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CONTENT

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ON KURATOWSKI MEASURE OF NONCOMPACTNESS IN $\mathbb{R}^2$ WITH THE RIVER METRIC

NERMIN OKIČIĆ AND AMRA REKIĆ-VUKOVIĆ

Abstract. In this paper we consider some properties of the Kuratowski measure of noncompatness on the space $(\mathbb{R}^2, d^*)$, where d^* is river metric. We prove the existence of the α-minimal sets in the given space, but also the strict minimalizability of the Kuratowski measure of noncompactness.

1. Introduction

One of the most usual techniques used to prove that some operator equations have a solution is the formulation of the problem in the way that it becomes the fixed-point problem. In that case, we consider solvability of the fixed-point problem, e.g. we search for a possible fixed point. The measures of noncompactness play important role in the fixed-point theory and we find them very applicable in the various branches of nonlinear analysis, including difference equations, integral equations, optimization problems, etc. Roughly speaking, the measure of noncompactness is the function which associates a real number with an arbitrary bounded set of a metric space, in the way that a real number is nonnegative and is considered as a characteristic that tells us to which measure the set is not totally bounded, e.g. to which measure the set is not relatively compact in the case when the completeness of the space is supposed. In other words, the measure of noncompactness is a function that is equal to zero in the family of all relatively compact sets. Kuratowski (see [7]) was the first one who introduced the concept of the measure of noncompactness in 1930. In addition to this measure, many other measures of noncompactness were defined and considered, including the Hausdorff measure of noncompactness (see [5]) and Istratescu measure of noncompactness (see [6]).

On the other hand, the notion of ϕ-minimal set for the measure of noncompactness ϕ was introduced in [3] because of the study of the connection between the condensing operators for the Kuratowski and Hausdorff measures of noncompactness. Besides,

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the concept of the ϕ-minimal sets was also used to get simpler expressions for some moduli of noncompact convexity that are, by definition, associated with the concrete measure of noncompact-ness. Some properties of the moduli of noncompact convexity associated to an arbitrary (strictly) minimalizable measure of noncompactness were given in [11] and [9]. We are able to distinguish some classes of the measures of noncompactness bearing in mind in which way we are choosing the minimal sets in the given metric space. Precisely, the measure of noncompactness shows different behavior in relation to the concept of minimal sets. "The most beautiful" behavior of the measure of noncompactness happens when every bounded set A contains the ϕ-minimal subset B with an equal measure of noncompactness ϕ, that is subset for which the equality $\phi(A) = \phi(B)$ holds. Such measure of noncompactness is actually strictly minimalizable. Using the notion of the ϕ-minimal set, a new modulus of the noncompact convexity was introduced (see [10]).

The aim of this paper is to show that in the metric space $(\mathbb{R}^2, d^*)$, where d^* is the river metric, there exist α-minimal sets and that the Kuratowski measure of noncompactness α is strictly minimalizable on the considered space. This result will give us the existence of the β-minimal sets and strict minimalizability of the Istratescu measure of noncompactness β.

2. Definitions and notation

In this paper (X, d) denotes the metric space with the defined metric d, $B(x, r)$ an open ball centered at x of radius r. If $B \subset X$, we denote by $\overline{B}$ the closure of a set B.

Definition 2.1. *Let $\mathcal{B}$ be a family of bounded subsets of the metric space (X, d). We call mapping $\phi : \mathcal{B} \to [0, \infty)$ the measure of noncompactness defined on X if it satisfies the following:*

(1) $\phi(B) = 0 \Leftrightarrow B$ is a relatively compact set
(2) $\phi(B) = \phi(\overline{B})$, for all $B \in \mathcal{B}$
(3) $\phi(B_1 \cup B_2) = \max\{\phi(B_1), \phi(B_2)\}$, for all $B_1, B_2 \in \mathcal{B}$.

The Kuratowski measure of nonocompactness α is a mapping that associates the infimum of all numbers $\varepsilon > 0$ such that the set B can be covered by finite number of sets of the diameter less than $\varepsilon > 0$, with the bounded set B.

The Hausdorff measure of noncompactness χ associates the infimum of all numbers $\varepsilon > 0$ such that the set B can be covered by an finite number of balls of the diameter less than $\varepsilon > 0$, with the bounded set B.

