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CONTENT

The Appendix

In honor of the first Doctor of Mathematical Sciences Acad. Blagoj Popov, a mathematician dedicated to differential equations, the idea of holding the "Day of Differential Equations" was born, prompted by Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski, and Prof. Ph.D. Lazo Dimov. Acad. Blagoj Popov presented his doctoral dissertation on 05.05.1952 in the field of differential equations. This is the main reason for holding the " Day of Differential Equations" at the beginning of May.

This year on May 10th, the "Day of Differential Equations" was held for the fifth time at the Faculty of Computer Sciences at "Goce Delcev" University in Stip under the auspices of Dean Prof. Ph.D. Cveta Martinovska - Bande, organized by Prof. Ph.D. Biljana Zlatanovska.

Acknowledgments to Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski and Prof. Ph.D. Lazo Dimov for the wonderful idea and the successful realization of the event this year and in previous years.

Acknowledgments to the Dean of the Faculty of Computer Sciences, Prof. Ph.D. Cveta Martinovska - Bande for her overall support of the organization and implementation of the "Day of Differential Equations".

The papers that emerged from the "Day of Differential Equations" are in the appendix to this issue of BJAMI.

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ON SOME MODELS OF DIFFERENTIAL GAMES

Nevena Serafimova

Abstract. This article presents a brief review of differential game models, their characteristics and solutions, including evolutionary games. At the end, some upgrades of the classical models are discussed.

1. Introduction

Game theory has its reflections everywhere: in living systems, in everyday life, in different domains of the human society. It can be applied for the purpose of prediction (what will happen), explanation (why something happened), or prescription (how to play). But in many of the observed encounters, the decision sequences and the strategic spaces are too big, which renders the problem intractable. To make such situations calculable, it is convenient to apply continuous decision - making and obtain differential game models. Differential equations in this respect are no more than a practical procedure.

The main research focus of the differential games' theory are problems in which players know the state of the game, while their moves are governed by a given control functions. In such circumstances, it is natural to define game strategies as continuous functions of time and state, with compact sets as their domains. The initial research has been predominantly directed towards military issues such as, for example, determining the best strategy for the pursuit – evasion game in the aerial combat problems. This game includes elements such as variable speed of the combatants, movement through the three-dimensionality of physical space, and the combat problem, while the solution includes the optimal missile guidance (pursuing strategy), the optimal maneuver for avoiding of missile (evasion strategy), and the guaranteed missing distance (value of game). Pursuit – evasion games are a special type of *zero-sum* games. The pursuer's strategy aims at minimizing the time of capture, while the evader seeks to maximize the capture time or, better yet, to make this time infinite, meaning that capture never occurs.

Today, there are many peaceful applications of differential games, mainly in economic problems. In all of them, one player tries to maximize, while the other minimizes a certain quantity (gain, payoff) that is received at the end of the game. In the computer community, there have been numerous efforts for solving a differential game by genetic algorithms / genetic programming, reinforced learning and artificial neural networks (ANN), which were mainly successful. One positive example of applying a genetic algorithm to ANN is the determination of optimal strategies for the before - mentioned pursuit-evasion game with two players. Another example is the application of reinforced learning to a differential game of Markov decision process, where the pursuit is simulated through a projectile launch towards a plane which tries to evade. The purpose is to find an optimal control strategy and in this case, it was determined that the learned strategy converged to a known optimal strategy. Nevertheless, these successful examples do not ensure an understanding of the mathematical apparatus that lies underneath.

When speaking of games in general, the concept of solution (or the best answer to the problem for that matter), is not simple. Game theory has embraced a versatile approach to the solution problem. In zero-sum games, the natural solution concept is the saddle point. But in games that are not zero-sum, or have more than two players, there are many solution concepts. They can be provisionally divided into two major classes: cooperative and non-cooperative, the difference between them being less clear when a dynamic process is observed. Even though most of the results from the general differential games theory refer to non-cooperative situations, there are some initial theories in the richer area of cooperative solutions as well.

In this overview, we will be mostly concerned with the Nash equilibrium as a solution concept to a differential game model. In general, a Nash equilibrium may not exist or may not be unique. Also, different Nash equilibria can yield different payoffs to each player. Specifically, in a game with two players, the Nash equilibrium is a state in which each of the participants is playing the best response to the adversary's strategies. If the game is played only once (a 'one-shot' or *normal form* game) with each of the players being rational, have common knowledge and complete information about the game, the outcome of the interaction will be a Nash equilibrium. The question is, how is this situation achievable in reality?

