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CONTENT

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EXAMPLES OF GROUP $\exp(tA), (t \in \mathbb{R})$ OF 2×2 REAL MATRICES IN CASE MATRIX A DEPENDS ON SOME REAL PARAMETERS

RAMIZ VUGDALIĆ

Abstract. In this paper of the given 2×2 matrix A, which depends on some real parameters, we obtain the multiplicative groups $\exp(tA)$ $(t \in \mathbb{R})$ of 2×2 real matrices.

1. Introduction

One can find some preliminaries about the matrix theory in [1, 2, 4], and other preliminaries about the semigroup or group theory of linear and bounded operators in Banach spaces in [3, 5]. Here we give some necessary preliminaries. Matrices constitute the fundamental analytic apparatus for the study of linear operators in an $n-$ dimensional vector space ($n \in \mathbb{N}$). For every real or complex matrix $n \times n$ matrix A, the matrix exponential of matrix A is defined as the matrix $\exp(A) = e^A := \sum\limits_{k=0}^{\infty} \frac{A^k}{k!}$. Namely, it is known that this series converges for every square matrix A. Also, it holds that
1. $\exp(0) = I$, I is the identity matrix;
2. $\exp((t+s)A) = \exp(tA)\exp(sA)$ $(t, s \in \mathbb{R})$;
3. If for square matrices A, B holds $AB = BA$, then $\exp(A+B) = \exp(A)\exp(B)$.

By defining any norm $\|\cdot\|$ on $M_n(\mathbb{C})$ (the vector space of all $n \times n$ matrices over the field of complex numbers) it is known that the matrix exponential $\exp(tA) = \sum\limits_{k=0}^{\infty} \frac{t^k A^k}{k!}$ is convergent, bounded and $\|\exp(tA)\| \le e^{t\|A\|}$ $(t \in \mathbb{R})$. Matrix exponentials are important in the solution of systems of ordinary differential equations (for example, see [1]). Namely, for every constant $n \times n$ matrix A, the solution of Cauchy problem $\frac{d}{dt}y(t) = Ay(t)$, $y(0) = y_0$, is given by $y(t) = e^{tA}y_0$. The matrix exponential can also be used to solve the inhomogenous differential equation

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$\frac{d}{dt}y(t) = Ay(t) + z(t)$, $y(0) = y_0$. From $I = \exp(0) = \exp(A + (-A)) = \exp(A)\exp(-A)$ it follows that every matrix exponential $\exp(A)$ is always an invertible matrix and $[\exp(A)]^{-1} = \exp(-A)$. The matrix exponential is a map $\exp : M_n(\mathbb{C}) \to GL(n, \mathbb{C})$, where $GL(n, \mathbb{C})$ denotes the general linear group of degree n, i.e. the multiplicative group of all $n \times n$ complex invertible matrices. This map is surjective. That means that every invertible matrix can be written as an exponential of some matrix from $M_n(\mathbb{C})$. The map $t \longmapsto \exp(tA)$ for some fixed matrix $A \in M_n(\mathbb{C})$ and $t \in \mathbb{R}$, is an one-parameter subgroup of the general linear group $GL(n, \mathbb{C})$ since $\exp(0) = I$ and $\exp((t+s)A) = \exp(tA)\exp(sA)$ $(t, s \in \mathbb{R})$. The group of matrices $T(t) := \exp(tA)$ $(t \in \mathbb{R})$ represents the group of linear and bounded operators in a complex Banach space $\mathbb{C}^n$ relative to a basis of $\mathbb{C}^n$. Then, the matrix A represents the infinitesimal generator of that group of operators relative to the same basis of $\mathbb{C}^n$. Namely, it holds the next definition.

