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CONTENT

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GROUPS OF OPERATORS IN $\mathbb{C}^2$ DETERMINED BY SOME COSINE OPERATOR FUNCTIONS IN $\mathbb{C}^2$

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ABSTRACT. In this paper we obtain all the groups of linear and bounded operators in $\mathbb{C}^2$ over the field $\mathbb{C}$ of complex numbers, determined by some concretely cosine operator functions in $\mathbb{C}^2$.

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1. INTRODUCTION AND SOME PRELIMINARIES

Many mathematicians have investigated the theory of semigroups and groups of linear and bounded operators in Banach space, or the theory of cosine operator functions in Banach space. For examples, see some of the references below. There exists a close correlation between the group of linear and bounded operators and the cosine operator function in the same Banach space.

Definition 1. The one-parameter family of linear and bounded operators $T(t)$ $(t \in \mathbb{R})$ defined from a Banach space X into X, which satisfies $T(0) = I$, the identity operator on X, and $T(t+s) = T(t)T(s)$ on X, for every $t, s \in \mathbb{R}$ is called the group of linear operators on X.

Definition 2. The one-parameter family of linear and bounded operators $C(t)$ $(t \in \mathbb{R})$ defined from a Banach space X into X, which satisfies $C(0) = I$, the identity operator on X, and $C(t+s) + C(t-s) = 2C(t)C(s)$ on X, for every $t, s \in \mathbb{R}$, is called the cosine operator function on X.

If in Definition 1 we have also $s-\lim_{t\to 0} T(t)x = x$ for every $x \in X$, then $T(t)$ $(t \in \mathbb{R})$ is called strongly continuous or C_0-group of linear operators on X. Here $s-\lim$ denotes the limit in strong operator topology in Banach space X. Analogously, if in Definition 2 we have also $s-\lim_{t\to 0} C(t)x = x$ for every $x \in X$, then $C(t)$ $(t \in \mathbb{R})$ is called strongly continuous or C_0- cosine operator function on X. The group of operators $T(t)$ $(t \in \mathbb{R})$ is the solution of the abstract Cauchy problem $T'(t) = AT(t),\ T(0) = I$. Here A is a linear and closed operator defined as an infinitesimal generator of $T(t)$ $(t \in \mathbb{R})$ as follows: The domain of A is the set

$$D(A) = \left\{ x \in X : \lim_{h\to 0} \frac{T(h)x - x}{h} \text{ exists} \right\}, \text{ and for every } x \in D(A),\ Ax := \lim_{h\to 0} \frac{T(h)x - x}{h}.$$

Obviously, it holds $A = T'(0)$ on $D(A)$. The cosine operator function $C(t)$ $(t \in \mathbb{R})$ is the solution of the abstract Cauchy problem $C''(t) = AC(t),\ C(0) = I, C'(0) = 0$. Here A is a linear and closed operator defined as an infinitesimal generator of $C(t)$ $(t \in \mathbb{R})$ as follows: The domain of A is the set

$$D(A) = \left\{ x \in X : \lim_{h\to 0} \frac{2\left(C(h)x - x\right)}{h^2} \text{ exists} \right\},$$

and for every $x \in D(A)$,

$$Ax := \lim_{h\to 0} \frac{2\left(C(h)x - x\right)}{h^2}.$$

Obviously, it holds $A = C''(0)$ on $D(A)$. If A is the infinitesimal generator of the cosine operator function $C(t)$ $(t \in \mathbb{R})$, then the square root of A is the infinitesimal generator of group $T(t) := C(t) + \sqrt{A}\, S(t)$ $(t \in \mathbb{R})$. Operator $\sqrt{A}$ is a linear and closed operator which satisfies $\left(\sqrt{A}\right)^2 = A$, and $S(t)$ $(t \in \mathbb{R})$ is an associated sine operator function on X defined as $S(t) = \int_0^t C(u)du$ $(t \in \mathbb{R})$. On the other hand, if operator A is the infinitesimal generator of group $T(t)$ $(t \in \mathbb{R})$, then $C(t) := \frac{1}{2}\left[T(t) + T(-t)\right]$ is a cosine operator function with infinitesimal generator A^2.

