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In honor of the first Doctor of Mathematical Sciences Acad. Blagoj Popov, a mathematician dedicated to differential equations, the idea of holding the "Day of Differential Equations" was born, prompted by Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski, and Prof. Ph.D. Lazo Dimov. Acad. Blagoj Popov presented his doctoral dissertation on 05.05.1952 in the field of differential equations. This is the main reason for holding the " Day of Differential Equations" at the beginning of May.

This year on May 10th, the "Day of Differential Equations" was held for the fifth time at the Faculty of Computer Sciences at "Goce Delcev" University in Stip under the auspices of Dean Prof. Ph.D. Cveta Martinovska - Bande, organized by Prof. Ph.D. Biljana Zlatanovska.

Acknowledgments to Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski and Prof. Ph.D. Lazo Dimov for the wonderful idea and the successful realization of the event this year and in previous years.

Acknowledgments to the Dean of the Faculty of Computer Sciences, Prof. Ph.D. Cveta Martinovska - Bande for her overall support of the organization and implementation of the "Day of Differential Equations".

The papers that emerged from the "Day of Differential Equations" are in the appendix to this issue of BJAMI.

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MATHEMATICAL MODELS WITH STOCHASTIC DIFFERENTIAL EQUATIONS

Marija Miteva and Limonka Koceva Lazarova

Abstract. In this paper we consider mathematical models that describe physical processes, applying stochastic differential equations. Modelling with ordinary differential equations usually results with a deterministic process, but the model will be more real if we do not know the future state of the process exactly, but we know it with certain probability. Then we use stochastic differential equations.

1. Introduction

We know that the different processes in nature and society are due to some changes of the parameters that describe a certain system that is involved in the considered process.

Starting with the definition of function's derivative and differentiation, we know that the derivative characterizes certain change, thus the rates of changes in the physical and society processes can be described mathematically by the derivative of a function. This means that different mathematical models of those processes can be expressed by differential equations.

Mathematical models obtained using ordinary differential equations are usually deterministic models, i.e. the model allows us to determine the behavior of the system at each moment of the process that is described with the considered model (that is the process in which the considered system is involved). But, a mathematical model will be closer to the real model if we allow randomness in its describing. It means that in the real processes, we usually do not know the exact state that the process will enter, but we know it with certain probability. In order to construct a mathematical model then, we should use stochastic differential equations. Actually, modeling with stochastic differential equations is generalization of the modeling with ordinary differential equations, when the parameters which characterize the system are changing randomly. The differential equation that describes the process does not determine the change of the parameters exactly, but it determines it with certain probability. Modeling with stochastic differential equations is not a much-explored topic. We can read about it in [1-5].

While modeling with stochastic differential equations, we usually consider a discrete case first, describing the changes in the process that occur in a small time interval Δt, then we let $\Delta t \to 0$ and obtain a continuous model.

Actually, models with stochastic differential equations are developed for dynamical systems when the impact of randomness is considered and we obtain those models mainly with the next steps: discrete stochastic model is obtained at the beginning, i.e. for a small interval Δt we determine all possible changes with an appropriate probability; then we estimate the expected change (mathematical expectation) and the covariance matrix for the discrete stochastic process; at the end, using these data, we determine a stochastic differential equation as a mathematical model for the considered dynamical process.

In continuation we will consider examples of mathematical models described with stochastic differential equations.

2. Mathematical models of dynamical systems – general case

Let us consider a dynamical system with two states, A and B. We will denote with $A(t)$ and $B(t)$ the value of each state at the moment t. We will suppose that for a small-time interval Δt the state A can change its value for a and we will consider a change $-a$ when the value is decreasing and a change $+a$ when the value is increasing ($a > 0$). If there is no change in the value of the state a, it will mean that the change is 0. Similarly, the value of the state B can be changed for $-b$, $+b$ ($b > 0$) or 0 (if there is no change). Such a model is described in [1].

