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Address of the editorial office
Goce Delcev University – Stip
Faculty of philology
Krste Misirkov 10-A
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ABOUT ONE B.S. POPOV'S RESULT

BORO M. PIPEREVSKI AND BILJANA ZLATANOVSKA

Abstract. In this paper, a hypergeometric homogeneous linear differential equation of second order is considered. The application of the transformation method yields conditions for reductability according to Frobenius, formulas for the general solution, as well as the corresponding systems of first-order differential equations.

1. Introduction

In this paper, we consider a class of linear homogeneous differential equations of the second order of type

$$Ay'' + By' + Cy = 0, \tag{1.1}$$

where

$$A = a_2x^2 + a_1x + a_0, \quad B = b_1x + b_0, \quad C = c_0, \quad a_2, a_1, a_0, b_1, b_0, c_0 \in \mathbb{R}.$$

Considering a homogeneous linear differential equation of the second order of type

$$P_2(x)y'' + P_1(x)y' + \lambda_n P_0(x)y = 0$$

where $P_i(x), (i = 0,1,2)$ are polynomials and λ_n is a parameter. According to Brenke [11], this equation will have polynomial solutions of degree n for each $n \in \mathbb{N}$ with an appropriate value of the parameter λ_n, if $P_2(x), P_1(x)$, and $P_0(x)$ are polynomials of second, first and zero degree respectively. Also, the general formula for the series of polynomial solutions of the equation as well as certain conditions for their orthogonality with appropriate weight are proved.

Note that in the case when all members of the sequence $(\lambda_n), n = 0,1,2,\ldots$, are different, then λ_n are called their own values, and the polynomials y_n own functions.

Special cases of such known orthogonal polynomials are the polynomials of Legendre, Jacobi, Tschebyscheff, Hermite, Laguerre and others, which are used in numerical mathematics.

The special class of differential equations (1.1) is obtained when the Laplace partial differential equation is converted into spherical coordinates with request the solution to be a product of functions that depend on only one variable.

Let us mention the classic results regarding polynomial solutions of the very important hypergeometric differential equation, as an equation with polynomial coefficients. Its solutions are special functions, especially the Jacobi, Legendre, Tschebyscheff polynomials, which belong to the class of classical orthogonal polynomials for which there are corresponding formulas, based on Rodrigues' famous formula.

In fact, this formula was obtained by Rodrigues O. in 1814 for a polynomial solution of a special differential equation of Legendre, but there are the other classical polynomials that are expressed in a similar way.

In the literature, the necessary and sufficient condition for which the equation (1.1) has a particular solution as a polynomial of degree n is known.

Lema 1.1. Equation (1.1) has a polynomial solution of degree n, if and only if there exists a natural number n, the smaller if there are two, which satisfies the condition

$$n(n-1)a_2 + nb_1 + c_0 = 0.$$

In that case, the polynomial solution of degree n is given by the formula

$$P_n = Ae^{-\int\frac{B}{A}dx}(A^{n-1}e^{\int\frac{B}{A}dx})^{(n)},$$

This formula is called the Rodrigues formula. The general solution will be given by the formula

$$y = C_1Ae^{-\int\frac{B}{A}dx}(A^{n-1}e^{\int\frac{B}{A}dx})^{(n)} + C_2Ae^{-\int\frac{B}{A}dx}(A^{n-1}e^{\int\frac{B}{A}dx}\int A^{n-1}e^{\int\frac{B}{A}dx}dx)^{(n)}$$

where C_1, C_2 are arbitrary constants.

Remark 1.1. The term reductability of linear homogeneous differential equations has two interpretations. Reduction in a wider system is a reduction of an equation of a system of linear homogeneous differential equations of lower order and that reduction can be more significant, i.e. it can be reduced to multiple classes of linear homogeneous differential equations systems of a lower order.

Definition 1.1. (Frobenius): A linear homogeneous differential equation whose coefficients are unambiguous functions is called more predictable according to Frobenius if there is no common solution with a linear homogeneous differential equation with coefficients unambiguous lower order functions. Otherwise, it is called reductive, according to Frobenius.

Let the differential equation (1.1) have one particular solution F. In [3] it is shown that the equation (1.1) is reductive according to Frobenius and it comes down to the system of first-order differential equations,

$$Fy' - F'y = z$$

$$Az' + Bz = 0 \qquad (1.3)$$

Let us now consider the class of differential equations (1.1) where the coefficient $A = a_2x^2 + a_1x + a_0$ has two real and different roots $x_1, x_2, x_1 \neq x_2$ and let $a_2 = 1$.