Definition 1. *The one-parameter family of linear and bounded operators $T(t)$ $(t \in \mathbb{R})$ defined from a Banach space X into X, which satisfies $T(0) = I$, the identity operator on X, and $T(t+s) = T(t)T(s)$ on X, for every $t, s \in \mathbb{R}$, is called the group of linear and bounded operators on X. If, in addition, we also have $s - \lim\limits_{t \to 0} T(t)x = x$ for every $x \in X$, in the strong operator topology, then $T(t)$ $(t \in \mathbb{R})$ is called a strongly continuous or C_0-group of linear and bounded operators on X. The operator A is defined as follows: The domain of A is the set $D(A) = \left\{ x \in X : \lim\limits_{h \to 0} \frac{T(h)x - x}{h} \text{ exists} \right\}$, and for every $x \in D(A)$, $Ax := \lim\limits_{h \to 0} \frac{T(h)x - x}{h}$, is called the infinitesimal generator of $T(t)$ $(t \in \mathbb{R})$.*

The group of operators $T(t)$ $(t \in \mathbb{R})$ on X is the solution of the abstract Cauchy problem $T'(t) = AT(t)$, $T(0) = I$. Here A is a linear and closed operator and the infinitesimal generator of $T(t)$ $(t \in \mathbb{R})$. Obviously, it holds $A = T'(0)$ on $D(A)$. It is known that a linear operator A is the infinitesimal generator of an uniformly continuous group $T(t)$ $(t \in \mathbb{R})$ (or semigroup $T(t)$ $(t \geq 0)$) of linear operators on a Banach space X if and only if A is a bounded linear operator ([5]). If X is a finite-dimensional Banach space, then any strongly continuous group of operators is an uniformly continuous group. In this case, $T(t) = \exp(tA) = \sum\limits_{k=0}^{\infty} \frac{t^k A^k}{k!}$ $(t \in \mathbb{R})$.

2. Results

In this section, for several given matrices $A \in M_2(\mathbb{R})$ we construct the groups of matrices $T(t) = \exp(tA) = \sum\limits_{k=0}^{\infty} \frac{t^k A^k}{k!}$ $(t \in \mathbb{R})$ in $M_2(\mathbb{R})$ (i.e. the groups of linear and bounded operators in $\mathbb{R}^2$). For this goal we use the matrix exponential, i.e.

Taylor's power series of the given matrix A. Since the given matrix A depends on one or more real parameters, therefore the obtained group also depends on these parameters. So we get the classes of groups that depend on one or more real parameters. We give our result in the form of the following theorem.

Theorem 2.1. *Real matrix:*

$$a)\ A=\begin{bmatrix}0 & a\\ 0 & 0\end{bmatrix};\ b)\ A=\begin{bmatrix}0 & 0\\ a & 0\end{bmatrix};\ c)\ A=\begin{bmatrix}a & a\\ a & a\end{bmatrix};$$

$$d)\ A=\begin{bmatrix}a & 2a\\ 4a & -a\end{bmatrix};$$

$$e)\ A=\begin{bmatrix}\pm a & 0\\ 0 & \pm a\end{bmatrix};\quad f)\ A=\begin{bmatrix}0 & b\\ \frac{a^2}{b} & 0\end{bmatrix}\ (a,b\neq 0);$$

$$g)\ A=\begin{bmatrix}a & b\\ \frac{c^2-a^2}{b} & -a\end{bmatrix}\ (b,c\neq 0);\quad h)\ A=\begin{bmatrix}\pm a & 0\\ b & \mp a\end{bmatrix}\ (a\neq 0);$$

$$i)\ A=\begin{bmatrix}\pm a & b\\ 0 & \mp a\end{bmatrix}\ (a\neq 0);\ j)\ A=\begin{bmatrix}0 & b\\ \frac{-a^2}{b} & 0\end{bmatrix}\ (a,b\neq 0);$$

$$k)\ A=\begin{bmatrix}a & b\\ \frac{-c^2-a^2}{b} & -a\end{bmatrix}\ (b,c\neq 0);$$

generates the group of real matrices $T(t)=\exp(tA)$ $(t\in\mathbb{R})$ *(i.e. the group of linear and bounded operators):*

$$a)\ T(t)=\begin{bmatrix}1 & at\\ 0 & 1\end{bmatrix};\ b)\ T(t)=\begin{bmatrix}1 & 0\\ at & 1\end{bmatrix};\ c)\ T(t)=\begin{bmatrix}\frac{1}{2}\left(e^{2t}+1\right) & \frac{1}{2}\left(e^{2t}-1\right)\\ \frac{1}{2}\left(e^{2t}-1\right) & \frac{1}{2}\left(e^{2t}+1\right)\end{bmatrix};$$

