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CONTENT

The Appendix

In honor of the first Doctor of Mathematical Sciences Acad. Blagoj Popov, a mathematician dedicated to differential equations, the idea of holding the "Day of Differential Equations" was born, prompted by Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski, and Prof. Ph.D. Lazo Dimov. Acad. Blagoj Popov presented his doctoral dissertation on 05.05.1952 in the field of differential equations. This is the main reason for holding the " Day of Differential Equations" at the beginning of May.

This year on May 7th, the "Day of Differential Equations" was held for the sixth time under the auspices of the Faculty of Computer Sciences at "Goce Delcev" University in Stip and Dean Prof. Ph.D. Cveta Martinovska - Bande, organized by Prof. Ph.D. Biljana Zlatanovska. The event was organized online via the platform Microsoft Teams and with the selfless help and support of Prof. Ph.D. Natasa Stojkovik, Ass. Prof. Ph.D. Limonka Koceva Lazarova, Ass. Prof. Ph.D. Marija Miteva, Ass. Prof. Ph.D. Mirjana Kocaleva, Ass. Prof. Ph.D. Aleksandra Stojanova.

The participants of this event were:

1. Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Sanja Atanasova and Stefan Boshkovski (student) from the Faculty of Electrical Engineering and Information Technology at Ss. Cyril and Methodius, University in Skopje;
2. Prof. Ph.D. Aleksa Malcheski from the Faculty of Mechanical engineering at Ss. Cyril and Methodius, University in Skopje;
3. Prof. Ph.D. Slagjana Brsakoska from the Faculty of Natural Sciences and Mathematics at Ss. Cyril and Methodius, University in Skopje;
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Acknowledgments to Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski and Prof. Ph.D. Lazo Dimov for the wonderful idea and the successful realization of the event this year and in previous years.

Acknowledgments to the Dean of the Faculty of Computer Sciences, Prof. Ph.D. Cveta Martinovska - Bande for her overall support of the organization and implementation of the "Day of Differential Equations".

The papers that emerged from the "Day of Differential Equations" are in the appendix to this issue of BJAMI.

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DYNAMICAL ANALYSIS OF A THIRD-ORDER AND A FOURTH-ORDER SHORTENED LORENZ SYSTEMS

BILJANA ZLATANOVSKA AND BORO PIPEREVSKI

Abstract. In [1], a Modified Lorenz system of the seventh-order is defined. In [2], from the Modified Lorenz system, shortened Lorenz systems of lower order are obtained. Between them, the third-order and fourth-order shortened Lorenz systems with graphical presentations for their local behavior are found. In this paper, the dynamical analysis of these systems according to [3] will be done via: symmetry of the systems, dissipative of the systems, finding of the fixed point, analysis of the behavior of the systems in a neighborhood of the fixed point and defining of Lyapunov function, which gives us the conditions for the stability and the asymptotical stability of the fixed point.

1. Introduction

The Lorenz system of differential equations is a part of the group of chaotic systems for which the explicit solutions are not known. In numerous works in mathematical literatures (as an example [4-14]), its behavior is analyzed. As a chaotic system, the Lorenz system is most often analyzed via graphical visualizations (see [9-14]).

The Lorenz system has a form

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= x(r-z)-y \\ \dot{z} &= xy-bz \end{aligned}$$

with parameters $\sigma > 0, r > 0, b > 0$ and initial values $x_0 = x(0), y_0 = y(0)$, $z_0 = z(0)$.

In [1], the Modified Lorenz system of the seventh-order with the initial values $a_0 = x(0), b_0 = y(0), c_0 = z(0), c_p = \overset{(p)}{z}(0), p \in \{1,2,3,4\}$ and the expressions $A = 1+\sigma+b, B = \sigma(r-z_0)-x_0^2, C = \sigma x_0 y_0, D = -\sigma^2 y_0^2,$

Date: November 18, 2021.

Keywords. Lorenz system, third-order shortened Lorenz systems, fourth-order shortened Lorenz systems, Lyapunov function, dissipative of the system.

$$\begin{aligned}
\dot{x} &= \sigma(y-x) \\
\dot{y} &= x(r-z)-y \\
\overset{(5)}{z} &= -(A+b)\overset{(4)}{z} + (B-Ab)\overset{(3)}{z} - (C-Bb)\overset{(2)}{z} + (D-Cb)\dot{z} + Dbz
\end{aligned}$$

is presented.