Every cosine operator function generates more groups of linear and bounded operators. We illustrate this in particular for the following cosine operator functions in $\mathbb{C}^2$,

$$C_1(t) = \begin{bmatrix} \cos at & 0 \\ 0 & \cos at \end{bmatrix} \text{ and } C_2(t) = \begin{bmatrix} \cosh at & 0 \\ 0 & \cosh at \end{bmatrix} \ (t \in \mathbb{R})\ (a \in \mathbb{R};\ a \neq 0).$$

2. RESULTS

In this main section, by using the relation $T(t)=C(t)+\sqrt{A}\,S(t)$ $(t\in\mathbb{R})$, we want to obtain all the groups of linear operators in $\mathbb{C}^2$ from the appropriate cosine operator functions defined in $\mathbb{C}^2$. Every linear operator in $\mathbb{C}^2$ can be identified by using the matrix of that operator. We cannot always use the relation above. For example, $C(t)=\begin{bmatrix}1 & 0\\ \frac{t^2}{2} & 1\end{bmatrix}$ $(t\in\mathbb{R})$ is a cosine operator function on C^2 with infinitesimal generator $A=C''(0)=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$. However, this matrix has no square root, so we cannot construct the group $T(t)=C(t)+\sqrt{A}\,S(t)$ $(t\in\mathbb{R})$. By using the group $T(t)=\begin{bmatrix}1 & 0\\ t & 1\end{bmatrix}$ $(t\in\mathbb{R})$, we can obtain the cosine operator function $C(t)=\frac{1}{2}[T(t)+T(-t)]=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$ $(t\in\mathbb{R})$, and this is a trivial example of a cosine operator function on $\mathbb{C}^2$. Also, $T(t)\equiv I$ is a trivial example of the group of linear operators on $\mathbb{C}^2$.

Now, consider the following cosine operator functions in $\mathbb{C}^2$:

$$C_1(t)=\begin{bmatrix}\cos at & 0\\ 0 & \cos at\end{bmatrix} \text{ and } C_2(t)=\begin{bmatrix}\cosh at & 0\\ 0 & \cosh at\end{bmatrix} \ (t\in\mathbb{R})\ (a\in\mathbb{R};\ a\neq 0).$$

Their infinitesimal generators are

$$A_1=C_1''(0)=\begin{bmatrix}-a^2 & 0\\ 0 & -a^2\end{bmatrix} \text{ and } A_2=C_2''(0)=\begin{bmatrix}a^2 & 0\\ 0 & a^2\end{bmatrix}.$$

Now, we need to calculate all the square roots of these operators.

Lemma 1. All the square roots of $A_1=\begin{bmatrix}-a^2 & 0\\ 0 & -a^2\end{bmatrix}$ $(a\in\mathbb{R};\ a\neq 0)$ have one of the forms:

$$\begin{bmatrix}\pm i\cdot a & 0\\ 0 & \pm i\cdot a\end{bmatrix},\ \begin{bmatrix}0 & B\\ \frac{-a^2}{B} & 0\end{bmatrix}\ (B\in\mathbb{C};\ B\neq 0),\ \begin{bmatrix}A & B\\ \frac{-a^2-A^2}{B} & -A\end{bmatrix},\ (A,B\in\mathbb{C};\ A,B\neq 0),$$

$$\begin{bmatrix}\pm i\cdot a & 0\\ C & \mp i\cdot a\end{bmatrix}\ (C\in\mathbb{C}) \quad\text{and}\quad \begin{bmatrix}\pm i\cdot a & B\\ 0 & \mp i\cdot a\end{bmatrix}\ (B\in\mathbb{C}).$$

Proof. Let $M=\sqrt{A_1}=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. From $M^2=\begin{bmatrix} A & B \\ C & D \end{bmatrix}^2=\begin{bmatrix} -a^2 & 0 \\ 0 & -a^2 \end{bmatrix}$ we obtain the system of equations:

$$\begin{aligned} &A^2+BC=-a^2 && (1)\\ &B(A+D)=0 && (2)\\ &C(A+D)=0 && (3)\\ &D^2+BC=-a^2 && (4) \end{aligned}$$

From (1) and (4) we conclude that $D^2=A^2$.

a) If we assume $D=A\neq 0$, then from (2) and (3) we have $B=C=0$. Therefore, $D=A=\pm i\cdot a$, i.e. $M=\begin{bmatrix} \pm i\cdot a & 0 \\ 0 & \pm i\cdot a \end{bmatrix}$.

b) If $D=A=0$, then, from $BC=-a^2$, we have $M=\begin{bmatrix} 0 & B \\ \frac{-a^2}{B} & 0 \end{bmatrix}$ $(B\in\mathbb{C};\ B\neq 0)$.