The Istratescu measure of noncompactness β is the infimum of all numbers $r > 0$ such that the bounded set B does not have an infinite r-separation. Let us note that the set B is r-separated if $d(x, y) \geq r$ for all $x, y \in B$ such that $x \neq y$.

For more on measures of noncompactness see [1], [2], [8].

Definition 2.2. *Let (X, d) be the metric space and $\mathcal{B}$ be the family of all bounded subsets of X. An infinite subset $A \in \mathcal{B}$ is said to be minimal for the measure ϕ, or in short ϕ-minimal, if $\phi(A) = \phi(B)$ for every infinite subset B of A.*

Since we know how it is possible to choose minimal sets in the metric space, we distinguish some special classes of measures of noncompactness. In particular, there are two important classes.

Definition 2.3. *Let ϕ be a measure of noncompactness defined onto the family $\mathcal{B}$ of all bounded subsets of a metric space (X, d). We say that*

a) ϕ is the minimalizable measure of noncompactness if for every infinite set $A \in \mathcal{B}$ and for all $\varepsilon > 0$, there exists a ϕ-minimal set $B \subset A$, such that $\phi(B) \geq \phi(A) - \varepsilon$,

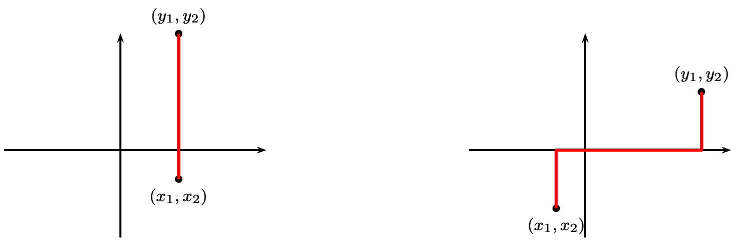
b) ϕ is a strictly minimalizable measure of noncompactness if for every infinite set $A \in \mathcal{B}$ there exists a ϕ-minimal set $B \subset A$ such that $\phi(B) = \phi(A)$.

Obviously, every strictly minimalizable measure of noncompactness is minimalizable.

We will consider the metric space $(\mathbb{R}^2, d^*)$ where d^* is the river metric defined by

$$d^*(v_1, v_2) = \begin{cases} |y_1 - y_2| & , \quad x_1 = x_2, \\ |y_1|+|y_2|+|x_1 - x_2| & , \quad x_1 \neq x_2, \end{cases}$$

where $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in \mathbb{R}^2$.



(a) Equal first coordinates (b) Different first coordinates

The unit ball in $(\mathbb{R}^2, d^*)$ is given by

$$B((0,0),1) = \left\{ (x, y) \in \mathbb{R}^2 : \begin{cases} |y| \leq 1 & , \quad x = 0 \\ |x|+|y| \leq 1 & , \quad x \neq 0 \end{cases} . \right\}.$$

Let $v = (x^*, 0)$, $x^* \in \mathbb{R}$. The ball centered at v, of radius r is given by

$$B(v, r) = \left\{ (x, y) \in \mathbb{R}^2 : \begin{cases} |y| \leq r & , \quad x = x^* \\ |x - x^*|+|y| \leq r & , \quad x \neq x^* \end{cases} . \right\}.$$

If $v = (0, y^*)$, $y^* \in \mathbb{R}$, $|y^*| < r$, then the ball centered at v, of the radius r is

$$B(v, r) = \left\{ (x, y) \in \mathbb{R}^2 : \begin{cases} |y - y^*| \leq r & , \quad x = 0 \\ |x|+|y| \leq r - |y^*| & , \quad x \neq 0 \end{cases} . \right\},$$

while for $|y^*|\geq r$ is

$$B(v,0)=\{(x,y)\in\mathbb{R}^2 : x=0 \ \wedge \ |y-y^*|\leq r\}.$$

If $v=(x^*,y^*)$, $x^*,y^*\in\mathbb{R}\setminus\{0\}$, then the ball centered at v, of the radius r, for $|y^*|< r$, is given with

$$B(v,r)=\left\{(x,y)\in\mathbb{R}^2 : \left\{\begin{array}{cc} |y-y^*|\leq r & , \quad x=x^* \\ |x-x^*|+|y|\leq r-|y^*| & , \quad x\neq x^* \end{array}\right.\right\},$$