The third equation of the Modified Lorenz system is the fifth-order homogeneous linear differential equation with constant coefficients and its characteristic equation has the form

$$k^5 + (A+b)k^4 - (B-Ab)k^3 + (C-Bb)k^2 - (D-Cb)k - Db = 0$$

with solutions $k_1 = -b, k_{2/3/4/5} = k(A,B,C,D,b)$.

By using one real solution from the characteristic equation, the seventh-order Modified Lorenz system can transform in a third-order subsystem of differential equations

$$\begin{aligned}
\dot{x} &= \sigma(y-x) \\
\dot{y} &= x(r-z)-y \\
\dot{z} &= kz
\end{aligned} \tag{1.1}$$

with initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0)$.

By using two solutions from the solutions $k_{1/2/3/4/5}$ of the characteristic equation, the seventh-order Modified Lorenz system can transform into fourth-order subsystems of differential equations

$$\begin{array}{lll}
\dot{x} = \sigma(y-x) & \dot{x} = \sigma(y-x) & \dot{x} = \sigma(y-x) \\
\dot{y} = x(r-z)-y & \dot{y} = x(r-z)-y & \dot{y} = x(r-z)-y \\
\dot{z} = u & \dot{z} = u & \dot{z} = u \\
\dot{u} = (k_1+k_2)u - k_1k_2z & \dot{u} = 2ku - k^2z & \dot{u} = 2\alpha u - (\alpha^2+\beta^2)z \\
\quad k_1, k_2 \in R & \quad k \in R & \quad \alpha, \beta \in R
\end{array}$$

which are marked as (1.2'), (1.2") and (1.2"') respectively with initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0), z_1 = \dot{z}(0) = u(0)$.

The systems (1.2'), (1.2") and (1.2"') can be present in a system that has a form

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(r-z)-y\\ \dot{z} &= u\\ \dot{u} &= Su-Pz\end{aligned} \tag{1.2}$$

where
- for the system (1.2'), the variables , $S = k_1 + k_2, P = k_1k_2$ are taken;
- for the system (1.2"), the variables , $S = 2k, P = k^2$ are taken;
- for the system (1.2"'), the variables , $S = 2\alpha, P = \alpha^2 + \beta^2$ are taken.

Remark 1.1. The system (1.2) is a simpler form of record for systems (1.2'), (1.2") and (1.2"') . Where necessary, each system will be considered separately.

The third-order and fourth-order subsystems (1.1) and (1.2) of the Modified Lorenz system will be called the third-order and the fourth-order shortened Lorenz systems respectively. In [1], for the Modified Lorenz system an explicit solution is given, which can be applied as an explicit solution for the third-order and fourth-order shortened Lorenz systems.

In this paper, we will give a simple dynamical analysis of these systems (1.1) and (1.2) via: symmetry of the systems, dissipative of the systems and finding of the fixed point according to [3]. The behavior of the systems in a neighborhood of the fixed point will be evaluated via the sign of eigenvalues for the matrix of systems (1.1) and (1.2) according to [15] and [16]. Finally, the Lyapunov function will be defined, which gives us the conditions for the stability and the asymptotical stability of the fixed point. In numerous works in mathematical literatures (as an example [17], [18], [19]), this research is carried out.

2. Third-order shortened Lorenz system

In this part, the dynamical analysis of the third-order shortened Lorenz system (1.1) will be reviewed.

Symmetry: It is easy to see that the system (1.1) has a symmetry by using the following transformation:

$$(x, y, z) \to (-x, -y, z)$$

If $(x(t), y(t), z(t))$ is a solution of the system (1.1) then $(-x(t), -y(t), z(t))$ is a solution of the system (1.1).

Dissipativity: In this part it will be shown that the system (1.1) is dissipative, i.e that trajectories remain in one compact set ellipsoid via the following Theorem 2.1.

Theorem 2.1. *The system (1.1) is dissipative for* $k < \sigma + 1$.

Proof. The system (1.1) has

$$divV = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -\sigma - 1 + k = -(\sigma + 1 - k) < 0 \ (k < \sigma + 1)$$

where V is the volume. The system (1.1) is dissipative with an exponent contraction rate of

$$\frac{dV}{dt} = \dot{V} = -(\sigma + 1 - k)$$
$$V = V_0 e^{-(\sigma+1-k)}, V_0 = V(0)$$

When the time t increases from 0 to infinity, then the volume V lowers, i.e. $\lim_{t\to\infty} V(t) = 0$ □

Fixed point: By Theorem 2.2, the fixed point for the system (1.1) will be obtained.