$$d)\ T(t)=\begin{bmatrix}\cosh(3at)+\frac{1}{3}\sinh(3at) & \frac{2}{3}\sinh(3at)\\ \frac{4}{3}\sinh(3at) & \cosh(3at)-\frac{1}{3}\sinh(3at)\end{bmatrix};\ e)\ T(t)=\begin{bmatrix}e^{\pm at} & 0\\ 0 & e^{\pm at}\end{bmatrix};$$

$$f)\ T(t)=\begin{bmatrix}\cosh(at) & \frac{b}{a}\sinh(at)\\ \frac{a}{b}\sinh(at) & \cosh(at)\end{bmatrix};$$

$$g)\ T(t)=\begin{bmatrix}\cosh(ct)+\frac{a}{c}\sinh(ct) & \frac{b}{c}\sinh(ct)\\ \frac{c^2-a^2}{bc}\sinh(ct) & \cosh(ct)-\frac{a}{c}\sinh(ct)\end{bmatrix};$$

$$h)\ T(t)=\begin{bmatrix}e^{\pm at} & 0\\ \frac{b}{a}\sinh(at) & e^{\mp at}\end{bmatrix};\ i)\ T(t)=\begin{bmatrix}e^{\pm at} & \frac{b}{a}\sinh(at)\\ 0 & e^{\mp at}\end{bmatrix};$$

$$j)\ T(t)=\begin{bmatrix}\cos(at) & \frac{b}{a}\sin(at)\\ \frac{-a}{b}\sin(at) & \cos(at)\end{bmatrix};\ k)\ T(t)=\begin{bmatrix}\cos(ct)+\frac{a}{c}\sin(ct) & \frac{b}{c}\sin(ct)\\ \frac{-c^2-a^2}{bc}\sin(ct) & \cos(ct)-\frac{a}{c}\sin(ct)\end{bmatrix}.$$

Proof. Denote: $T(t)=\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}$.

a) For $A=\begin{bmatrix}0 & a\\ 0 & 0\end{bmatrix}$ it is $A^2=0$. Therefore,

$$T(t)=\exp(tA)=I+tA=\begin{bmatrix}1 & at\\ 0 & 1\end{bmatrix}.$$

b) If $A = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$, then $A^2 = 0$. Hence,

$$T(t) = \exp(tA) = I + tA = \begin{bmatrix} 1 & 0 \\ at & 1 \end{bmatrix}.$$

c) For $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$ we have:$A^2 = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 4a^3 & 4a^3 \\ 4a^3 & 4a^3 \end{bmatrix}$, and so on. In general, it is $A^k = \begin{bmatrix} 2^{k-1}a^k & 2^{k-1}a^k \\ 2^{k-1}a^k & 2^{k-1}a^k \end{bmatrix}$ $(k \in \mathbb{N})$. Therefore,

$$T_{11} = T_{22} = 1+\sum_{k=1}^{\infty} \frac{t^k}{k!}\left(2^{k-1}a^k\right) = 1+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(2at)^k}{k!} = 1+\frac{1}{2}\left(e^{2at}-1\right) = \frac{1}{2}\left(e^{2at}+1\right).$$

$$T_{12} = T_{21} = \sum_{k=1}^{\infty} \frac{t^k}{k!}\left(2^{k-1}a^k\right) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(2at)^k}{k!} = \frac{1}{2}\left(e^{2at}-1\right).$$

Hence,

$$T(t) = \begin{bmatrix} \frac{1}{2}\left(e^{2t}+1\right) & \frac{1}{2}\left(e^{2t}-1\right) \\ \frac{1}{2}\left(e^{2t}-1\right) & \frac{1}{2}\left(e^{2t}+1\right) \end{bmatrix}.$$

d) If $A = \begin{bmatrix} a & 2a \\ 4a & -a \end{bmatrix}$, then $A^2 = \begin{bmatrix} 9a^2 & 0 \\ 0 & 9a^2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 9a^3 & 18a^3 \\ 36a^3 & -9a^3 \end{bmatrix}$, $A^4 = \begin{bmatrix} 81a^4 & 0 \\ 0 & 81a^4 \end{bmatrix}$, and so on. One can prove that

$$A^{2k-1} = \begin{bmatrix} 3^{2k-2}a^{2k-1} & 2\cdot 3^{2k-2}a^{2k-1} \\ 4\cdot 3^{2k-2}a^{2k-1} & -3^{2k-2}a^{2k-1} \end{bmatrix} = 3^{2k-2}a^{2k-1}\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \quad (k \in \mathbb{N})$$

and

$$A^{2k} = \begin{bmatrix} 3^{2k}a^{2k} & 0 \\ 0 & 3^{2k}a^{2k} \end{bmatrix} \quad (k \in \mathbb{N}).$$

