ \boldsymbol{P}^*=\begin{bmatrix} -Ad & bB \\ Ac & -aB \end{bmatrix},\boldsymbol{M}^*=\begin{bmatrix} AB+(n-1)d'A & -(n-1)b'B \\ -(n-1)c'A & AB+(n-1)a'B \end{bmatrix}.$$

With the substitution $\boldsymbol{Z}=\boldsymbol{T_1}\cdot\boldsymbol{U}$ (Lema 2.1),

$$\boldsymbol{U}=\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},\boldsymbol{T}_1=\begin{bmatrix} B(kb+a) & 0 \\ 0 & A(kd+c) \end{bmatrix},\ k=-\frac{nc'}{B+nd'}=-\frac{A+na'}{nb'},$$

this equation is transformed into the equation $\boldsymbol{P_1^*}\cdot\boldsymbol{U'}-\boldsymbol{M_1^*}\cdot\boldsymbol{U}=\boldsymbol{0}$, where

$$\boldsymbol{P}_1^*=\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix},\ \boldsymbol{M}_1^*=\begin{bmatrix} aB+Ad+(n-1)(ad-bc)' & 0 \\ 0 & aB+Ad+(n-1)(ad-bc)' \end{bmatrix}.$$

A particular solution is

$$\boldsymbol{U}_1=\frac{(ad-bc)^{n-1}}{f(t)}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

On the other hand, if we differentiate (2.3) $n$-1 times and use the substitution $\boldsymbol{Z^{(n-1)}}=\boldsymbol{T^{-1}}\cdot\boldsymbol{X}$**,** where

$$\boldsymbol{T}=\begin{bmatrix} f(t) & 0 \\ 0 & f(t) \end{bmatrix},f(t)=e^{-\int\frac{aB+Ad}{ad-bc}dt},$$

then we easily obtain the same equation (1.3). Similarly, applying the substitutions, we get the particular polynomial solution by formula (2.2).

**Remark 2.1.** We consider a special case when $b'=0$, or $c'=0$, or $A+na'=0$, or $B+nd'=0$. That definition 1 is not valid.

Now, consider the matrix differential equation (1.3) and the appropriate system

$$ax_1'+bx_2'+Ax_1=0, \qquad (2.4)$$
$$cx_1'+dx_2'+Bx_2=0,$$

where

$$a=a_1t+a_2,\qquad b=b_1t+b_2,\qquad c=c_1t+c_2, d=d_1t+d_2,$$
$$A,B,a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2\in R,\quad x_1'(t)=\frac{dx_1}{dt},x_2'(t)=\frac{dx_2}{dt},$$

By method of differentiation and transformation the last system (2.4) can be transformed into the following differential equations of a second order:

$$b(ad-bc)x_1''+[ba'd+bad'-b^2c'+Adb-b'ad+abB]x_1'+$$
$$+A(d'b-b'd+bB)x_1=0, \qquad (2.5)$$

$$c(ad-bc)x_2''+[ad'c-b'c^2+a'dc-adc'+Acd+acB]x_2'+$$

$$+B(Ac-c'a+a'c)x_2=0\,. \tag{2.6}$$

**Theorem 2.2**: The equation (2.5) with condition $a\cdot b\cdot c\cdot d\cdot(ad-bc)\cdot A\cdot B\neq 0$ has a polynomial solution of degree *n*, and another polynomial solution of degree *k*, *k*<*n*, does not exist, if only there exists a positive integer *n* such that the conditions

$$(a'd'-b'c')n^2+(Ad'+Ba')n+AB=0,$$
$$b'\neq 0, c'\neq 0, A+na'\neq 0, B+nd'\neq 0, \tag{2.7}$$

(*n* is the smaller one if both roots of the first condition are natural numbers) are satisfied.

The polynomial solution will be given by the formula

$$x_1(t)=P_n(t)=f(t)\frac{d^{n-1}}{dz^{n-1}}\left[B(kb+a)\frac{(ad-bc)^{n-1}}{f(t)}\right]$$
$$f(t)=e^{-\int\frac{aB+Ad}{ad-bc}dt}\,,\ k=-\frac{nc'}{B+nd'}=-\frac{A+na'}{nb'}\,.$$

Proof. With the application of Theorem 2.1. the conditions (2.7) are equivalent to conditions (2.1).

**Theorem 2.3**: The equation (2.6) with condition $a\cdot b\cdot c\cdot d\cdot(ad-bc)\cdot A\cdot B\neq 0$ has a polynomial solution of degree *n*, and another polynomial solution of degree *k*, *k*<*n*, does not exist, if only there exists a positive integer *n* such that the conditions

$$(a'd'-b'c')n^2+(Ad'+Ba')n+AB=0,$$
$$b'\neq 0, c'\neq 0, A+na'\neq 0, B+nd'\neq 0, \tag{2.7}$$

(*n* is the smaller one if both roots of the first condition are natural numbers) are satisfied.

The polynomial solution will be given by the formula

$$x_2(t)=Q_n(t)=f(t)\frac{d^{n-1}}{dz^{n-1}}\left[A(kd+c)\frac{(ad-bc)^{n-1}}{f(t)}\right]$$
$$f(t)=e^{-\int\frac{aB+Ad}{ad-bc}dt}\,,\ k=-\frac{nc'}{B+nd'}=-\frac{A+na'}{nb'}\,.$$

Proof. With the application of Theorem 2.1. the conditions (2.7) are equivalent to conditions (2.1).

**References**

[1] Piperevski Boro M.: Sur une formule de solution polynomme d' une classe d' equations differentielles lineares du duxieme ordre, Bulletin Mathematique de la SDM de SRM, tome 7-8, 10-15, 1983/84.
[2] Piperevski Boro: On complex polynomials orthogonal on a circular arc (in Macedonian); Proceedings of the 7MSDR Ohrid,Republic of Macedonia, 21-26, 2002, http://www.cim.feit.ukim.edu.mk
[3] Walter Gautschi, Henry J. Landau, and Gradimir V. Milovanovic: Polynomials Orthogonal on the Semicircle, II; Constructive Approximation 3: 389-404, (1987).
[4] De Bruin M.G.: Polynomials orthogonal on a circular arc; Journal of Computational and Applied Mathematics 31 (2) 253 – 256 (1990).
[5] Gradimir V. Milovanovic, Predrag M. Rajkovic: On polynomials orthogonal on a circular arc; Journal of Computational and Applied Mathematics 51 1 – 13, (1994).
[6] Boro M. Piperevski: On a Class of Matrix Differential Equations with Polynomial Coefficients, International Conference Approximation & Computation (APP&COM 2008), Dedicated to 60th anniversary of Professor Gradimir V. Milovanovic, University of Nis, Faculty of Electronic Engineering, August 25-29, 2008, Nis, Serbia. Springer optimization and its applications, 42, 2010, 385-390

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