In some situations, the outcome of NE can be regulated by previous agreements that are established prior to the encounter. Another possibility is to rely on the influence of social norms, since they act as a powerful tool in behavior modeling. In addition, there is another, nature - inspired approach that comes in the form of *evolutionary games* (games of evolution), which can formalize the process of arriving to a state of a NE.

Even though we will not be entering into a deeper discussion on game theory concepts, it is worth noting at this point that the concept of information is very important in a game, in that that a different information structure can result in a distinctively altered game situation.

2. Games, continuity and optimal control

In the theory of optimization, the goal is to find a value of the argument x that maximizes (or minimizes) the value of a given univariate function P(x). Typically, P is a continuous function and the maximum is sought over a closed, possibly unbounded domain $X \subseteq \mathbf{R}^m$. An extensive mathematical theory is currently available on the existence of the maximum, on necessary and sufficient conditions for optimality, and on computational methods. Interpreting P(x) as a *payoff* function, one can regard the problem of optimization as a decision problem. The decision is to choose, among all possible choices $x \in X$, the one that provides the maximum possible payoff.

The interactive environment of the game theory determines the payoff function P with at least two variables. In contrast to the optimization problem, here it is not generally possible to find an ordered pair (x_1, x_2) which will maximize the payoffs of both players at the same time. For this reason**,** various alternative solution concepts have been proposed in the literature. Each of them can be relevant in different gaming situations, depending on the information available to the players and their ability or incentive to cooperate. Specifically, if players have no means to communicate and do not cooperate, then an appropriate concept of solution is the Nash equilibrium, defined as a fixed point of the best reply map [1].

In the control theory, the system is modeled by an ordinary differential equation,

$$\begin{cases} \dot{x}(t) = f(x(t), \alpha(t)), \quad t > 0 \\ x(0) = x_0 \end{cases}$$

where $x(t)$ is the state of the system which changes in time t, $\alpha(t)$ is the control and f is the evolution function of the system. Given an initial state $x(t_0) = x_0$, the future system's evolution is thus determined by the input $\alpha(t)$, which acts as an optimizator. The received payoff $P(\alpha)$ can be calculated according to:

$$P(\alpha) := \int_0^T r\big(x(t), \alpha(t)\big) dt + g\big(x(T)\big)$$

where r is the running payoff and g is the terminal payoff. The main problem in the control theory is to find an optimal control $\alpha^*(t)$ that maximizes the payoff P, i.e. such that

$$P\big(\alpha^*(t)\big) \geq P\big(\alpha(t)\big)$$

holds for every $\alpha(t) \in A$, the set of allowed controls.

From a mathematical perspective, the key questions related to the behavior of the controlled system are: Is there an optimal control? How can the optimal control be described? How can the optimal control be constructed? For a given initial state x_0 and a target set $S \subset \mathbf{R}^n$, is there a control management that will guide the system into S, in finite time?

Differential games are a natural extension of the control model to cases where two or more decision-makers are present, each one of them seeking to maximize the individual payoff. The theory of differential games was first developed by R. Isaacs [2]. Some initial results were also given by other authors, such as in [3,4]. In a two-player game, the model is given by:

$$\begin{cases} \dot{x}(t) = f(x(t), \alpha(t), \beta(t)) & (t > 0) \\ x(0) = x^0 \end{cases}$$

$$\max P_i = \int_0^T r_i(x(t), \alpha(t), \beta(t))dt + g_i(x(T)),\ \ i = 1, 2$$

Similarly, as before, $x(t)$ is the state of the system, $\alpha(t)$ and $\beta(t)$ are the controls of Player 1 ($i = 1$) and Player 2 ($i = 2$) respectively, f is the evolution function, r is the running payoff, and g is the gain that is received at the end of the game. In some models, g is interpreted as a cost, in which case it should be subtracted instead of added to the final payoff P. In a game environment, the controls α and β are actually the strategies that the players are applying, in order to maximize their respective payoffs.

The strategies in a differential game are mappings, resulting from the response that a given player chooses in advance, to all of the controls that are available to the adversary. Concerning the information structure of the game, it is agreed that each player has perfect knowledge of the evolution function f of the system, the control sets of the players, the payoff functions, the moment of time t and the initial state x_0. Also, it is assumed that the future states of the system are not visible to the players. During the course of the game, the task of each player is not to choose a control function, but a response to the controls of the other player(s), i.e. a mapping from the adversary's control set to his control set.