We will denote the change in the state A with ΔA, the change in the state B with ΔB. The change in the state of the considered dynamical process S, in a small interval Δt will be denoted with $\Delta S = \begin{bmatrix} \Delta A \\ \Delta B \end{bmatrix}$. The state of the system may be changed in eight ways, depending on the changes in both states A and B, and the new state which is the same with the previous one, when there are no changes in the interval Δt, is the ninth way. We will list all possible changes:

- The change in the process S, if there is a change in the value of the state A for $-a$, and there is no change in the value of the state B, will be denoted with $\Delta S_1 = \begin{bmatrix} -a \\ 0 \end{bmatrix}$;
- The change in the process S, when there is a change in the value of the state A for a, and there is no change in the value of the state B, will be denoted with $\Delta S_2 = \begin{bmatrix} a \\ 0 \end{bmatrix}$;
- In a similar way we define the other changes in the process: $\Delta S_3 = \begin{bmatrix} 0 \\ -b \end{bmatrix}$ means there is no change in the state A and the value of the state B has been decreased for $-b$.

  -We will list the other possible changes: $\Delta S_4 = \begin{bmatrix} 0 \\ b \end{bmatrix}$; $\Delta S_5 = \begin{bmatrix} -a \\ b \end{bmatrix}$; $\Delta S_6 = \begin{bmatrix} a \\ -b \end{bmatrix}$; $\Delta S_7 = \begin{bmatrix} -a \\ -b \end{bmatrix}$; $\Delta S_8 = \begin{bmatrix} a \\ b \end{bmatrix}$; $\Delta S_9 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The process will enter in a certain state at the moment t, with a probability which depends on t and will be proportional with the length of the interval Δt in which the change is happening. We will denote with p_i the probability $p(\Delta S_i)$, for $i = 1,2,...,9$ and thus we have: $p_i = k_i \Delta t$ where each k_i depends on t and the states A and B, i.e. $k_i = k_i(t, A, B)$.

Let us estimate the expected change $E(\Delta S)$.

$$E(\Delta S)=\sum_{i=1}^{9} p_i \Delta S_i = k_1\Delta t\begin{bmatrix}-a\\0\end{bmatrix}+k_2\Delta t\begin{bmatrix}a\\0\end{bmatrix}+k_3\Delta t\begin{bmatrix}0\\-b\end{bmatrix}$$
$$+k_4\Delta t\begin{bmatrix}0\\b\end{bmatrix}+k_5\Delta t\begin{bmatrix}-a\\b\end{bmatrix}+k_6\Delta t\begin{bmatrix}a\\-b\end{bmatrix} \tag{2.1}$$
$$+k_7\Delta t\begin{bmatrix}-a\\-b\end{bmatrix}+k_8\Delta t\begin{bmatrix}a\\b\end{bmatrix}+k_9\Delta t\begin{bmatrix}0\\0\end{bmatrix}$$

$$E(\Delta S)=\begin{bmatrix}(-k_1+k_2-k_5+k_6-k_7+k_8)a\\(-k_3+k_4+k_5-k_6-k_7+k_8)b\end{bmatrix}\cdot\Delta t \tag{2.2}$$

Now we will estimate the covariance matrix. We estimate first:

$$E\left[\Delta S(\Delta S)^T\right]=\sum_{i=1}^{9} p_i\left[\Delta S_i(\Delta S_i)^T\right]=$$
$$=\begin{bmatrix}(k_1+k_2+k_5+k_6+k_7+k_8)a^2 & (-k_5-k_6+k_7+k_8)ab\\(-k_5-k_6+k_7+k_8)ab & (k_3+k_4+k_5+k_6+k_7+k_8)b^2\end{bmatrix}\cdot\Delta t$$

We will use the notation $k_5+k_6+k_7+k_8=k'$ and $-k_5-k_6+k_7+k_8=k''$ and then write:

$$E\left[\Delta S(\Delta S)^T\right]=\begin{bmatrix}(k_1+k_2+k')a^2 & k''ab\\k''ab & (k_3+k_4+k')b^2\end{bmatrix}\cdot\Delta t \tag{2.3}$$

The expectation vector μ is $\mu(t,A,B)=\dfrac{E(\Delta S)}{\Delta t}$ and the covariance matrix V is the matrix $V(t,A,B)=\dfrac{E\left[\Delta S(\Delta S)^T\right]}{\Delta t}$. We will denote with $D(t,A,B)$ a matrix such that $\left[D(t,A,B)\right]^2=V(t,A,B)$.

The notation $p(t,x_1,x_2)$ represents the probability for $A=x_1$ and $B=x_2$ at the moment t.