Therefore,

$$\begin{aligned} T_{11} &= 1+\sum_{k=1}^{\infty}\frac{t^{2k}}{(2k)!}\left(3^{2k}a^{2k}\right)+\sum_{k=1}^{\infty}\frac{t^{2k-1}}{(2k-1)!}\left(3^{2k-2}a^{2k-1}\right) \\ &= 1+\sum_{k=1}^{\infty}\frac{(3at)^{2k}}{(2k)!}+\frac{1}{3}\sum_{k=1}^{\infty}\frac{(3at)^{2k-1}}{(2k-1)!} = \cosh(3at)+\frac{1}{3}\sinh(3at), \end{aligned}$$

$$\begin{aligned} T_{22} &= 1+\sum_{k=1}^{\infty}\frac{t^{2k}}{(2k)!}\left(3^{2k}a^{2k}\right)-\sum_{k=1}^{\infty}\frac{t^{2k-1}}{(2k-1)!}\left(3^{2k-2}a^{2k-1}\right) \\ &= 1+\sum_{k=1}^{\infty}\frac{(3at)^{2k}}{(2k)!}-\frac{1}{3}\sum_{k=1}^{\infty}\frac{(3at)^{2k-1}}{(2k-1)!} = \cosh(3at)-\frac{1}{3}\sinh(3at), \end{aligned}$$

$$T_{12} = 2\sum_{k=1}^{\infty} \frac{t^{2k-1}}{(2k-1)!}\left(3^{2k-2}a^{2k-1}\right) = \frac{2}{3}\sinh(3at) \text{ and } T_{21} = 2T_{12} = \frac{4}{3}\sinh(3at).$$

Therefore,

$$T(t) = \begin{bmatrix} \cosh(3at) + \frac{1}{3}\sinh(3at) & \frac{2}{3}\sinh(3at) \\ \frac{4}{3}\sinh(3at) & \cosh(3at) - \frac{1}{3}\sinh(3at) \end{bmatrix}.$$

e) For $A = \begin{bmatrix} \pm a & 0 \\ 0 & \pm a \end{bmatrix}$ we have $A^{2k} = \begin{bmatrix} a^{2k} & 0 \\ 0 & a^{2k} \end{bmatrix}$ and $A^{2k-1} = \begin{bmatrix} \pm a^{2k-1} & 0 \\ 0 & \pm a^{2k-1} \end{bmatrix}$ $(k \in \mathbb{N})$. Therefore, $T_{21} = T_{12} = 0$,

$$T_{11} = T_{22} = 1 + \sum_{k=1}^{\infty} \frac{a^{2k}t^{2k}}{(2k)!} \pm \sum_{k=1}^{\infty} \frac{a^{2k-1}t^{2k-1}}{(2k-1)!} = \cosh(at) \pm \sinh(at) = e^{\pm at} \text{ and}$$

$$T(t) = \begin{bmatrix} e^{\pm at} & 0 \\ 0 & e^{\pm at} \end{bmatrix}.$$

f) For $A = \begin{bmatrix} 0 & b \\ \frac{a^2}{b} & 0 \end{bmatrix}$ $(a, b \neq 0)$ it holds $A^2 = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & a^2 b \\ \frac{a^4}{b} & 0 \end{bmatrix}$, $A^4 = \begin{bmatrix} a^4 & 0 \\ 0 & a^4 \end{bmatrix}$, $A^5 = \begin{bmatrix} 0 & a^4 b \\ \frac{a^6}{b} & 0 \end{bmatrix}$, $A^6 = \begin{bmatrix} a^6 & 0 \\ 0 & a^6 \end{bmatrix}$, and so on. It is easy to prove that

$$A^{2k} = \begin{bmatrix} a^{2k} & 0 \\ 0 & a^{2k} \end{bmatrix} \text{ and } A^{2k-1} = \begin{bmatrix} 0 & a^{2k-2}b \\ \frac{a^{2k}}{b} & 0 \end{bmatrix} \ (k \in \mathbb{N}).$$