In differential game models, various types of strategies can exist. *Open-loop* strategies have a form of $\alpha = \alpha(t)$, i.e. they depend only on time. *Feedback* strategies on the other hand can depend both on time and on the current state of the system, and are defined by $\alpha = \alpha(t,x)$. Clearly, in situations where only the initial state of the system can be observed, the natural approach is to consider open-loop strategies, while in situations where the current states of the system are observable, it is more appropriate to apply feedback strategies.

A pair of open-loop strategies $t \to (\alpha^*(t), \beta^*(t))$ is a Nash equilibrium, if it holds that:

1. α^* is a solution of the optimal control problem for Player 1,

$$\max_{\alpha} P_1(\alpha, \beta^*) = g_1(x(T)) + \int_0^T r_1\left(t, x(t), \alpha(t), \beta^*(t)\right)dt$$

for system dynamics given by

$$\dot{x}(t) = f(t, x, \alpha, \beta^*), \quad x(0) = x_0,\ \alpha(t) \in A,\ t \in [0, T]\,.$$

2. β^* is a solution of the optimal control problem for Player 2,

$$\max_{\beta} P_2(\alpha^*, \beta) = g_2(x(T)) + \int_0^T r_2\left(t, x(t), \alpha^*(t), \beta(t)\right)dt$$

for system dynamics given by

$$\dot{x}(t) = f(t, x, \alpha^*, \beta), \quad x(0) = x_0, \beta(t) \in B, t \in [0, T].$$

In order to find the Nash equilibrium, it is necessary to solve both optimal control problems simultaneously. This is done by introducing the optimal solution of the first problem as a parameter in the second problem, and vice versa. Under the assumption that all of the functions *f*, g_1, g_2, r_1, r_2 are continuously differentiable, the necessary conditions for optimality are given by the Pontryagin's Maximum Principle (PMP). It should be taken into account that these conditions are not sufficient for optimality. In other words, each pair of open-loop strategies that are a Nash equilibrium will be a solution to the system of equations, but not vice-versa: each solution of this system need not be a Nash equilibrium. However, under a suitable (very restrictive) concavity condition, it turns out that every solution satisfying the PMP is optimal [1].

In the special cases where an encounter of total conflict takes place, the appropriate model is a zero-sum game. The idea is that the two players control the dynamics of the given system, with one of them trying to maximize, while at the same time, the other tries to minimize the payoff functional P, depending on the trajectory of *f*:

$$\begin{cases} \dot{x}(t) = f(x(t), \alpha(t), \beta(t)) & (\mathrm{t} > 0) \\ x(0) = x^0 \end{cases}$$

$$P_{x,t}[\alpha(\cdot), \beta(\cdot)] = \int_s^T r(x(s), \alpha(s), \beta(t))ds + g(x(T)), \; i = 1, 2$$

In static games, the existence of a Nash equilibrium results from the Kakutani's Fixed Point Theorem for multivalued functions. In the special case of zero-sum games, some basic results of von Neumann can be obtained as corollaries. On the other hand, the equilibrium feedback strategies are best analyzed by observing a system of Hamilton-Jacobi-Bellman (HJB) PDEs for the individual payoff functions of the players, according to the principles of dynamic programming.

2.1. Information background of the game

Relative to the information structure of the differential game, different types of strategies can occur. When players base their decision only on time and on an initial condition, the open loop strategy is applied. If on the other hand players use the position of the game (t, x(t)) as an information basis, they will apply the feedback (Markovian) type of strategy. Strategies of a non-Markovian type are applied when players use history during the strategic choice.

The analysis of a differential game is extensively based on the concepts and techniques of the Optimal Control Theory. For example, the open-loop equilibrium strategies can be obtained by solving a two-point boundary value problem for an ordinary differential equation, resulting from the PMP. Below, we will discuss in short some of the strategic types that can occur in a differential game.

Open-loop strategies. The player *i* can access only the initial information about the system's state and is not able to observe further development and the strategies of the other player. In this case, only open-loop strategies can be applied, i.e. strategies that are time-dependent. This implies that the strategic set S_i of *i* consists of measurable functions $t \to u_i(t)$, from [0,T] to U_i.