c) For $D=-A\neq 0$ we get $BC=-a^2-A^2$. Then,

$$M=\begin{bmatrix} A & B \\ \frac{-a^2-A^2}{B} & -A \end{bmatrix} \quad (A,B\in\mathbb{C};\ A,B\neq 0).$$

d) If $D=-A\neq 0$, $B=0$ or $C=0$, we get

$$M=\begin{bmatrix} \pm i\cdot a & 0 \\ C & \mp i\cdot a \end{bmatrix} \ (C\in\mathbb{C}) \quad \text{or} \quad M=\begin{bmatrix} \pm i\cdot a & B \\ 0 & \mp i\cdot a \end{bmatrix} \ (B\in\mathbb{C}) \ . \ \blacksquare$$

Lemma 2. All the square roots of $A_2=\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$ $(a\in\mathbb{R};\ a\neq 0)$ have one of the forms:

$$\begin{bmatrix} \pm a & 0 \\ 0 & \pm a \end{bmatrix},\ \begin{bmatrix} 0 & B \\ \frac{a^2}{B} & 0 \end{bmatrix} (B\in\mathbb{C};\ B\neq 0),\ \begin{bmatrix} A & B \\ \frac{a^2-A^2}{B} & -A \end{bmatrix} (A,B\in\mathbb{C};\ A,B\neq 0),$$

$$\begin{bmatrix} \pm a & 0 \\ C & \mp a \end{bmatrix} (C\in\mathbb{C}) \quad \text{and} \quad \begin{bmatrix} \pm a & B \\ 0 & \mp a \end{bmatrix} (B\in\mathbb{C}).$$

Proof. Let $M=\sqrt{A_2}=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. From $M^2=\begin{bmatrix} A & B \\ C & D \end{bmatrix}^2=\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$ we obtain the system of equations:

$$A^2 + BC = a^2 \qquad (1')$$
$$B(A + D) = 0 \qquad (2')$$
$$C(A + D) = 0 \qquad (3')$$
$$D^2 + BC = a^2 \qquad (4')$$

From (1') and (4') we conclude that $D^2 = A^2$.

a) If we assume $D = A \neq 0$, then from (2') and (3') we have $B = C = 0$. Therefore, $D = A = \pm a$, i.e. $M = \begin{bmatrix} \pm a & 0 \\ 0 & \pm a \end{bmatrix}$.

b) If $D = A = 0$, then, from $BC = a^2$, we have $M = \begin{bmatrix} 0 & B \\ \frac{a^2}{B} & 0 \end{bmatrix}$ $(B \in \mathbb{C};\ B \neq 0)$.

c) For $D = -A \neq 0$ we get $BC = a^2 - A^2$. Then,

$$M = \begin{bmatrix} A & B \\ \frac{a^2 - A^2}{B} & -A \end{bmatrix} \quad (A, B \in \mathbb{C};\ A, B \neq 0).$$

d) If $D = -A \neq 0$, $B = 0$ or $C = 0$, we get

$$M = \begin{bmatrix} \pm a & 0 \\ C & \mp a \end{bmatrix} \ (C \in \mathbb{C}) \ \text{ or } \ M = \begin{bmatrix} \pm a & B \\ 0 & \mp a \end{bmatrix} \ (B \in \mathbb{C}). \ \blacksquare$$

Theorem 1. All the groups $T(t)$ $(t \in \mathbb{R})$ of linear and bounded operators in $\mathbb{C}^2$ determined by cosine operator function

$$C(t) = \begin{bmatrix} \cos at & 0 \\ 0 & \cos at \end{bmatrix} \ (t \in \mathbb{R}) \ (a \in \mathbb{R};\ a \neq 0)$$

have one of the forms:

$$\begin{bmatrix} e^{\pm i \cdot at} & 0 \\ 0 & e^{\pm i \cdot at} \end{bmatrix}, \quad \begin{bmatrix} \cos at & \frac{B}{a} \sin at \\ \frac{-a}{B} \sin at & \cos at \end{bmatrix} \ (B \in \mathbb{C}; B \neq 0),$$

$$\begin{bmatrix} \cos at + \frac{A}{a} \sin at & \frac{B}{a} \sin at \\ \frac{-a^2 - A^2}{aB} \sin at & \cos at - \frac{A}{a} \sin at \end{bmatrix} \ (A, B \in \mathbb{C};\ A, B \neq 0),$$

$$\begin{bmatrix} e^{\pm i \cdot at} & 0 \\ \frac{C}{a} \sin at & e^{\mp i \cdot at} \end{bmatrix} \ (C \in \mathbb{C}) \ \text{ and } \ \begin{bmatrix} e^{\pm i \cdot at} & \frac{B}{a} \sin at \\ 0 & e^{\mp i \cdot at} \end{bmatrix} \ (B \in \mathbb{C}).$$