Theorem 2.2. *The system (1.1) has an unique fixed point* $O(0,0,0)$.

Proof. For the system (1.1), from

$$\begin{aligned} \dot{x} &= 0 \qquad \sigma(y - x) = 0 \\ \dot{y} &= 0 \Leftrightarrow x(r - z) - y = 0 \\ \dot{z} &= 0 \qquad kz = 0 \end{aligned}$$

the fixed point is obtained. □

In Theorem 2.3, the eigenvalues of the matrix of the system (1.1) are obtained. They depend on the parameters of the system (1.1).

Theorem 2.3. *The characteristic equation of the system (1.1) in a neighborhood of the fixed point* $O(0,0,0)$ *has a form*

$$(\lambda - k)[\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r)] = 0 \tag{2.1}$$

with solutions

$$\lambda_{1,2} = \frac{-(\sigma+1) \pm \sqrt{(\sigma+1)^2 + 4\sigma(r-1)}}{2}, \lambda_3 = k \tag{2.2}$$

Proof. For the system (1.1), the matrix $A = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ 0 & 0 & k \end{bmatrix}$

in the neighborhood of the fixed point is $A = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & k \end{bmatrix}$.

From $det(A - \lambda E) = 0$ (the matrix E is the identity matrix of size 3) the characteristic equation (2.1) is obtained. By solving the characteristic equation, the solutions (2.2) as eigenvalues for the matrix of the system (1.1) are obtained. □

The behavior of the system (1.1) in a neighborhood of the fixed point depends on the sign of eigenvalues (2.2) for the matrix of the system (1.1). For their sign, we know that

$$\lambda_{1/2} = \frac{-(\sigma+1) \pm \sqrt{(\sigma+1)^2 + 4\sigma(r-1)}}{2} = \frac{-(\sigma+1) \pm \sqrt{(\sigma-1)^2 + 4\sigma r}}{2} \in R$$

and

$$\lambda_2 = \frac{-(\sigma+1) - \sqrt{(\sigma+1)^2 + 4\sigma(r-1)}}{2} < 0$$

for every $r > 0$.

In Theorem 2.4, the behavior of the system (1.1) in a neighborhood of the fixed point $O(0,0,0)$ will be given.

Theorem 2.4. *In a neighborhood of the fixed point O (0,0,0) for the system (1.1), the dynamics of the attractor is:*

- a fixed point for $0 < r < 1$ *and* $k < 0$;

- a chaos C^1 *for* $r > 1$ *and* $k = 0$;

- a torus T^2 *for* $r = 1$ *and* $k = 0$.

Proof. The sign of the eigenvalues (2.2) of the matrix of system (1.1) will be considered.

- For $0 < r < 1$ and $k < 0$, the sign of the eigenvalues (2.2) are $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$. This implies a form $(-,-,-)$, which shows that the attractor is a fixed point.

- For $r > 1$ and $k = 0$, the sign of the eigenvalues (2.2) are $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 = 0$. This implies a form $(+,0,-)$, which shows that the attractor is a chaos C^1.

- For $r = 1$ and $k = 0$, the sign of the eigenvalues (2.2) are $\lambda_1 = 0, \lambda_2 < 0, \lambda_3 = 0$. This implies a form $(0,0,-)$, which shows that the attractor is a torus T^2.

□

The Lyapunov function: The stability of the dynamical system with the Lyapunov function is proved. Therefore, its constructing is called the Lyapunov's second method for the stability of a dynamical system. For the stability of the system (1.1) at the fixed point O(0, 0, 0) via the following Theorem 2.5, the Lyapunov function is defined.

Theorem 2.5. *Let* $O(0,0,0)$ *be a fixed point for the system (1.1). Let the Lyapunov function as a continuously differentiable function for the system (1.1) be defined with* *the following function:*

$$L(x,y,z)=\frac{x^2}{2\sigma}+\frac{y^2}{2}+\frac{z^2}{2}$$

where $L(x,y,z)>0,\forall(x,y,z)\neq(0,0,0)$ *and* $L(0,0,0)=0$. *Then the fixed point* $O(0,0,0)$ *is stable when* $0<r<1$ *and* $k<0$. *If* $\dot{L}(x,y,z)<0,\forall(x,y,z)\neq(0,0,0)$ *then the fixed point* $O(0,0,0)$ *is asymptotically stable for* $0<r<1$ *and* $k<0$.