while for $|y^*|\geq r$, we have

$$B(v,r)=\{(x,y)\in\mathbb{R}^2 : x=x^* \ \wedge \ |y-y^*|\leq r\}.$$

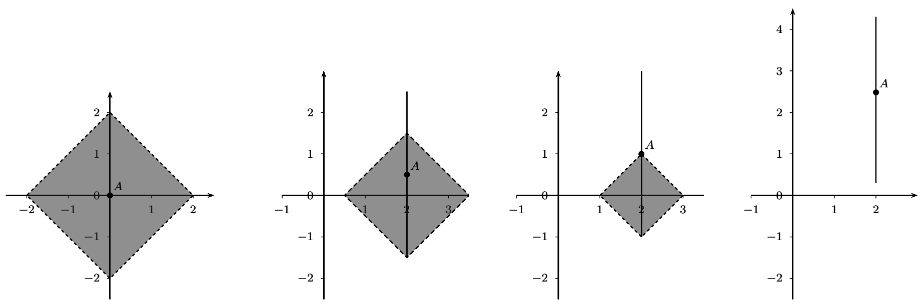


FIGURE 1. The ball $B(A,2)$ in the river metric, depending on the center A.

3. Main Results

Lemma 3.1. *The metric d^* is not equivalent (uniformly equivalent) to the standard metric d_p ($1\leq p\leq+\infty$) on $\mathbb{R}^2$.*

Proof. Consider the sequence given by

$$(x_n,y_n)=\left(1-\frac{1}{n},1-\frac{1}{n}\right),\ n\in\mathbb{N}.$$

Let $n,m\in\mathbb{N}$, $n\neq m$. In $(\mathbb{R}^2,d^*)$ we have

$$\begin{aligned} d^*((x_n,y_n),(x_m,y_m)) &= \left\{\begin{array}{ccc} |y_n-y_m| & ; & x_n=x_m \\ |y_n|+|y_m|+|x_n-x_m| & ; & x_n\neq x_m \end{array}\right. \\ &= \left\{\begin{array}{ccc} |\frac{1}{n}-\frac{1}{m}| & ; & \frac{1}{n}=\frac{1}{m} \\ |1-\frac{1}{n}|+|1-\frac{1}{m}|+|\frac{1}{n}-\frac{1}{m}| & ; & \frac{1}{n}\neq\frac{1}{m} \end{array}\right. \\ &= \left|1-\frac{1}{n}\right|+\left|1-\frac{1}{m}\right|+\left|\frac{1}{n}-\frac{1}{m}\right| \ \to \ 2 \ \ (n,m\to\infty). \end{aligned}$$

Thus, the sequence $(x_n, y_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence in $(\mathbb{R}^2, d^*)$. So, the sequence is not convergent either.

The given sequence is obviously convergent in $(\mathbb{R}^2, d_p)$ $(1 \leq p \leq +\infty)$, i.e. $(x_n, y_n) \to (1, 1) \in \mathbb{R}^2 \quad (n \to \infty)$. □

Lemma 3.2. *The metric space* $(\mathbb{R}^2, d^*)$ *is complete.*

Proof. Let $(x_n, y_n)_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $\mathbb{R}^2$, that is

$$d^*((x_n, y_n), (x_m, y_m)) \to 0 \ , (n, m \to \infty).$$

Let us prove that the given sequence is convergent. Assume that almost all points of the sequence have equal first coordinates, that is, there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0}$, for all $n \geq n_0$. Then

$$d^*((x_n, y_n), (x_m, y_m)) = |y_n - y_m|, \quad n, m \geq n_0 \ , \ n \neq m,$$

hence

$$|y_n - y_m| \to 0 \quad (n, m \to \infty).$$

We conclude that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and since $\mathbb{R}$ is a complete metric space, we realize that $(y_n)_{n\in\mathbb{N}}$ is a convergent sequence. So, there exists $y^* \in \mathbb{R}$ such that

$$|y_n - y^*| \to 0 \quad (n \to \infty).$$

If we put $x^* = x_{n_0}$, the following holds

$$d^*((x_n, y_n), (x^*, y^*)) = |y_n - y^*| \to 0, \quad (n \to \infty) \ .$$

Therefore, the sequence $(x_n, y_n)_{n\in\mathbb{N}}$ is convergent in $(\mathbb{R}^2, d^*)$.