Markovian type strategies. Suppose that in each moment $t \in [0,T]$, player *i* can observe the current state $x(t)$ of the system, but has no additional information on the strategies of the other players. Specifically, the player cannot predict future actions of the other players. In this case, a feedback strategy (i.e. of a Markovian type) can be applied, where the control $u_i = u_i(t,x)$ depends on time *t* and the current state *x*. The strategic set S_i available to *i* therefore consists of measurable functions $(t,x) \to u_i(t,x)$ from $[0,T]\times\mathbf{R}^n$ to U_i.

Hierarchical game. At the beginning, before the game starts, the leading player (Player 1) announces his strategy, which can be either of an open-loop $\alpha^o(t)$ or a feedback $\alpha^o(t, x)$ type. As a result, the game transforms into an optimal control problem for the other player (Player 2), given by:

$$\max_{\beta} \left\{ g_2(x(T)) - \int_0^T r_2\left(t, x(t), \alpha^o(t, x(t)), \beta(t)\right) dt \right\}$$

$$\text{when} \quad \dot{x}(t) = f(t, x, \alpha^o(t, x(t)), \beta), \quad x(0) = x_0, \; \beta(t) \in B.$$

We observe that in this case, the knowledge of the initial point x_0 together with the evolution of the differential equation, gives Player 2 complete information about the state of the system, for each $t \in [0,T]$. At the same time, the goal of Player 1 is to choose a strategy $\alpha = \alpha^o(t, x)$ that will guarantee him the best possible payoff, in relation to the response β applied by Player 2.

Information delay. Suppose that none of the players can observe the system's state $x(t)$, but at the same time they have information about the actions undertaken by the other players, with a certain time delay of $\delta > 0$. Put differently, let us suppose that in each moment $t > 0$, player i learns the strategy $\{u_j(s) |\, s \in [0, t-\delta]\}$ previously applied by player j. In this case, there is a possibility for setting up cooperative agreements between the two players, which are reinforced by initiating a punishment for the deviating player (breaking the deal). For example, if two players agree on playing a pair of Pareto-optimal strategies $t \to \alpha^{\Diamond}(t)$, $t \to \beta^{\Diamond}(t)$ (i.e. there is no other outcome that makes at least one player better off and no player worse off) and upon time τ, Player 2 decides to deviate and starts to play $t \to \beta^{\dagger}(t) \neq \beta^{\Diamond}(t)$, $t > \tau$ which increases his payoff, then after the moment $t = \tau+\delta$, Player 1 learns of this deviation and could decide to punish Player 2 by choosing another strategy $t \to \alpha^{\dagger}(t)$ that yields a much lower payoff for Player 2.

3. Evolutionary differential games

Deterministic models of practical systems have limited applicability, due to their predominantly linear nature, with only a limited number of cases including non-linear models. One way to approach this problem are evolutionary models, which have already been successfully applied in different areas. The evolutionary algorithm is an adaptive method that imitates the process of biological evolution through the mechanisms of natural selection and genetic diversity, guided by the principle of "survival of the most fit". These algorithms are characterized by self-adaptiveness, self-organization, self-learning, parallelism and generality, features that can prove to be very helpful in attacking problems of optimization control, machine learning, parallel processing, economic forecasting etc. In cases of limited information, the evolutionary approach allows that in some stages of the traditional modeling process such as hypotheses development, model construction, calculations, and even complete modeling, the human (rational) intelligence is replaced by computational intelligence.

The classical game theory deals with a number (usually two) of rational players who fully understand the game situation and have unlimited computational possibilities at hand. Looking into a different perspective, the evolutionary game theory deals with a population of players who do not even need to know they are participating in a game. The interactions take place between randomly chosen participants of these populations and the outcome of the interaction, together with the information related to the behavior of the other side, influence future behavior. Populations use strategies such as imitation, natural selection, mutation and other, to improve their fitness, i.e. their chances of survival. This process gradually leads to the stabilization of the strategic choice, which results in a strategic profile that should be a Nash equilibrium. The question is, to which degree do these models support the existence of a Nash equilibrium?

In order to be applied to the game theory, the formal evolutionary structures that are adopted from biology need to be additionally adjusted and extended. The genetic mechanism and natural selection that biologists are dealing with, allow only for a simple, canonical dynamic presentation (Malthusian dynamic). In other areas, such as economy for example, the main focus is on the processes of learning and imitation, which to a great extent are context dependent. Furthermore, most of the biological models deal with evolutive processes within a single species, while ignoring the coevolution of multiple species. They also incorporate linear specifications that include random interactions between two individuals, whereas an interaction of the whole population would necessitate an introduction of nonlinearity in the model.