Proof. The system (1.1) at the fixed point $O(0,0,0)$ is stable when for a continuously differentiable function $L:D\to R, D\subset R^3$ exactly is $\frac{dL}{dt}\leq 0$, where $L(x,y,z)>0,\forall(x,y,z)\neq(0,0,0)$ and $L(0,0,0)=0$. We differentiate the Lyapunov function:

$$\dot{L}(x,y,z)=\frac{dL}{dt}=\frac{x\dot{x}}{\sigma}+y\dot{y}+z\dot{z}=-[x-\frac{r+1-z}{2}y]^2-[1-(\frac{r+1-z}{2})^2]y^2-(-k)z^2<0$$

The inequality implies that $r<1+z$. As we observe in the neighborhood of the fixed point $O(0,0,0)$, then $|z|<c, c\in(0,\varepsilon)$ for given $\varepsilon>0$. This indicates that the system (1.1) at the fixed point $O(0,0,0)$ is stable when $0<r<1$ and $k<0$. When $\dot{L}(x,y,z)<0$ then the fixed point $O(0,0,0)$ is asymptotically stable for $0<r<1$ and $k<0$. □

3. Fourth-order shortened Lorenz system

In this part, the dynamical analysis of the fourth-order shortened Lorenz system (1.2) will be reviewed.

Symmetry: It is easy to see that the system (1.2) has a symmetry by using the following transformations: $(x,y,z,u)\to(-x,-y,z,u)$ and $(x,y,z,u)\to(-x,-y,-z,-u)$. If $(x(t),y(t),z(t),u(t))$ is a solution of the system (1.2) then $(-x(t),-y(t),z(t),u(t)),(-x(t),-y(t),-z(t),-u(t))$ are solutions of the system (1.2).

Dissipativity: In this part it will be shown that the system (1.2) is dissipative, i.e the trajectories remain in one compact set ellipsoid via the following Theorem 3.1.

Theorem 3.1. *The system (1.2) is dissipative* $S<\sigma+1$.

Proof. The system (1.2) has

$$divV=\frac{\partial\dot{x}}{\partial x}+\frac{\partial\dot{y}}{\partial y}+\frac{\partial\dot{z}}{\partial z}+\frac{\partial\dot{u}}{\partial u}=-\sigma-1+S=-(\sigma+1-S)<0\,(S<\sigma+1)$$

where V is the volume. The system (1.2) is dissipative with an exponent contraction rate of

$$\frac{dV}{dt}=\dot{V}=-(\sigma+1-S)$$

$$v=V_0e^{-(\sigma+1-S)t}, V_0=V(0)$$

When the time t increases from 0 to infinity, then volume V lowers, i.e. $\lim_{t\to\infty} V(t) = 0$. □

From $S < \sigma + 1$, we can review the dissipative for every a system (1.2'), (1.2") and (1.2"') individually via Corollary 3.1.

Corollary 3.1. *The systems (1.2'), (1.2") and (1.2"') are dissipative for* $k_1 + k_2 < \sigma + 1, k < \frac{\sigma+1}{2}$ *and* $\alpha < \frac{\sigma+1}{2}$ *respectively.*

Proof. In the proof of Theorem 3.1, we will take the appropriate change for S for every system individually.

For the system (1.2'),$S = k_1 + k_2$ and $S < \sigma + 1$ we obtained $k_1 + k_2 < \sigma + 1$.
For the system (1.2"), $S = 2k$ and $S < \sigma + 1$ we obtained $k < \frac{\sigma+1}{2}$.
For the system (1.2"'), $S = 2\alpha$ and $S < \sigma + 1$ we obtained $\alpha < \frac{\sigma+1}{2}$. □

Fixed point: By Theorem 3.2, the fixed point for the system (1.2) will be obtained.

Theorem 3.2. *The system (1.2) has an unique fixed point* $O(0, 0, 0, 0)$.

Proof. For the system (1.2), from

$$\begin{aligned} \dot{x} &= 0 & \sigma(y - x) &= 0 \\ \dot{y} &= 0 \Leftrightarrow & x(r - z) - y &= 0 \\ \dot{z} &= 0 & u &= 0 \\ \dot{u} &= 0 & Su - Pz &= 0 \end{aligned}$$

the fixed point is obtained.