For the resulting equation

$$y'' + (\frac{p}{x-x_1} + \frac{q}{x-x_2})y' + \frac{r}{(x-x_1)(x-x_2)}y = 0 \qquad (1.4)$$

or

$$(x-x_1)(x-x_2)y'' + [(p+q)x - px_2 - qx_1]y' + ry = 0,$$

where

$$p = \frac{b_1x_1 + b_0}{x_1 - x_2}, q = \frac{b_1x_2 + b_0}{x_2 - x_1}, r = c_0$$

in [6] by the method of transformations, the following two theorems are obtained.

Theorem 1.1. Let a differential equation (1.4) be given. Let $t = a$ be the root of the characteristic equation $t^2 + (p+q-1)t + r = 0$. If the root satisfies one of the conditions:

1^0 $a \in \mathbb{N}$ is the smaller root if both roots are natural numbers;
2^0 $a + p - 1 \in \mathbb{N}$ or $-(a+q) \in \mathbb{N}$ is the smaller root if both roots are natural numbers;
3^0 $a + q - 1 \in \mathbb{N}$ or $-(a+1) \in \mathbb{N}$ is the smaller root if both roots are natural numbers;
4^0 $a + p + q - 2 \in \mathbb{N}$ or $-(a+q) \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

then the equation (1.4) can be integrated into a closed form.

Theorem 1.2. If one of the conditions 1^0 to 4^0 of Theorem 1.1. for the Equation (1.4) is satisfied, then the general solution is

$$1^0\ y = (x-x_1)^{1-p}(x-x_2)^{1-q} \cdot \{(x-x_1)^{n+p-1}(x-x_2)^{n+q-1}[C_1 + C_2\int (x-x_1)^{-n-p}(x-x_2)^{-n-q}dx]\}^{(n)}$$

where $n = a \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

$$2^0\ y = (x-x_2)^{1-q}\{(x-x_1)^{n-p+1}(x-x_2)^{n+q-1}[C_1 + C_2\int (x-x_1)^{-n+p-2}(x-x_2)^{-n-q}dx]\}^{(n)}$$

where $n = a + p - 1 \in \mathbb{N}$ or $n = -(a+q) \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

$$3^0\ y = (x-x_1)^{1-p}\{(x-x_1)^{n+p-1}(x-x_2)^{n-q+1}[C_1 + C_2\int (x-x_1)^{-n-p}(x-x_2)^{-n+q-2}dx]\}^{(n)}$$

where $n = a + q - 1 \in \mathbb{N}$ or $n = -(a+p) \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

$$4^0\ y = \{(x-x_1)^{n-p+1}(x-x_2)^{n-q+1}[C_1 + C_2\int (x-x_1)^{-n+p-2}(x-x_2)^{-n+q-2}dx]\}^{(n)}$$

where $n = a + p + q - 2 \in \mathbb{N}$ or $n = -(a+1) \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

for C_1, C_2 are arbitrary constants.

In case 1^0, the differential equation (1.4) has one polynomial solution which is given by Rodrigues' formula. From the formula (1.2), the general solution is obtained.

A special type of the equation (1.4) is the hypergeometric differential equation

$$x(x-1)y^n + [(\alpha+\beta+1)x-\gamma]y' + \alpha\beta y = 0 \quad (1.5)$$

where

$x_1 = 1, x_2 = 0, b_1 = \alpha+\beta+1, b_0 = -\gamma, c_0 = \alpha\beta, p = \alpha+\beta+1-\gamma, q = \gamma, r = \alpha\beta.$

In [7], B.S Popov examines the equation (1.5) and, by applying Mitrinovic's method with operator equations, he gets general conditions for its reductive according to Frobenius.

Theorem 1.3. The equation (1.5) is reducible according to Frobenius if and only if

$$\alpha \in \mathbb{Z} \text{ or } \beta \in \mathbb{Z} \text{ or } \gamma-\alpha \in \mathbb{Z} \text{ or } \gamma-\beta \in \mathbb{Z} \quad (1.6)$$

Frobenius, Picard, and Goursat [8],[9],[10] have obtained the same result for the hypergeometric equation in another way.