The evolutionary dynamics of one population game with a total of n pure strategies (actions) can be expressed as a differential equation:

$$\dot{x} = (x, u(x)),$$

where $x_i(t)$ is the part of the population that plays strategy i in time t, the population's state in time t is given by $x(t) = (x_1(t), .., x_n(t))$, $S = \Delta(I) = \{x \in R_+^n \mid \Sigma_i x_i = 1\}$ is the state space, $u_i(x)$ is the payoff function for the strategy i in state x, and the total payoff function for the game is $u(x) = (u_1(x), \ldots, u_n(x))$.

When studying behavior patterns, it is useful to consider different models of evolutionary dynamics, which apply different rules of optimization and control processes. Still, there is no reason to expect any particular rule when modeling social adaptation phenomena. Instead, it is better to consider large dynamic classes such as the myopic adaptive dynamics, the payoff functional dynamics, the sign-preserving dynamics and others. The main question for all of them is, do they converge? Below, some dynamical models of evolutionary games are stated and discussed in short.

The Replicator's Dynamic (REP) (Taylor, Jonker, 1978) results from the assumption that the payoff from the strategy i in the game is the growth rate (per capita) of the total of individuals that play i. If we set the fitness of the strategy i to be $C + u_i(x)$, where C is the background fitness of the population, then the density $X_i(t)$ of the individuals that play i is governed according to the differential equation:

$$\dot{X}_i = X_i\left[C + u_i(x)\right].$$

Hence, the frequency $x_i = X_i / \Sigma_i X_i$ at which i is played will follow the *replicator dynamics (REP)*:

$$\dot{x}_i = x_i\left[u_i(x) - x \cdot u(x)\right]$$

where $x \cdot u(x) = \Sigma_i x_i u_i(x)$ is the average payoff for the set of strategies. The right hand side of this equation is Lipschitz continuous in x, so there will be a unique solution for every initial condition. Furthermore, this solution is interior when $x_i(t) > 0$ for all $i \in I$ and all $t \in \mathbf{R}$. Since the faces of the strategic simplex are invariant in REP, this boils down to the initial condition being interior i.e. $x_i(0) > 0$ for all $i \in I$.

For interactions of random pairs we have $u_i(x) = (Ax)_i$, therefore:

$$\dot{x}_i = x_i\left[(Ax)_i - x \cdot Ax\right]$$

When a player that plays strategy i is paired with a random, identically distributed adversary, the latter will choose strategy j if and only if $u_j > u_i$, with a probability that is proportional to $u_j - u_i$. Therefore, we have:

$$b_{ij} = x_j \cdot \max\{0, u_j - u_i\}.$$

This leads to the replicator's dynamics of a form:

$$\dot{x}_i = x_i\left(u_i - \sum_j x_j u_j\right).$$

Replicator's Dynamic between two populations. When two populations interact, the evolution takes place through a finite, not necessarily symmetric game with two players. All interactions are in pairs, without any auto-interactions. Let I = {1, …, n} and J = {1, …, m} be the sets of pure strategies for the two populations, $x(t) = (x_1(t), .., x_n(t))$ and $y(t) = (y_1(t), .., y_m(t))$ are populations' profiles, and the payoffs for the strategy i of the populations are $(Ay)_i$ and $(Bx)_i$, accordingly. Then the replicator's dynamics is given by the system of differential equations:

$$\dot{x}_i = x_i\left[(Ay)_i - x \cdot Ay\right]$$
$$\dot{y}_i = y_i\left[(Bx)_i - y \cdot Bx\right]$$

Social dynamics (revision protocol). Revision protocols in games describe when and how players decide to switch strategies, specifying the information they use in making decisions. Together, a population game and a revision protocol generate a differential equation called 'the mean dynamic', to describe the evolution of aggregate behavior when the revision protocol is employed during the recurrent play of the game. Let's assume a large (possibly infinite) population of players, who revise their choice of strategies in small time intervals. Suppose that the player who plays i moves to j with a rate b_{ij}. This defines the dynamics of the system by the equation:

$$x_i = \sum_j \lambda x_j b_{ji} - \lambda x_i \sum_j b_{ij} .$$

Different values of b_{ij} will define a different dynamics.