□

In Theorem 3.3, the eigenvalues of the matrix of systems (1.2) are obtained. They depend on the parameters of the system (1.2).

Theorem 3.3. *The characteristic equation of the system (1.2) in a neighborhood of the fixed point* $O(0, 0, 0, 0)$ *has a form*

$$(\lambda^2 - S\lambda + P)[\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r)] = 0 \tag{3.1}$$

with solutions

$$\lambda_{1,2} = \frac{-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 + 4\sigma(r - 1)}}{2}, \lambda_{3/4} = \frac{S \pm \sqrt{S^2 - 4P}}{2} \tag{3.2}$$

Proof. For the system (1.2), the matrix $A = \begin{bmatrix} -\sigma & \sigma & 0 & 0 \\ r-z & -1 & -x & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -P & S \end{bmatrix}$ in the neighborhood of the fixed point is $A = \begin{bmatrix} -\sigma & \sigma & 0 & 0 \\ r & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -P & S \end{bmatrix}$.

From $det(A - \lambda E) = 0$ (the matrix E is the identity matrix of size 4) the characteristic equation (3.1) is obtained. By solving the characteristic equation, the solutions (3.2) as eigenvalues for the matrix of the system (1.2) are obtained. □

By Theorem 3.3, eigenvalues for the matrix of the system (1.2) are obtained, where the system (1.2) presents a simpler record for the systems (1.2'), (1.2") and (1.2"'). For this goal in the following Corollary 3.2, the eigenvalues for the matrix for every system individually will be given.

Corollary 3.2. *The characteristic equation for the systems (1.2'), (1.2") and (1.2"') in a neighborhood of the fixed point O (0, 0, 0, 0) have a form*

$$(\lambda - k_1)(\lambda - k_2)[\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r)] = 0 \tag{3.3}$$

$$(\lambda - k)^2[\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r)] = 0 \tag{3.4}$$

$$(\lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2)[\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r)] = 0 \tag{3.5}$$

with solutions

$$\lambda_{1,2} = \frac{-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 + 4\sigma(r - 1)}}{2}, \lambda_3 = k_1, \lambda_4 = k_2 \tag{3.6}$$

$$\lambda_{1,2} = \frac{-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 + 4\sigma(r - 1)}}{2}, \lambda_{3/4} = k \tag{3.7}$$

$$\lambda_{1,2} = \frac{-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 + 4\sigma(r - 1)}}{2}, \lambda_{3/4} = \alpha \pm \beta i \tag{3.8}$$

respectively.

Proof. By using $S = k_1 + k_2, P = k_1 k_2, k_1, k_2 \in R$ for the system (1.2') $S = 2k, P = k^2, k \in R$ for the system (1.2") and $S = 2\alpha, P = \alpha^2 + \beta^2, \alpha, \beta \in R$ for the system (1.2"') in the characteristic equation (3.1) from Theorem 3.3, the characteristic equations (3.3), (3.4) and (3.5) are obtained. By their solving, their solutions (3.6), (3.7) and (3.8) as eigenvalues for the matrix of systems (1.2'), (1.2") and (1.2"') respectively are obtained. □

The same as for the system (1.1), the behavior of the system (1.2) in a neighborhood of the fixed point depends on the sign of the eigenvalues (3.6), (3.7), and (3.8) for the matrix of systems (1.2'), (1.2") and (1.2"') respectively. Because the eigenvalues $\lambda_{1/2}$ are the same as for the system (1.1), we already saw that $\lambda_{1/2} \in R$ and $\lambda_2 < 0$ for every $r > 0$. In Theorem 3.4, the behavior of the systems (1.2'), (1.2") and (1.2"') in a neighborhood of the fixed point $O(0,0,0,0)$ will be given.