In [7], operator equations are obtained. For these, there is no explicit formula for a particular solution.

2. Main result

Theorem 2.1. The equation (1.4) is reducible according to Frobenius if and only if the conditions of Theorem 1.1 are fulfilled. According to this, the following appropriate reductive systems of differential equations are obtained

$$1^0\{(x-x_1)^{1-p}(x-x_2)^{1-q}\cdot[(x-x_1)^{n+p-1}\cdot(x-x_2)^{n+q-1}]^{(n)}\}\cdot y' +$$
$$\{(x-x_1)^{1-p}(x-x_2)^{1-q}\cdot[(x-x_1)^{n+p-1}\cdot(x-x_2)^{n+q-1}]^{(n)}\}'\cdot y = z$$
$$(x-x_1)(x-x_2)z' + [(p+q)x - px_2 - qx_1]z = 0$$

$$2^0\{(x-x_2)^{1-q}\cdot[(x-x_1)^{n-p+1}\cdot(x-x_2)^{n+q-1}]^{(n)}\}\cdot y' +$$
$$\{(x-x_2)^{1-q}\cdot[(x-x_1)^{n-p+1}\cdot(x-x_2)^{n+q-1}]^{(n)}\}'\cdot y = z$$
$$(x-x_1)(x-x_2)z' + [(p+q)x - px_2 - qx_1]z = 0$$

$$3^0\{(x-x_1)^{1-p}\cdot[(x-x_1)^{n+p-1}\cdot(x-x_2)^{n-q+1}]^{(n)}\}\cdot y' +$$
$$\{(x-x_1)^{1-p}\cdot[(x-x_1)^{n+p-1}\cdot(x-x_2)^{n-q+1}]^{(n)}\}'\cdot y = z$$
$$(x-x_1)(x-x_2)z' + [(p+q)x - px_2 - qx_1]z = 0$$

$$4^0 \quad \{[(x-x_1)^{n-p+1}\cdot(x-x_2)^{n-q+1}]^{(n)}\}\cdot y' + \{[(x-x_1)^{n-p+1}\cdot(x$$
$$-x_2)^{n-q+1}]^{(n)}\}'\cdot y = z$$
$$(x-x_1)(x-x_2)z' + [(p+q)x - px_2 - qx_1]z = 0.$$

Proof. According to Theorem 1.2., the formula for one particular solution is obtained and, in accordance with the system (1.3), the corresponding reductive systems differential equations are obtained.

Let the equation (1.5) be given. Next, we will show that the result obtained in [7] can be obtained in another way, different from the ways known in the literature. In addition, the formulas for the general solution are additionally obtained, as well as the corresponding reductive systems, first-order differential equations.

Theorem 2.2. The hypergeometric equation (1.5) can be integrated into a closed form if the following conditions are satisfied:
1^0 $-\beta \in \mathbb{N}$or $-\alpha \in \mathbb{N}$ is the smaller root if both roots are natural numbers;
2^0 $\alpha - \gamma \in \mathbb{N}$ or $\beta - \gamma \in \mathbb{N}$ is the smaller root if both roots are natural numbers;
3^0 $\gamma - \alpha - 1 \in \mathbb{N}$ or $\gamma - \beta - 1 \in \mathbb{N}$ is the smaller root if both roots are natural numbers;
4^0 $\alpha - 1 \in \mathbb{N}$ or $\beta - 1 \in \mathbb{N}$ is the smaller root if both roots are natural numbers.
.

Proof. If we put in Theorem 1.1.

$$x_1 = 1, x_2 = 0, b_1 = \alpha + \beta + 1, b_0 = -\gamma, c_0 = \alpha\beta, p = \alpha + \beta + 1 - \gamma, q = \gamma, r = \alpha\beta$$
$$t^2 + (\alpha + \beta)t + \alpha\beta = 0, t_1 = -\alpha, t_2 = -\beta$$

then the theorem is proved.

Consequence 2.1. The equation (1.5) can be integrated into a closed form if the conditions (1.6) are satisfied.

Really, Theorem 2.2. (properties 1^0 and 4^0) implies $\alpha, \beta \in \mathbb{Z}$ and Theorem 2.2. (properties 2^0 and 3^0) implies $\gamma - \alpha, \gamma - \beta \in \mathbb{Z}$. So, we get the condition (1.6) which is obtained in [7],[8],[9],[10] as a condition for reduction according to Frobenius.