Proof. We use the square roots obtained in Lemma 1. Also, we use the formula $T(t)=C(t)+\sqrt{A}\,S(t)$ $(t\in\mathbb{R})$ and obtain:

a) $T(t)=\begin{bmatrix}\cos at & 0\\ 0 & \cos at\end{bmatrix}+\begin{bmatrix}\pm i\cdot a & 0\\ 0 & \pm i\cdot a\end{bmatrix}\cdot\frac{1}{a}\begin{bmatrix}\sin at & 0\\ 0 & \sin at\end{bmatrix}=\begin{bmatrix}e^{\pm i\cdot at} & 0\\ 0 & e^{\pm i\cdot at}\end{bmatrix}.$

b) $T(t)=\begin{bmatrix}\cos at & 0\\ 0 & \cos at\end{bmatrix}+\begin{bmatrix}0 & B\\ \frac{-a^2}{B} & 0\end{bmatrix}\cdot\frac{1}{a}\begin{bmatrix}\sin at & 0\\ 0 & \sin at\end{bmatrix}=\begin{bmatrix}\cos at & \frac{B}{a}\sin at\\ \frac{-a}{B}\sin at & \cos at\end{bmatrix}$

$(B\in\mathbb{C};\ B\neq 0)$. Specially, in $\mathbb{R}^2$, for $B=\pm a$, this group is the group of rotations.

c) $T(t)=\begin{bmatrix}\cos at & 0\\ 0 & \cos at\end{bmatrix}+\begin{bmatrix}A & B\\ \frac{-a^2-A^2}{B} & -A\end{bmatrix}\cdot\frac{1}{a}\begin{bmatrix}\sin at & 0\\ 0 & \sin at\end{bmatrix}$

$=\begin{bmatrix}\cos at+\frac{A}{a}\sin at & \frac{B}{a}\sin at\\ \frac{-a^2-A^2}{aB}\sin at & \cos at-\frac{A}{a}\sin at\end{bmatrix}$ $(A,B\in\mathbb{C};\ A,B\neq 0)$.

d) $T(t)=\begin{bmatrix}\cos at & 0\\ 0 & \cos at\end{bmatrix}+\begin{bmatrix}\pm i\cdot a & 0\\ C & \mp i\cdot a\end{bmatrix}\cdot\frac{1}{a}\begin{bmatrix}\sin at & 0\\ 0 & \sin at\end{bmatrix}=\begin{bmatrix}e^{\pm i\cdot at} & 0\\ \frac{C}{a}\sin at & e^{\mp i\cdot at}\end{bmatrix}$ $(C\in\mathbb{C})$

and $T(t)=\begin{bmatrix}\cos at & 0\\ 0 & \cos at\end{bmatrix}+\begin{bmatrix}\pm i\cdot a & B\\ 0 & \mp i\cdot a\end{bmatrix}\cdot\frac{1}{a}\begin{bmatrix}\sin at & 0\\ 0 & \sin at\end{bmatrix}=\begin{bmatrix}e^{\pm i\cdot at} & \frac{B}{a}\sin at\\ 0 & e^{\mp i\cdot at}\end{bmatrix}$ $(B\in\mathbb{C})$.

It is easy to prove that in all cases it holds $T(0)=I$ and $T(t+s)=T(t)T(s)$ for all $t,s\in\mathbb{R}$. ■

Note that $C(t)=\begin{bmatrix}\cos at & 0\\ 0 & \cos at\end{bmatrix}$ can be obtained now in all cases as $C(t)=\frac{1}{2}[T(t)+T(-t)]$ $(t\in\mathbb{R})$.