We are interested in finding a mathematical model that can tell us which the state of the system at a certain future moment $t+\Delta t$ will be, if we know the state at the moment t. Such model can be constructed using *forward Kolmogorov equations* (we can read about it in [6] and [7]). According to these equations, for the future moment $t+\Delta t$ we have the probability:

$$p(t+\Delta t,x_1,x_2)=p(t,x_1,x_2)+\Delta t\sum_{i=1}^{10}K_i \tag{2.4}$$

where:

$$K_1=p(t,x_1,x_2)\left(-k_1(t,x_1,x_2)-k_2(t,x_1,x_2)-k_3(t,x_1,x_2)-k_4(t,x_1,x_2)\right)$$

$$K_2=p(t,x_1,x_2)\left(-k'(t,x_1,x_2)\right)$$

$$K_3=p(t,x_1+a,x_2)k_1(t,x_1+a_1,x_2)$$

$$K_4 = p(t, x_1 - a, x_2) k_2(t, x_1 - a, x_2)$$

$$K_5 = p(t, x_1, x_2 - b) b_2(t, x_1, x_2 - b)$$

$$K_6 = p(t, x_1, x_2 + b) d_2(t, x_1, x_2 + b)$$

$$K_7 = p(t, x_1 + a, x_2 - b) m_{12}(t, x_1 + a, x_2 - b)$$

$$K_8 = p(t, x_1 - a, x_2 + b) m_{21}(t, x_1 - a, x_2 + b)$$

$$K_9 = p(t, x_1 + a, x_2 + b) m_{11}(t, x_1 + a, x_2 + b)$$

$$K_{10} = p(t, x_1 - a, x_2 - b) m_{22}(t, x_1 - a, x_2 - b)$$

If we expand each K_i, for $i = \overline{3,10}$ in Taylor series at (t, x_1, x_2) and substitute those terms in the above sum, we will obtain the equation:

$$\frac{p(t+\Delta t, x_1, x_2) - p(t, x_1, x_2)}{\Delta t} = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[\mu_i(t, x_1, x_2) p(t, x_1, x_2)\right] + \\ + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j}\left[\sum_{k=1}^{2} d_{i,k}(t, x_1, x_2) d_{j,k}(t, x_1, x_2) p(t, x_1, x_2)\right] \quad (2.5)$$

where $d_{i,j}$ is the element in the i - th row and j - th column of the matrix D.

Let $\Delta t \to 0$ in the equation (2.5), and then we will obtain:

$$\frac{\partial p(t, x_1, x_2)}{\partial t} = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[\mu_i(t, x_1, x_2) p(t, x_1, x_2)\right] \\ + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j}\left[\sum_{k=1}^{2} d_{i,k}(t, x_1, x_2) d_{j,k}(t, x_1, x_2) p(t, x_1, x_2)\right] \quad (2.6)$$

In the equation (2.6) μ_i is the i - th component of the expectation vector μ.

The function $p(t, x_1, x_2)$ (the probability density) that satisfies the previous equation is identical to the probability density that is the solution of the following system of stochastic differential equations:

$$\begin{cases} dS(t) = \mu(t, A, B) dt + D(t, A, B) dW(t) \\ S(0) = S_0 \end{cases} \quad (2.7)$$

where $W(t) = \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix}$ is Brownian motion (Wiener process). We can read about Wiener process in [8] and [9]. We can also see the application of stochastic differential equations in dynamical systems in [10] and [11].

3. Mathematical models of dynamical systems – examples

Using the previously described general case, we will make a mathematical model considering a certain university as a system with active and non-active students (students whose studies are temporarily on pause). We will denote the number of active students at the moment t with x_1 and the number of non-active students with x_2. The number of active students can be changed in a way that it is decreasing when a student is graduating, or giving up studies, and it is increasing when students are starting for the first time, or a student is transferring from another university. These changes in the number of active students do not cause a change in the number of non-active students. We will denote the first one of the above changes with $\Delta S_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and the second one with $\Delta S_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Each change represents the change for one student in the first state, and will be proportional to the rate of the appropriate change and the number of students x_1. The number of non-active students can be changed in a way that a non-active student is leaving the studies. We will denote this change in the state with $\Delta S_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$; it is proportional to the rate of giving up studies and the number of non-active students x_2.