Imitative dynamics are the most thoroughly studied dynamics in the evolutionary game theory. They are derived from protocols under which agents consider switching to the strategy of a randomly sampled opponent, relative to payoff difference considerations. Hence, the functional form of the imitative dynamics expresses proportionality between the strategies' absolute growth rates and their current levels of use:

$$\dot{x}_i = x_i \sum_{j \in S} x_j \left(r_{ji}(P(x), x) - r_{ij}(P(x), x) \right).$$

Here, i and j are the applied strategies, P is the payoff function and the coefficients r_{ij} are called *conditional imitation rates.*

Smith's dynamics occurs when players employ imitation of successful strategies with repeated sampling, for positive payoffs. If the probability of the playing strategy j is 1/N (instead of x_j), we have:

$$b_{ij} = 1/N \cdot \max\{0, u_j - u_i\}, \text{ or}$$
$$b_{ij} = \max\{0, u_j - u_i\}.$$

This defines the Smith's dynamics [5]:

$$\dot{x}_i = \sum_j x_j \cdot \max\{u_i - u_j\} - x_i \sum_j x_j \cdot \max\{u_i - u_j\} .$$

Here, the occurrence of new (innovative) strategies is possible. The stationary points are (symmetric) Nash Equilibria.

Best response dynamics. The dynamics of best response (BR) is obtained under the assumption that in each small time interval, a part of the population revises its strategy and switches, rationally or myopically, to the best response of the current population profile. Since the best response need not be unique, this leads to a differential inclusion, instead of an equation.

Let $BR(x)$ be the set of the best (mixed) responses to the probability profile x:

$$BR(x) := \{y \in S \mid y \cdot Ax = \max_{z \in S} z \cdot Ax\} .$$

Since the best replies are in general not unique, this is a differential inclusion rather than a differential equation. Now, let us assume that in each small time interval, a part of the population revises its strategy and chooses the best response to the momentary average behavior. This leads to the best response dynamics (BRD), whose differential equation is given by:

$$\dot{x} \in BR(x) - x\,.$$

The solution of (BRD) is an absolutely continuous function that satisfies the inclusion, for almost all t. This problem always has a solution, but it might not be unique, i.e. more than one solution is possible for the same initial condition.

3.1. The evolutionary folk theorem

In mathematics, the term *folk theorem* refers generally to any theorem that is believed and discussed, but has not been published. In game theory, folk theorems are a class of theorems about possible Nash equilibrium payoff profiles in repeated games [6]. More specifically, the folk theorem of the evolutionary game theory states the following:

- If a stationary population distribution is dynamically stable, then it constitutes a Nash equilibrium;
- If the population process converges from an interior initial state, then the limit distribution is a Nash equilibrium;
- If the population process starts from an interior state, then iteratively strictly dominated strategies will be asymptotically eliminated.

A population is considered to be in an evolutionarily stable state if its genetic composition is restored by a selection after a disturbance. Evolutionarily stable states have close relationship with the concept of the Nash equilibrium and the folk theorem for infinitely repeated games. The reason why they are characterized by the folk theorem is because exact solutions to the replicator equation are difficult to obtain. It is generally assumed that the folk theorem, which is the fundamental theorem for non-cooperative games, defines all Nash equilibria in infinitely repeated games. However, it can be proved that Nash equilibria that are not characterized by the folk theorem do exist. By adopting specific reactive strategies, a group of players can be better off by coordinating their actions in repeated games [7].

Here, we will state the evolutionary folk theorem for replicator's dynamics. Let (x,y) be the state in REP with two populations.

- If (x,y) is an interior rest point, then it is a Nash equilibrium.
- If (x,y) is Lyapunov stable, then it is a Nash equilibrium.
- If (x,y) is the limit point of an inner point's trajectory $(x(t), y(t))$ when $t\to\infty$, then it is a Nash equilibrium.

In dynamic systems, Lyapunov functions are a tool for proving the stability of an equilibrium point. Some properties of Nash equilibria, such as existence and stability, arise naturally from the theory of Lyapunov. Lyapunov stability is weaker than the asymptotic stability, which is always robust to small perturbations of the dynamics. In this sense, an asymptotically stable Nash equilibrium permits the definition of a Lyapunov function. Also, if there is a Lyapunov function, it converges to a Nash (Lyapunov) equilibrium [8].