Theorem 3.4. *In a neighborhood of the fixed point $O(0,0,0,0)$ for the systems (1.2'), (1.2") and (1.2"'), the dynamics of the attractor is:*

- a fixed point if $0 < r < 1$ and $k_{1/2} < 0$ for the system (1.2'); if $0 < r < 1$ and $k < 0$ for the system (1.2"); if $0 < r < 1$ and $\alpha < 0$ for the system (1.2"');

- a hypertorus T^3 if $r = 1$ and $k_{1/2} = 0$ for the system (1.2'); if $r = 1$ and $k = 0$ for the system (1.2");

- a chaos on T^3 if $r = 1$, $k_1 = 0 (k_2 = 0)$ and $k_2 > 0 (k_2 > 0)$ orr > 1 and $k_{1/2} = 0$ for the system (1.2'); if $r > 1$ and $k = 0$ for the system (1.2");

- a hyperchaos C^2 if $r > 1, k_1 = 0 (k_2 = 0)$ and $k_2 > 0 (k_2 > 0)$ for the system (1.2'); if $r = 1$ and $k > 0$ for the system (1.2"); if $r = 1$ and $\alpha > 0$ for the system (1.2"').

Proof. The sign of the eigenvalues (3.6), (3.7) and (3.8) of the matrix of systems (1.2'), (1.2") and (1.2"') respectively will be considered.

- The system (1.2'): For $0 < r < 1$ and $k_{1/2} < 0$, the sign of the eigenvalues (3.6) are $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0, \lambda_4 < 0$. This implies a form $(-,-,-,-)$, which shows that the attractor is a fixed point.

The system (1.2"): For $0 < r < 1$ and $k < 0$, the sign of the eigenvalues (3.7) are $\lambda_1 < 0, \lambda_2 < 0, \lambda_{3/4} < 0$. This implies a form $(-,-,-,-)$, which shows that the attractor is a fixed point.

The system (1.2"'): For $0 < r < 1$ and $\alpha < 0$, the sign of the eigenvalues (3.8) are $\lambda_1 < 0, \lambda_2 < 0, Re\lambda_{3/4} < 0$. This implies a form $(-,-,-,-)$, which shows that the attractor is a fixed point.

- The system (1.2'): For $r = 1$ and $k_{1/2} = 0$, the signs of the eigenvalues (3.6) are $\lambda_1 = 0, \lambda_2 = -(\sigma+1) < 0, \lambda_{3/4} = 0$. This implies a form $(0,0,0,-)$, which shows that the attractor is a hypertorus T^3.

The system (1.2"): For $r = 1, k_1 = 0 (k_2 = 0)$ and $k_2 > 0 (k_2 > 0)$ or $r > 1$ and $k_{1/2} = 0$, the signs of the eigenvalues (3.7) are $\lambda_1 = 0, \lambda_2 = -(\sigma+1) < 0, \lambda_{3/4} = 0$. This implies a form $(0,0,0,-)$, which shows that the attractor is a hypertorusT^3.

- The system (1.2'): For $r = 1$ and $k_{1/2} = 0$, the signs of the eigenvalues (3.6) are $\lambda_1 = 0, \lambda_2 = -(\sigma+1) < 0, \lambda_3 = 0 (\lambda_4 = 0), \lambda_4 > 0 (\lambda_3 > 0)$ This implies a form $(+,0,0,-)$, which shows that the attractor is a chaos on T^3.

The system (1.2"): For $r > 1$ and $k = 0$, the signs of the eigenvalues (3.7) are $\lambda_1 > 0, \lambda_2 < 0, \lambda_{3/4} = 0$. This implies a form $(0,0,0,-)$, which shows that the attractor is a chaos on T^3.

- The system (1.2'): For $r > 1, k_1 = 0(k_2 = 0)$ and $k_2 > 0(k_2 > 0)$, the signs of the eigenvalues (3.6) are $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 = 0(\lambda_4 = 0), \lambda_4 > 0(\lambda_3 > 0)$ This implies a form $(+,+,0,-)$, which shows that the attractor is a hyperchaos C^2.

The system (1.2"): For $r = 1$ and $k > 0$, the signs of the eigenvalues (3.7) are $\lambda_1 = 0, \lambda_2 = -(\sigma+1) < 0, \lambda_3 = 0(\lambda_4 = 0), \lambda_4 > 0(\lambda_3 > 0)$. This implies a form $(+,+,0,-)$, which shows that the attractor is a hyperchaos C^2.

The system (1.2"'): For $r = 1$ and $\alpha > 0$, the signs of the eigenvalues (3.8) are $\lambda_1 = 0, \lambda_2 = -(\sigma+1) < 0, Re\lambda_{3/4} > 0$. This implies a form $(+,+,0,-)$, which shows that the attractor is a hyperchaos C^2.