Theorem 2. All the groups $T(t)$ $(t\in\mathbb{R})$ of linear and bounded operators in $\mathbb{C}^2$ determined by cosine operator function

$$C(t)=\begin{bmatrix}\cosh at & 0\\ 0 & \cosh at\end{bmatrix}\ (t\in\mathbb{R})\ (a\in\mathbb{R};\ a\neq 0)$$

have one of the forms:

$$\begin{bmatrix}e^{\pm at} & 0\\ 0 & e^{\pm at}\end{bmatrix},\quad \begin{bmatrix}\cosh at & \frac{B}{a}\sinh at\\ \frac{a}{B}\sinh at & \cosh at\end{bmatrix}\ (B\in\mathbb{C};\ B\neq 0),$$

$$\begin{bmatrix}\cosh at+\frac{A}{a}\sinh at & \frac{B}{a}\sinh at\\ \frac{a^2-A^2}{aB}\sinh at & \cosh at-\frac{A}{a}\sinh at\end{bmatrix}\ (A,B\in\mathbb{C};A,B\neq 0),$$

$$\begin{bmatrix}e^{\pm at} & 0\\ \frac{C}{a}\sinh at & e^{\mp at}\end{bmatrix}\ (C\in\mathbb{C})\ \text{ and }\ \begin{bmatrix}e^{\pm at} & \frac{B}{a}\sinh at\\ 0 & e^{\mp at}\end{bmatrix}\ (B\in\mathbb{C}).$$

Proof. We use the square roots obtained in Lemma 2. Also, we use the formula $T(t)=C(t)+\sqrt{A}\,S(t)\ (t\in\mathbb{R})$ and obtain:

a) $$T(t)=\begin{bmatrix}\cosh at & 0\\ 0 & \cosh at\end{bmatrix}+\begin{bmatrix}\pm a & 0\\ 0 & \pm a\end{bmatrix}\cdot\frac{1}{a}\begin{bmatrix}\sinh at & 0\\ 0 & \sinh at\end{bmatrix}=\begin{bmatrix}e^{\pm at} & 0\\ 0 & e^{\pm at}\end{bmatrix}.$$

b) $$T(t)=\begin{bmatrix}\cosh at & 0\\ 0 & \cosh at\end{bmatrix}+\begin{bmatrix}0 & B\\ \frac{a^2}{B} & 0\end{bmatrix}\cdot\frac{1}{a}\begin{bmatrix}\sinh at & 0\\ 0 & \sinh at\end{bmatrix}=\begin{bmatrix}\cosh at & \frac{B}{a}\sinh at\\ \frac{a}{B}\sinh at & \cosh at\end{bmatrix}$$

$(B\in\mathbb{C};\ B\neq 0)$.

c) $$T(t)=\begin{bmatrix}\cosh at & 0\\ 0 & \cosh at\end{bmatrix}+\begin{bmatrix}A & B\\ \frac{a^2-A^2}{B} & -A\end{bmatrix}\cdot\frac{1}{a}\begin{bmatrix}\sinh at & 0\\ 0 & \sinh at\end{bmatrix}$$

$$=\begin{bmatrix}\cosh at+\frac{A}{a}\sinh at & \frac{B}{a}\sinh at\\ \frac{a^2-A^2}{aB}\sinh at & \cosh at-\frac{A}{a}\sinh at\end{bmatrix}\ (A,B\in\mathbb{C};\ A,B\neq 0).$$

d) $$T(t)=\begin{bmatrix}\cosh at & 0\\ 0 & \cosh at\end{bmatrix}+\begin{bmatrix}\pm a & 0\\ C & \mp a\end{bmatrix}\cdot\frac{1}{a}\begin{bmatrix}\sinh at & 0\\ 0 & \sinh at\end{bmatrix}=\begin{bmatrix}e^{\pm at} & 0\\ \frac{C}{a}\sinh at & e^{\mp at}\end{bmatrix}\ (C\in\mathbb{C})$$

and $T(t)=\begin{bmatrix}\cosh at & 0\\ 0 & \cosh at\end{bmatrix}+\begin{bmatrix}\pm a & B\\ 0 & \mp a\end{bmatrix}\cdot\frac{1}{a}\begin{bmatrix}\sinh at & 0\\ 0 & \sinh at\end{bmatrix}=\begin{bmatrix}e^{\pm at} & \frac{B}{a}\sinh at\\ 0 & e^{\mp at}\end{bmatrix}$ $(B\in\mathbb{C})$.

It is easy to prove that in all cases it holds $T(0)=I$ and $T(t+s)=T(t)T(s)$ for all $t,s\in\mathbb{R}$.■

Note that $C(t)=\begin{bmatrix}\cosh at & 0\\ 0 & \cosh at\end{bmatrix}$ can be obtained now in all cases as $C(t)=\frac{1}{2}[T(t)+T(-t)]$ $(t\in\mathbb{R})$.

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