4. Briefly on some alternative approaches

Differential games as dynamic models describe interaction in continuous time, expressing it in a form of a differential equation. The main purpose of the differential approach is to simplify the analysis of games that incorporate many moves and states. In a differential game with a small number of players, each player has the power to modify the state of the system (influence the evolution of the differential equation) by applying individual control (strategy) in a form of a measurable function. For example, in an oligopolistic market with a small number of producers, each company can affect the market price by changing its own production level. But, when the number of players is very large, the analysis of the game becomes intractable and, at the same time, the individual influence of a player is very small.

Consequently, in games with many players, the state of the system is better determined by average behavior: no single player has the power to change the overall evolution of the system. This fact greatly simplifies the mathematical model and the search for Nash equilibrium solutions to the differential game, leading to the emergence of a theory of *mean field games.* This type of games is inspired by the mean field theory in physics and motivated by interaction models in economy, finance, network control etc., where a large number of agents with infinitesimal impact upon the system are present. Specifically, they are a class of non-cooperative stochastic differential games where individual players interact with the aggregate effect of all other players through a mean-field coupling term, expressed by the payoff (cost) function. In continuous time, a mean field game is a combination of the HJB equation that describes the optimal control problem of an individual, with a Fokker–Planck–Kolmogorov equation that describes the dynamics of the aggregate distribution of agents. Under fairly general assumptions, it can be proved that the equilibria of a class of mean field games are the limits of Nash equilibria of N-player games, as $N \to \infty$. [9]

Another interesting upgrade of differential games is to hybrid systems, which are helpful in understanding various aspects of a system separately at their natural level. While continuous dynamics are more natural for some aspects of the system, discrete dynamics may be more appropriate for others. Combining the characteristics of hybrid systems with those of differential games results in a unifying framework where discrete, continuous, and adversarial dynamics mix freely. The result is a model of *hybrid games*, which are games of two players on a hybrid system's discrete and continuous dynamics. Players have control over some discrete-time choices during the evolution of the system, but the continuous dynamics stays deterministic and its duration (time of evolution) is the only choice in the game. Hybrid games can model discrete aspects like decision or reaction delays, sporadic or discrete-time interactions, and games with different dynamics or different controls, structurally complex systems or dynamics in different modes of the system. Differential hybrid games also allow for the structural advantages of hybrid systems so that structurally more complex cases with different parts and subsystems can be modeled. [10]

Finally, concerning the computational complexity problems related to the practical application of differential games, neural networks combined with the theory of multiplayer intelligence control can be introduced to modeling problems of practical dynamic conflicts. This approach turns the problem of a differential game into a problem of maneuver identification and optimization control based on neural networks. It allows adding human experience and efficiently combines the qualitative with the quantitative analysis of the differential game. The adaptive distributed learning rate of neural networks considerably increases the learning speed and performance. [11]

References

[1] A. Bressan, Noncooperative Differential Games. A Tutorial. *Milan Journal of Mathematics* 79(2):357-427. January 2011

[2] R. Isaacs, *Differential Games*, Wiley, New York, 1965.

[3] A. Friedman, *Differential Games*, Wiley, New York, 1971.

[4] N. N. Krasovskii, A. I. Subbotin, Game-Theoretical Control Problems Springer-Verlag, Berlin, 1988

[5] SMITH, M. (1984). The Stability of a Dynamic Model of Traffic Assignment—An Application of a Method of Lyapunov. *Transportation Science, 18*(3), 245-252.

[6] Friedman, J. (1971), "A non-cooperative equilibrium for supergames", *Review of Economic Studies*, 38 (1): 1–12, doi: 10.2307/2296617

[7] Li J, Kendall G (2015) On Nash Equilibrium and Evolutionarily Stable States That Are Not Characterised by the Folk Theorem. PLoS ONE 10(8): e0136032.

[8] E. Somanathan, Evolutionary Stability of Pure-Strategy Equilibria in Finite Games, Games and Economic Behavior 21, 253-265, 1997

[9] Cardaliaguet, Pierre (September 27, 2013). "Notes on Mean Field Games", Lecture notes, Université Paris-Dauphine

[10] Lasry, Jean-Michel & Lions, Pierre-Louis. (2007). Mean Field Games. Japanese Journal of Mathematics. 2. 229-260. 10.1007/s11537-007-0657-8.

[11] Zhou, R. & Li, H.F. (2000). Application of neural networks in differential game. Beijing Hangkong Hangtian Daxue Xuebao/Journal of Beijing University of Aeronautics and Astronautics. 26. 666-668.

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