Now, assume that almost all points of the sequence have different first coordinates. Then

$$d^*((x_n, y_n), (x_m, y_m)) = |y_n| + |y_m| + |x_n - x_m| \to 0, \quad (n, m \to \infty) \ ,$$

so, $\lim n \to y_n = 0$, i $|x_n - x_m| \to 0$, $(n, m \to \infty)$. This means that the sequence $(y_n)_{n\in\mathbb{N}}$ is a zero-sequence and $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Hence, a sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and there exists $x^* \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = x^*$. Notice that

$$d^*((x_n, y_n), (x^*, 0)) = \begin{cases} |y_n| & , \quad x_n = x^* \\ |y_n| + |x_n - x^*| & , \quad x_n \neq x^* \end{cases} .$$

Therefore, $d^*((x_n, y_n), (x^*, 0)) \to 0, \quad (n \to \infty)$. So, a sequence $(x_n, y_n)_{n\in\mathbb{N}}$ is convergent.

We actually proved the completeness of the space in relation to the metric d^*, since in both cases we showed that an arbitrary Cauchy sequence is convergent in the given space. □

The proof of the completeness of the metric space $(\mathbb{R}^2, d^*)$ gives us the description of Cauchy sequences in the concrete space. Namely, the following lemma holds.

Lemma 3.3. *The sequence* $((x_n, y_n))_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $(\mathbb{R}^2, d^*)$ *if and only if* $(x_n)_{n\in\mathbb{N}}$ *is convergent and the sequence* $(y_n)_{n\in\mathbb{N}}$ *is a zero-sequence, or if and only if the sequence* $(x_n)_{n\in\mathbb{N}}$ *is a constant sequence starting from some index and the sequence* $(y_n)_{n\in\mathbb{N}}$ *is convergent in* $\mathbb{R}$.

We also had the confirmation of this statement in the proof of the Lemma 3.1 when we proved that the sequence $\left(\left(1-\frac{1}{n}, 1-\frac{1}{n}\right)\right)_{n\in\mathbb{N}}$ is not a Cauchy sequence, where for the sequence of the first coordinates and the sequence of the second coordinates holds $1-\frac{1}{n} \to 1$ $(n \to \infty)$ in $\mathbb{R}$ with standard metric. Notice that the sequences of the coordinates do not satisfy neither one nor the other condition of the Lemma 3.3. D. Bugajewski and E. Grzelaczyk (see [4]) gave a concrete expression for the Kuratowski measure of noncompactness α of any subset of the space $(\mathbb{R}^2, d^*)$, with the river metric d^*.

Definition 3.1. *Let* D *be any bounded subset of* $\mathbb{R}^2$. *We say that* $y' \in \mathbb{R}$ *satisfies: the condition* $A^*(D)$, *if for every* $y < y'$ *there exists at least a countable (but not finite) number of points* $v_n = (x_n, y_n) \in D$ *such that*

$$x_n \neq y_n, \quad (n \neq m), \quad y < y_n \leq y', \quad (n \in \mathbb{N})$$

the condition $A_*(D)$, *if for all* $y > y'$ *there exists at least a countable (but not finite) number of points* $v_n = (x_n, y_n) \in D$ *such that*

$$x_n \neq y_n, \quad (n \neq m), \quad y > y_n \geq y', \quad (n \in \mathbb{N}).$$

Put $y^*(D) = \sup_{y'}|y'|$, *where* y' *satisfies* $A^*(D)$ *or* $A_*(D)$. *If there exists any number* y' *satisfying neither* $A^*(D)$ *nor* $A_*(D)$, *put* $y^*(D) = 0$.

Theorem 3.1. *For any bounded subset* $D \subset \mathbb{R}^2$ *with the river metric, we have*

$$\alpha(D) = 2y^*(D).$$

Further, we will check the existence of the α-minimal sets and the minimalizability of the Kuratowski measure α in the $(\mathbb{R}^2, d^*)$. It is clear that all relatively compact sets, i.e. sets with the Kuratowski measure α equal zero, are also α-minimal sets. The question arises: whether there are α-minimal sets of a positive measure in the space $\mathbb{R}^2$ with the river metric? The following example gives the answer to this question.

Example 3.1. *Let us consider the set*

$$A = \left\{(x_n, y_n) = \left(\frac{1}{n}, 1-\frac{1}{n}\right); n \in \mathbb{N}\right\}.$$

Since $y^*(A) = 1$, *we have* $\alpha(A) = 2$.