□

The Lyapunov function: The stability of the dynamical system with the Lyapunov function is proved. Therefore, its constructing is called the Lyapunov's second method for the stability of a dynamical system. For the stability of the system (1.2) at the fixed point $O(0,0,0,0)$ via the following Theorem 3.5, the Lyapunov function is defined.

Theorem 3.5. *Let $O(0,0,0)$ be a fixed point for the system (1.1). Let the Lyapunov function as a continuously differentiable function for the system (1.1) be defined with the following function*

$$L(x,y,z,u) = \frac{x^2}{2\sigma} + \frac{y^2}{2} + \frac{z^2}{2} + \frac{u^2}{2P} \tag{3.9}$$

where $P > 0, L(x,y,z,u) > 0, \forall(x,y,z,u) \neq (0,0,0,0)$ and $L(0,0,0,0) = 0$. Then the fixed point $O(0,0,0,0)$ is stable when $0 < r < 1$ and $S < 0$. If $\dot{L}(x,y,z,u) < 0, \forall(x,y,z,u) \neq (0,0,0,0)$ then the fixed point $O(0,0,0,0)$ is asymptotically stable for $0 < r < 1$ and $S < 0$.

Proof. The system (1.2) at the fixed point $O(0,0,0,0)$ is stable when for a continuously differentiable function $L : D \to R, D \subset R^4$ exactly is $\frac{dL}{dt} \leq 0$, where $L(x,y,z,u) > 0, \forall(x,y,z,u) \neq (0,0,0,0)$ and $L(0,0,0,0) = 0$. We differentiate the Lyapunov function:

$$\dot{L}(x,y,z,u) = \frac{dL}{dt} = \frac{x\dot{x}}{\sigma} + y\dot{y} + z\dot{z} + \frac{u\dot{u}}{P} = x(y-x) + y[x(r-z) - y] + \frac{S}{P}u^2$$

By using mathematical transformations and for $P > 0, \sigma > 0, r > 0$, we obtained

$$\dot{L}(x,y,z,u) = -[x - \frac{r+1-z}{2}y]^2 - [1 - (\frac{r+1-z}{2})^2]y^2 - (-\frac{S}{P})u^2 < 0$$

The inequality implies that $r < 1 + z$ and $-\frac{S}{P}?0$ i.e. for $P > 0$ and from $-S > 0 \Rightarrow S < 0$. As we observe in the neighborhood of the fixed point $O(0,0,0)$, then $|z| < c, c \in (0,\varepsilon)$ for the given $\varepsilon > 0$. This indicates that the system (1.2) at the fixed point $O(0,0,0,0)$ is stable when $0 < r < 1$ and $S < 0$. When $\dot{L}(x,y,z,u) < 0$ then the fixed point $O(0,0,0,0)$ is asymptotically stable for $0 < r < 1$ and $S < 0$. □

By Theorem 3.5 for the system (1.2), the Lyapunov function is defined. Because the system (1.2) presents a simpler record for systems (1.2'), (1.2") and (1.2"'), in the following Corollary 3.3 for the same Lyapunov function (3.9), the conditions for asymptotically stable of the fixed point $O(0,0,0,0)$ for every system individually will be given.

Theorem 3.6. *For the same Lyapunov function (3.9) where* $S = k_1 + k_2, P = k_1k_2, k_1, k_2 \in R$ *, for the system (1.2'),* $S = 2k, P = k^2, k \in R$ *for the system (1.2") and* $S = 2\alpha, P = \alpha^2 + \beta^2, \alpha, \beta \in R$ *for the system (1.2"'), the fixed point* $O(0,0,0,0)$ *is stable when* $0 < r < 1$ *and*
- $k_1, k_2 < 0$ *for the system (1.2');*
- $k < 0$ *for the system (1.2");*
- $\alpha < 0$ *for the system (1.2"'). If* $\dot{L}(x,y,z,u) < 0, \forall(x,y,z,u) \neq (0,0,0,0)$ *then the fixed point* $O(0,0,0,0)$ *is asymptotically stable for* $0 < r < 1$ *and*
- $k_1, k_2 < 0$ *for the system (1.2');*
- $k < 0$ *for the system (1.2");*
- $\alpha < 0$ *for the system (1.2"').*

Proof. By using $S = k_1 + k_2, P = k_1k_2, k_1, k_2 \in R$ for the system (1.2'), $S = 2k, P = k^2, k \in R$ for the system (1.2") and $S = 2\alpha, P = \alpha^2 + \beta^2, \alpha, \beta \in R$ for the system (1.2"'), the proof is identical with the proof of Theorem 3.5. □