Now we have

$$\begin{aligned} T_{11} &= T_{22} = 1 + \sum_{k=1}^{\infty} \frac{a^{2k}t^{2k}}{(2k)!} = \cosh(at),\ T_{12} = \sum_{k=1}^{\infty} \frac{a^{2k-2}bt^{2k-1}}{(2k-1)!} = \frac{b}{a}\sinh(at), \\ T_{21} &= \sum_{k=1}^{\infty} \frac{a^{2k}}{b}\frac{t^{2k-1}}{(2k-1)!} = \frac{a}{b}\sinh(at). \end{aligned}$$

Hence,

$$T(t) = \begin{bmatrix} \cosh(at) & \frac{b}{a}\sinh(at) \\ \frac{a}{b}\sinh(at) & \cosh(at) \end{bmatrix}.$$

g) If $A = \begin{bmatrix} a & b \\ \frac{c^2-a^2}{b} & -a \end{bmatrix}$ $(b, c \neq 0)$, then $A^2 = \begin{bmatrix} c^2 & 0 \\ 0 & c^2 \end{bmatrix}$, $A^3 = \begin{bmatrix} ac^2 & bc^2 \\ \frac{c^2(c^2-a^2)}{b} & -ac^2 \end{bmatrix}$, $A^4 = \begin{bmatrix} c^4 & 0 \\ 0 & c^4 \end{bmatrix}$, $A^5 = \begin{bmatrix} ac^4 & bc^4 \\ \frac{c^4(c^2-a^2)}{b} & -ac^4 \end{bmatrix}$, $A^6 = \begin{bmatrix} c^6 & 0 \\ 0 & c^6 \end{bmatrix}$, and so on. In general,

$$A^{2k} = \begin{bmatrix} c^{2k} & 0 \\ 0 & c^{2k} \end{bmatrix} \text{ and } A^{2k-1} = \begin{bmatrix} ac^{2k-2} & bc^{2k-2} \\ \frac{c^{2k-2}(c^2-a^2)}{b} & -ac^{2k-2} \end{bmatrix} \quad (k \in \mathbb{N}).$$

Therefore,

$$\begin{aligned}
T_{11} &= 1 + \sum_{k=1}^{\infty} \frac{c^{2k}t^{2k}}{(2k)!} + \sum_{k=1}^{\infty} \frac{ac^{2k-2}t^{2k-1}}{(2k-1)!} = \cosh(ct) + \frac{a}{c}\sinh(ct), \\
T_{22} &= 1 + \sum_{k=1}^{\infty} \frac{c^{2k}t^{2k}}{(2k)!} - \sum_{k=1}^{\infty} \frac{ac^{2k-2}t^{2k-1}}{(2k-1)!} = \cosh(ct) - \frac{a}{c}\sinh(ct), \\
T_{12} &= \sum_{k=1}^{\infty} \frac{bc^{2k-2}t^{2k-1}}{(2k-1)!} = \frac{b}{c}\sinh(ct), T_{21} = \sum_{k=1}^{\infty} \frac{c^{2k-2}\left(c^2-a^2\right)}{b} \frac{t^{2k-1}}{(2k-1)!} = \frac{c^2-a^2}{bc}\sinh(ct), \\
T(t) &= \begin{bmatrix} \cosh(ct) + \frac{a}{c}\sinh(ct) & \frac{b}{c}\sinh(ct) \\ \frac{c^2-a^2}{bc}\sinh(ct) & \cosh(ct) - \frac{a}{c}\sinh(ct) \end{bmatrix}.
\end{aligned}$$

h) For $A = \begin{bmatrix} \pm a & 0 \\ b & \mp a \end{bmatrix}$ $(a \neq 0)$ we have $A^2 = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$, $A^3 = \begin{bmatrix} \pm a^3 & 0 \\ a^2 b & \mp a^3 \end{bmatrix}$, $A^4 = \begin{bmatrix} a^4 & 0 \\ 0 & a^4 \end{bmatrix}$, and so on. It is easy to see that

$$A^{2k} = \begin{bmatrix} a^{2k} & 0 \\ 0 & a^{2k} \end{bmatrix} \text{ and } A^{2k-1} = \begin{bmatrix} \pm a^{2k-1} & 0 \\ a^{2k-2}b & \mp a^{2k-1} \end{bmatrix} \quad (k \in \mathbb{N}).$$