Theorem 2.3. Let the conditions from 1^0 to 4^0 of Theorem 2.2 for the Equation (1.5) be satisfied. Then the general solution is

1^0 $y = (x-1)^{1-p} \cdot x^{1-q} \cdot \{(x-1)^{n+p-1} \cdot x^{n+q-1} \cdot [C_1 + C_2 \int (x-1)^{-n-p} \cdot x^{-n-q} dx]\}^{(n)}$
where $n = -\beta \in \mathbb{N}$ or $n = -\alpha \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

2^0 $y = x^{1-q} \cdot \{(x-1)^{n-p+1} \cdot x^{n+q-1} \cdot [C_1 + C_2 \int (x-1)^{-n+p-2} \cdot x^{-n-q} dx]\}^{(n)}$
where $n = \alpha - \gamma \in \mathbb{N}$ or $n = \beta - \gamma \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

3^0 $y = (x-1)^{1-p} \cdot \{(x-1)^{n+p-1} \cdot x^{n-q+1} \cdot [C_1 + C_2 \int (x-1)^{-n-p} \cdot x^{-n+q-2} dx]\}^{(n)}$

where $n = \gamma - \alpha - 1 \in \mathbb{N}$ or $n = \gamma - \beta - 1 \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

4^0 $y = \{(x-1)^{n-p+1} \cdot x^{n-q+1} \cdot [C_1 + C_2 \int (x-1)^{-n+p-2} \cdot x^{-n+q-2} dx]\}^{(n)}$
where $n = \alpha - 1 \in \mathbb{N}$ or $n = \beta - 1 \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

for $p = \alpha + \beta + 1 - \gamma,\ q = \gamma$ and C_1, C_2 are arbitrary constants.

Proof. If it is put in Theorem 1.2.

$$x_1 = 1, x_2 = 0, b_1 = \alpha + \beta + 1, b_0 = -\gamma, c_0 = \alpha\beta, p = \alpha + \beta + 1 - \gamma, q = \gamma, r = \alpha\beta$$

then the claim is proved.

Theorem 2.4. Let the conditions from 1^0 to 4^0 of Theorem 2.2. for the Equation (1.5) be satisfied. Then it is reducible according to Frobenius and comes down to the next system of differential equations
1^0

$$P_n y' - P_n' y = z$$
$$x(x-1)z' + [(\alpha + \beta + 1)x - \gamma]z = 0$$

where $P_n = (x-1)^{1-p} \cdot x^{1-q} \cdot [(x-1)^{n+p-1} \cdot x^{n+q-1}]^{(n)}$ and $n = -\beta \in \mathbb{N}$ or $n = -\alpha \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

2^0

$$Q_n y' - Q_n' y = z$$
$$x(x-1)z' + [(\alpha + \beta + 1)x - \gamma]z = 0$$

where $Q_n = x^{1-q} \cdot [(x-1)^{n-p+1} \cdot x^{n+q-1}]^{(n)}$ $n = \alpha - \gamma \in \mathbb{N}$ or $n = \beta - \gamma \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

3^0

$$R_n y' - R_n' y = z$$
$$x(x-1)z' + [(\alpha + \beta + 1)x - \gamma]z = 0$$

where $R_n = (x-1)^{1-p} \cdot [(x-1)^{n+p-1} \cdot x^{n-q+1}]^{(n)}$ and $n = \gamma - \alpha - 1 \in \mathbb{N}$ or $n = \gamma - \beta - 1 \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

4^0

$$S_n y' - S_n' y = z$$
$$x(x-1)z' + [(\alpha + \beta + 1)x - \gamma]z = 0$$

where $S_n = [(x-1)^{n-p+1} \cdot x^{n-q+1}]^{(n)}$ and $n = \alpha - 1 \in \mathbb{N}$ or $n = \beta - 1 \in \mathbb{N}$ is the smaller root if both roots are natural numbers;

for $p = \alpha + \beta + 1 - \gamma, q = \gamma$.

Proof. According to Theorem 2.3., the formula for one particular solution is obtained. In accordance with the system (1.3), the corresponding reductive systems differential equations are obtained.