4. Conclusion

In this paper, the basic properties of the dynamical analysis for the third-order and the fourth-order shortened Lorenz systems as a symmetry of the systems, a dissipative of the systems, finding of the fixed point, analysis of the behavior of the systems in a neighborhood of the fixed point, and the defining of a Lyapunov function were given. This paper is a good base for the next research as numerical calculations and the graphical interpretations of the spectra of the Lyapunov exponents for the concrete values of the parameters of the systems, calculation of the largest Lyapunov exponent, 2D and 3D graphical presentations of the systems with an accent of the system (1.2"'). The graphical interpretations of a hypertorus T^3, a hyperchaos C^2 and chaos on T^3 for the concrete values for the parameters of the fourth-order shortened Lorenz systems will be interesting. But, these things will stay for future studies.

References

[1] *B. Zlatanovska, D. Dimovski* A Modified Lorenz system: Definition and solution, Asian-European Journal of Mathematics (2020) 2050164 (7 pages)

[2] *2 B. Zlatanovska, D. Dimovski*, Systems of differential equations approximating the Lorenz system, Proc. CMSM4 (dedicated to the centenary of Vladimir Andrunachievici 1917-1997), Chisiau, Republic of Moldova (2017), pp.359-362

[3] *B. Zlatanovska, B. Piperevski* , Dynamic analysis of the Lorenz system, Asian-European Journal of Mathematics (2020) 2050171 (12 pages)
[4] *K. T. Alliggood, T. D. Yorke* An Introduction to Dynamical Systems (Springer-Verlag, USA, 2000), pp. 359-370
[5] *R. Barrio* Performance of the Taylor series method for ODEs/DAEs, Computer Math. Appl. 163 (2012) pp. 525-545
[6] *M.A. Fathi* An analytical solution for the modified Lorenz system, in Proc, World Congress on Engineering, Vol. 1 (London, U.K., 2012), pp.230-233
[7] *M. W. Hirsch, S. Smale, R.L. Devaney* Differential Equations, Dynamical Systems ana an Introduction to Chaos (Elsevier, USA, 2004) pp. 303-324
[8] *L.S. Pontryagin* Ordinary Differential Equations, Russian edn. (Science, Moscow, 1970)
[9] *B. Zlatanovska, D. Dimovski* Systems of difference equations approximating the Lorentz system of differential equations, Contributions Sec. Math. Tech. Sci. Manu. XXXIII 1–2 (2012) 75–96
[10] *B. Zlatanovska, D. Dimovski* Systems of difference equations as a model for the Lorentz system, in Proc. 5th Int. Scientific Conf. FMNS, Vol. I (Blagoevgrad, Bulgaria, 2013), pp. 102–107
[11] *B. Zlatanovska, D. Dimovski* Models for the Lorenz system, Bull. Math. 42(2) (2018) 75–84
[12] *B. Zlatanovska, N. Stojkovic, M. Kocaleva, A. Stojanova, L. Lazarova, R. Gobubovski* Modeling of some chaotic systems with any logic software, TEM J. 7(2) (2018) 465–470
[13] *B. Zlatanovska* Numerical analysis of behavior for Lorenz system with Mathematica, Yearbook 2014 3(3) (2014) 63–71
[14] *B. Zlatanovska* Approximation for the solutions of Lorenz system with systems of differential equations, Bull. Math. 41(1) (2017) 51–61
[15] Lyapunov exponent, ScienceDirect, https://www.sciencedirect.com/topics/engineering/lyapunov-exponent
[16] *M. Sandri* Numerical calculations of Lyapunov exponents, Vol. 6, Issue 3 (1996), pp. 78-84
[17] *W. Guang-Yi, L. Jing-Biao, Z. Xin* Analysis and implementation of a new hiperchaotic system, Chin. Phys. Soc. IOP Publ. 16(8) (2007) 2278–2284
[18] *L. S. Tee, Z. Salleh* Dynamical analysis of a modified Lorenz system, J. Math. Hindawi Publ. Corp. ID 820946 (2013) 1–8
[19] *Chen Yong, Yang Yun-Qing* A new four-dimensional chaotic system, Chin. Phys. B, Vol.19, No.12 (2010), 120510, 5 pages.

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