Now we have

$$\begin{aligned}
T_{11} &= 1 + \sum_{k=1}^{\infty} \frac{a^{2k}t^{2k}}{(2k)!} \pm \sum_{k=1}^{\infty} \frac{a^{2k-1}t^{2k-1}}{(2k-1)!} = \cosh(at) \pm \sinh(at) = e^{\pm at}, \\
T_{12} &= 0, \, T_{21} = \sum_{k=1}^{\infty} \frac{a^{2k-2}bt^{2k-1}}{(2k-1)!} = \frac{b}{a}\sinh(at), \\
T_{11} &= 1 + \sum_{k=1}^{\infty} \frac{a^{2k}t^{2k}}{(2k)!} \mp \sum_{k=1}^{\infty} \frac{a^{2k-1}t^{2k-1}}{(2k-1)!} = \cosh(at) \mp \sinh(at) = e^{\mp at}, \\
T(t) &= \begin{bmatrix} e^{\pm at} & 0 \\ \frac{b}{a}\sinh(at) & e^{\mp at} \end{bmatrix}.
\end{aligned}$$

i) Analogously as in h), for $A = \begin{bmatrix} \pm a & b \\ 0 & \mp a \end{bmatrix}$ $(a \neq 0)$ we obtain $T(t) = \begin{bmatrix} e^{\pm at} & \frac{b}{a}\sinh(at) \\ 0 & e^{\mp at} \end{bmatrix}$.

j) If $A = \begin{bmatrix} 0 & b \\ \frac{-a^2}{b} & 0 \end{bmatrix}$ $(a, b \neq 0)$, then $A^2 = \begin{bmatrix} -a^2 & 0 \\ 0 & -a^2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & -a^2 b \\ \frac{a^4}{b} & 0 \end{bmatrix}$, $A^4 = \begin{bmatrix} a^4 & 0 \\ 0 & a^4 \end{bmatrix}$, $A^5 = \begin{bmatrix} 0 & a^4 b \\ \frac{-a^6}{b} & 0 \end{bmatrix}$, $A^6 = \begin{bmatrix} -a^6 & 0 \\ 0 & -a^6 \end{bmatrix}$, and so on. It can be proven that

$$A^{2k} = \begin{bmatrix} (-1)^k a^{2k} & 0 \\ 0 & (-1)^k a^{2k} \end{bmatrix} \text{ and } A^{2k-1} = \begin{bmatrix} 0 & (-1)^{k-1} a^{2k-2} b \\ \frac{(-1)^k a^{2k}}{b} & 0 \end{bmatrix} \ (k \in \mathbb{N}).$$

Therefore,

$$T_{11} = T_{22} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k a^{2k} t^{2k}}{(2k)!} = \cos(at),\ T_{12} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} a^{2k-2} b t^{2k-1}}{(2k-1)!} = \frac{b}{a}\sin(at),$$

$$T_{21} = \sum_{k=1}^{\infty} \frac{(-1)^k a^{2k}}{b} \frac{t^{2k-1}}{(2k-1)!} = \frac{-a}{b}\sin(at),\ T(t) = \begin{bmatrix} \cos(at) & \frac{b}{a}\sin(at) \\ \frac{-a}{b}\sin(at) & \cos(at) \end{bmatrix}.$$

k) For $A = \begin{bmatrix} a & b \\ \frac{-c^2-a^2}{b} & -a \end{bmatrix}$ $(b, c \neq 0)$ we have

$A^2 = \begin{bmatrix} -c^2 & 0 \\ 0 & -c^2 \end{bmatrix}$, $A^3 = \begin{bmatrix} -ac^2 & -bc^2 \\ \frac{c^2(c^2+a^2)}{b} & ac^2 \end{bmatrix}$, $A^4 = \begin{bmatrix} c^4 & 0 \\ 0 & c^4 \end{bmatrix}$,

$A^5 = \begin{bmatrix} ac^4 & bc^4 \\ \frac{-c^4(c^2+a^2)}{b} & -ac^4 \end{bmatrix}$, $A^6 = \begin{bmatrix} -c^6 & 0 \\ 0 & -c^6 \end{bmatrix}$, and so on.