The number of non-active and active students can be changed at the same time in a way that the first one is decreasing when a student is re-starting the studies, and increasing when an active student becomes non-active. The appropriate changes will be denoted with $\Delta S_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\Delta S_5 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The change ΔS_5 is proportional to the rate of the upcoming non-active students and its number x_1, and the change ΔS_4 is proportional to the rate of re-activating of the studies and the number of active students x_2.

The quantities in both states can be decreased at the same time in a way that an active student is graduating and a non-active student is leaving the studies. This change is denoted with $\Delta S_6 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. It is proportional to the rate of graduating and the rate of leaving students and with the number of both active and non-active students, i.e. $x_1 x_2$. The numbers in both states can be increased in a way that an active student becomes non-active, but there are new students coming from another university. This change is $\Delta S_7 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. It is proportional to the rate of putting studies temporarily on pause and to the number of active students x_1. We will denote $\Delta S_8 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ when there is no change in each state.

If we denote with p_i the probability $p(\Delta S_i)$, for $i=1,2,...,8$, we have: $p_i = k_i x_1 \Delta t$ for $i=1,2,5,7$; $p_i = k_i x_2 \Delta t$ for $i=3,4$ and $p_6 = k_6 x_1 x_2 \Delta t$. Each k_i depends on t and the states A and B, i.e. $k_i = k_i(t,A,B)$.

Let us estimate the expected change $E(\Delta S)$.

$$E(\Delta S) = \sum_{i=1}^{8} p_i \Delta S_i = k_1 x_1 \Delta t \begin{bmatrix} -1 \\ 0 \end{bmatrix} + k_2 x_1 \Delta t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_3 x_2 \Delta t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + k_4 x_2 \Delta t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + k_5 x_1 \Delta t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + k_6 x_1 x_2 \Delta t \begin{bmatrix} -1 \\ -1 \end{bmatrix} + k_7 x_1 \Delta t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + p_8 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.8}$$

$$E(\Delta S) = \begin{bmatrix} (-k_1 + k_2 - k_5 + k_7)x_1 + (k_4 - k_6 x_1)x_2 \\ (k_5 + k_7)x_1 + (-k_3 - k_4 - k_6 x_1)x_2 \end{bmatrix} \cdot \Delta t \tag{2.9}$$

The expectation vector is

$$\mu(t,A,B) = \frac{E(\Delta S)}{\Delta t} = \begin{bmatrix} (-k_1 + k_2 - k_5 + k_7)x_1 + (k_4 - k_6 x_1)x_2 \\ (k_5 + k_7)x_1 + (-k_3 - k_4 - k_6 x_1)x_2 \end{bmatrix} \tag{2.10}$$

Let us estimate now the covariance matrix. We estimate first:

$$E\left[\Delta S (\Delta S)^T\right] = \sum_{i=1}^{8} p_i \left[\Delta S_i (\Delta S_i)^T\right] =$$

$$= \begin{bmatrix} (k_1 + k_2 + k_5 + k_7)x_1 + (k_4 + k_6 x_1)x_2 & (-k_5 + k_7)x_1 + (-k_4 + k_6 x_1)x_2 \\ (-k_5 + k_7)x_1 + (-k_4 + k_6 x_1)x_2 & (k_5 + k_7)x_1 + (k_3 + k_4 + k_6 x_1)x_2 \end{bmatrix} \cdot \Delta t$$

The covariance matrix V is the matrix $V(t,A,B) = \dfrac{E\left[\Delta S (\Delta S)^T\right]}{\Delta t}$, i.e.

$$V(t,A,B) = \begin{bmatrix} (k_1 + k_2 + k_5 + k_7)x_1 + (k_4 + k_6 x_1)x_2 & (-k_5 + k_7)x_1 + (-k_4 + k_6 x_1)x_2 \\ (-k_5 + k_7)x_1 + (-k_4 + k_6 x_1)x_2 & (k_5 + k_7)x_1 + (k_3 + k_4 + k_6 x_1)x_2 \end{bmatrix}$$

According to the discussion in the previous section, the model that we have considered can be described by the differential equation:

$$dS(t) = \mu(t,A,B)dt + D(t,A,B)dW(t) \tag{2.11}$$

with the initial condition $S(0) = S_0$. In the equation (2.11), D is the matrix such that $V = D^2$ and $W(t) = \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix}$ is Wiener process. The equation (2.11) is describing the dynamics in the process of deactivating and activating studies by the students.

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