To illustrate equation (1.5), we will give the system obtained in [7] for $\alpha = -n$

$$x \cdot y' - n\frac{F(1-n, 1-\gamma-n, 1-\beta-n, \frac{1}{x})}{F(-n, 1-\gamma-n, 1-\beta-n, \frac{1}{x})} \cdot y = z$$

$$(x-1) \cdot z' + \left[(x-1)\left(lg\, F\,(-n, 1-\gamma-n, 1-\beta-n, \frac{1}{x})\right)' + \frac{n(x-1)}{x} + \frac{1-\gamma-(n-\beta)x}{x}\right] z = 0$$

where $F(\alpha, \beta, \gamma, x)$ is the known hypergeometric function.

Example 2.1. Let the differential equation be given in the form of

$$x(x-1)y'' - (6x+1)y' + 12y = 0$$

where $\alpha = -3, \beta = -4, \gamma = 1, n_1 = -\alpha = 3, n_2 = -\beta = 4, p = -7, q = 1.$

From the property 1^0 of Theorem 2.3, the general solution is

$$y = C_1(4x^3 + 18x^2 + 12x + 1) + C_2(x-1)^8 x^0\{(x-1)^{-5}[\int (x-1)^4 x^{-4} dx]\}'''$$

From Theorem 2.4. the system is

$$(4x^3 + 18x^2 + 12x + 1)y' - (12x^2 + 36x + 12)y = z$$
$$x(x-1)z' - (6x+1)z = 0$$

Example 2.2. Let the differential equation be given in the form of

$$x(x-1)y'' + (7x-1)y' + 8y = 0$$

where $\alpha = 2, \beta = 4, \gamma = 1, n_1 = 1, n_2 = 3, p = 6, q = 1.$
From the property 2^0 or 4^0 of Theorem 2.3., the general solution is

$$y = C_1 \frac{3x+1}{(x-1)^5} + C_2 \frac{x^3-9x^2-18x+14+18x \cdot \ln x + 6 \ln x}{(x-1)^5}.$$

From Theorem 2.4., the system is

$$\frac{3x+1}{(x-1)^5} y' - \left(\frac{3x+1}{(x-1)^5}\right)' y = z$$
$$x(x-1)z' + (7x-1)z = 0$$

or

$$\frac{3x+1}{(x-1)^5} y' + \frac{4(3x+2)}{(x-1)^6} y = z$$
$$x(x-1)z' + (7x-1)z = 0.$$

Example 2.3. Let the differential equation be given in the form of

$$x(x-1)y'' + (x - \frac{3}{2})y' - \frac{9}{4} y = 0$$

where $\alpha = -\frac{3}{2}, \beta = \frac{3}{2}, \gamma = \frac{3}{2}, n_1 = \gamma - \alpha - 1 = 2, n_2 = \gamma - \beta - 1 = -1, p = -\frac{1}{2}, q = \frac{3}{2}$.

From the property 3^0 of Theorem 2.3., the general solution is

$$y = C_1 \frac{8x^2 - 12x + 3}{\sqrt{x}} + C_2 (x-1)^{\frac{3}{2}} \{(x-1)^{\frac{1}{2}} \cdot x^{\frac{3}{2}} [\int (x-1)^{-\frac{3}{2}} \cdot x^{-\frac{5}{2}} dx]\}''$$

or

$$y = C_1 \frac{8x^2 - 12x + 3}{\sqrt{x}} + C_2 (x-1)^{\frac{3}{2}}$$

From Theorem 2.4., the system is

$$\frac{8x^2 - 12x + 3}{\sqrt{x}} y' - \left(\frac{8x^2 - 12x + 3}{\sqrt{x}}\right)' y = z$$
$$x(x-1)z' + (x - \frac{3}{2})z = 0$$

or

$$\frac{8x^2 - 12x + 3}{\sqrt{x}} y' - \frac{3(8x^2 - 4x - 1)}{2x\sqrt{x}} y = z$$
$$x(x-1)z' + (x - \frac{3}{2})z = 0$$

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Boro M. Piperevski
Ss Ciril and Methodius University
Faculty of Electrical Engineering and Information Technologies
Karpos bb, P.O.Box 574, 1000 Skopje,
Republic of North Macedonia
e-mail: borom@feit.ukim.edu.mk

Biljana Zlatanovska
Goce Delcev University
Faculty of Computer Science
"Krste Misirkov" No. 10-A, 2000 Stip,
Republic of North Macedonia
E-mail address: biljana.zlatanovska@ugd.edu.mk