In general, it holds

$$A^{2k} = \begin{bmatrix} (-1)^k c^{2k} & 0 \\ 0 & (-1)^k c^{2k} \end{bmatrix} \text{ and } A^{2k-1} = \begin{bmatrix} (-1)^{k-1} a c^{2k-2} & (-1)^{k-1} b c^{2k-2} \\ \frac{(-1)^k c^{2k-2}(c^2+a^2)}{b} & (-1)^k a c^{2k-2} \end{bmatrix} \ (k \in \mathbb{N}).$$

Now we have

$$T_{11} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k c^{2k} t^{2k}}{(2k)!} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} a c^{2k-2} t^{2k-1}}{(2k-1)!} = \cos(ct) + \frac{a}{c}\sin(ct),$$

$$T_{22} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k c^{2k} t^{2k}}{(2k)!} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1} a c^{2k-2} t^{2k-1}}{(2k-1)!} = \cos(ct) - \frac{a}{c}\sin(ct),$$

$$T_{12} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} b c^{2k-2} t^{2k-1}}{(2k-1)!} = \frac{b}{c} \sin(ct),$$

$$T_{21} = \sum_{k=1}^{\infty} \frac{(-1)^k c^{2k-2} \left(c^2 + a^2\right)}{b} \frac{t^{2k-1}}{(2k-1)!} = \frac{-c^2 - a^2}{bc} \sin(ct),$$

$$T(t) = \begin{bmatrix} \cos(ct) + \frac{a}{c}\sin(ct) & \frac{b}{c}\sin(ct) \\ \frac{-c^2-a^2}{bc}\sin(ct) & \cos(ct) - \frac{a}{c}\sin(ct) \end{bmatrix}.$$

□

Remark 2.1: It is known that from every group of operators $T(t)$ $(t \in \mathbb{R})$ one can obtain an appropriate cosine operator function $C(t)$ $(t \in \mathbb{R})$ in the same Banach space with $C(t) = \frac{1}{2}[T(t) + T(-t)]$ $(t \in \mathbb{R})$. Such family of operators satisfies $C(0) = I$ and $C(t+s) + C(t-s) = 2C(t)C(s)$ for every $t, s \in \mathbb{R}$. If A is generator of $T(t)$ $(t \in \mathbb{R})$, then A^2 is generator of $C(t)$ $(t \in \mathbb{R})$. Therefore we have the next corollary.

Corollary 2.1. *If the matrix A is the same as in Theorem 2.1, then the matrix A^2 generates the following cosine operator functions $C(t)$ $(t \in \mathbb{R})$ defined in $\mathbb{R}^2$:*

In a) and b), $C(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ (a trivial case of cosine operator function); in c), $C(t) = \begin{bmatrix} \frac{1}{2}(\cosh(2t)+1) & \frac{1}{2}(\cosh(2t)-1) \\ \frac{1}{2}(\cosh(2t)-1) & \frac{1}{2}(\cosh(2t)+1) \end{bmatrix}$; in d), $C(t) = \begin{bmatrix} \cosh(3at) & 0 \\ 0 & \cosh(3at) \end{bmatrix}$; in e), f), h) and i), $C(t) = \begin{bmatrix} \cosh(at) & 0 \\ 0 & \cosh(at) \end{bmatrix}$, in g), $C(t) = \begin{bmatrix} \cosh(ct) & 0 \\ 0 & \cosh(ct) \end{bmatrix}$, in j), $C(t) = \begin{bmatrix} \cos(at) & 0 \\ 0 & \cos(at) \end{bmatrix}$ and in k), $C(t) = \begin{bmatrix} \cos(ct) & 0 \\ 0 & \cos(ct) \end{bmatrix}$.

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