Let $B \subset A$ *be an arbitrary infinite set. Knowing that the Kuratowski measure of noncompactness* α *is a monotone function, we get*

$$\alpha(B) \leq \alpha(A) = 2\ .$$

Assume $\alpha(B) < 2$, that is $y^*(B) < 1$. Since $y^*(A) = 1$, we conclude that for all $y < 1$ there exist infinitely many elements of the set A, i. e. infinitely many members of the sequence $(x_n, y_n)_{n\in\mathbb{N}} \subset A$ such that $x_n \neq x_m$, $y < y_n \leq 1$. In other words, for $y^*(B) < 1$ there exists $n_0 \in \mathbb{N}$ such that

$$(x_n, y_n) \in A, \quad x_n \neq x_m, \quad y^*(B) < y_n \leq 1, \quad n \geq n_0.$$

Since the sequence $(y_n)_{n\in\mathbb{N}}$ of the second coordinates of the points from the set A is a monotonically growing sequence, we conclude that there is only a finite number of elements of the set A whose y coordinates are smaller or equal to $y^*(B)$. Thus, only a finite number of points, of the set B whose coordinates are less or equal to $y^*(B)$, can exist. But this is in contradiction with the definition of the number $y^*(B)$. So, $y^*(B) = 1$, that is $\alpha(B) = 2$.

This proves that the set A is α-minimal.

Theorem 3.2. *The Kuratowski measure of noncompactness is strictly minimalizable on the space $(\mathbb{R}^2, d^*)$.*

Proof. Let $D \subset \mathbb{R}^2$ be a bounded, infinite set. If $\alpha(D) = 0$, then D is a relatively compact set and also α-minimal. Since every infinite subset of the α-minimal set is also α-minimal, we conclude that, in this case, α is a strictly minimalizable measure.

Let $\alpha(D) = 2y^*(D) > 0$. Without losing generality we assume that $y^*(D)$ is such that it satisfies the condition $A^*(D)$, that is for all $y < y^*(D)$ there exists at least a countable (but not finite) number of points $v_n = (x_n, y_n) \in D$ such that

$$x_n \neq x_m, \quad (n \neq m), \quad y < y_n \leq y^*(D), \quad (n \in \mathbb{N}).$$

Let $\varepsilon > 0$ be arbitrary. For $y^*(D) - \varepsilon < y^*(D)$ there exists a countable number of points $(x_n, y_n) \in D$ such that

$$x_n \neq x_m, \quad (n \neq m), \quad y^*(D) - \varepsilon < y_n \leq y^*(D), \quad (n \in \mathbb{N}).$$

Let us define the set A in the way that it contains exactly those points, precisely, at least a countable number of those points v_n, but also contains other points from the set D whose first coordinates are different from each other, and the second coordinates are from the interval $(y^*(D) - \varepsilon, y^*(D)]$. So, the set A is given by

$$A = \{(x, y) \in D : y^*(D) - \varepsilon < y \leq y^*(D)\}.$$

Consider the sets

$$A_k = \left\{(x, y) \in A : y^*(D) - \frac{\varepsilon}{k} < y \leq y^*(D) - \frac{\varepsilon}{k+1}\right\}, \quad k \in \mathbb{N} .$$

Choose one point (x_k, y_k) from each set A_k, $k \in \mathbb{N}$, and using these points let us construct the following set

$$X = \{(x_k, y_k) \in A_k : k \in \mathbb{N}\}.$$

The set X is infinite. Indeed, if X is a finite set, this would mean that only a finite number of sets A_k is nonempty, that is

$$A_k \neq \emptyset, \quad k = n_1, n_2, ..., n_j.$$

Let $n^* = \max\{n_1, n_2, ..., n_j\}$. Then, the set of points (x, y) whose first coordinates are different from each other, and the values of the second coordinates belong to the interval $\left(y^*(D) - \frac{\varepsilon}{n^*+1}, y^*(D)\right)$ is an empty set. This is in contradiction with the definition of the number $y^*(D)$, because for y such that

$$y^*(D) - \frac{\varepsilon}{n^*+1} < y < y^*(D),$$

there should exist at least a countable number of points from the set D with the mentioned property. Hence, the set X is an infinite set.

Since $X \subset D$ and the Kuratowski measure of noncompactness α is monotone, we have

$$\alpha(X) \leq \alpha(D),$$

i.e.

$$y^*(X) \leq y^*(D).$$

Let us prove the equality $y^*(X) = y^*(D)$. Suppose that the inequality $y^*(X) < y^*(D)$ holds. By the definition of the number $y^*(X)$, that is by the fact that all the elements of the set X have y coordinate less or equal to $y^*(X)$, there would exist the number $n_0 \in \mathbb{N}$ such that all the sets A_k, $k > n_0$, would be empty. In other words, we would have infinitely many empty sets A_k. This would be in contradiction with the definition of the set X. Therefore, the equality $y^*(X) = y^*(D)$ holds, and we have

$$\alpha(X) = \alpha(D).$$

In this way we showed that for the arbitrary infinite and bounded set D there exists a α-minimal subset with the same measure of noncompactness. This proves that the Kuratowski measure of noncompactness is a strictly minimalizable on the space $(\mathbb{R}^2, d^*)$. □

We know the relation between α-minimal and β-minimal sets. Namely, every α-minimal subset A of the metric space (X, d) is also β-minimal and $\alpha(A) = \beta(A)$. Furthermore, if (X, d) is a complete metric space, then the Istratescu measure of noncompactness β is a minimalizable, but in general it is not strictly minimalizable, see e.g. [2]. Therefore, in the space $(\mathbb{R}^2, d^*)$ there exist β-minimal sets. The values of the Istratescu and Kuratowski measure of noncompactness are equal on those sets. Notice that, if the set $A \subset \mathbb{R}^2$ is a bounded and infinite set, then there exists the α-minimal set $B \subset A$ such $\alpha(B) = \alpha(A)$, since the Kuratowski measure of noncompactness is strictly minimalizable on the space $(\mathbb{R}^2, d^*)$. Then, the set B is also a β-minimal and $\beta(B) = \alpha(B)$. Since $B \subset A$ and the measure β is a monotone function, we have $\beta(B) \leq \beta(A)$. On the other hand, $\beta(B) \leq \alpha(B) = \alpha(A)$. So, for the arbitrary set A and the measure of noncompactness β, there exists a β-minimal set $B \subset A$ such that $\beta(B) = \beta(A)$. This proves the following fact.

Corollary 3.1. *The Istratescu measure of noncompactness β is strictly minimalizable on the space $(\mathbb{R}^2, d^*)$.*

References

[1] *Akhmerov R.R., Kamenskii M.I., Potapov A.S., Rodkina A.E., Sadovskii B.N.*: (1992). Measures of noncompactness and condensing operators: Birkhäuser, book.
[2] *Ayerbe Toledano J.M., Benavides T.D., Acedo G.L.*: (1997). Measures of noncompactness in metric fixed point theory: Springer, book.
[3] *Benavides T.D.*: (1986). Some properties of the set and ball measures of non-compactness and applications: Journal of the London Mathematical Society, 1 (2), 120 - 128.
[4] *Bugajewski D., Grzelaczyk E.*: (1998). On the measures of noncompactness in some metric spaces: New Zealand Journal of Mathematics, New Zealand Mathematical Society and Department of Mathematics & Statistics, University of Auckland, 2 (27), 177 - 182.
[5] *Gohberg I., Goldenstein L.S., Markus A.S.*: (1570). Investigations of some properties of bounded linear operators with their q-norms: Ucenie Zapiski, Kishinevskii Gosuniversitet, (29), 29 - 36.
[6] *Istratescu V.*: (1972). On a measure of noncompactness: Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie, 2 (16), 195 - 197.
[7] *Kuratowski K.*: (1930). Sur les espaces complets: Fundamenta Mathematicae, 1 (15), 301-309.
[8] *Rakočević V.*: (1998). Measures of noncompactness and some applications: University of Nis, Filomat: Faculty of Sciences and Mathematics, 2 (12), 87 - 120.
[9] *Rekić-Vuković A., Okičić N., Pašić V., Arandjelović I.*: (2015). Continuity of modulus of noncompact convexity for minimalizable measures of noncompactness: Functional Analysis, Approximation and Computation, 3 (7), 39 - 46.
[10] *Rekić-Vuković A., Okičić N.*: (2020). Another look at the continuity of moduli of noncompact convexity: Palestine Journal of Mathematics, 2 (9), 771 - 778.
[11] *Rekić-Vuković A., Okičić N., Arandjelović I.*: (2014). On modulus of noncompact convexity for a strictly minimalizable measure of noncompactness: Gulf Journal of Mathematics, 3 (2), 38 - 45.

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