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CONTENT

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WORKING WITH MATHEMATICALLY GIFTED STUDENTS AGED 16-17

KATERINA ANEVSKA, VALENTINA GOGOVSKA, RISTO MALCHESKI

Abstract. The goals and activities of almost all educational systems state that working with gifted students is of special interest and that it will be given a special priority. However, it seems that this is only a declarative commitment since the care for these students comes down to organizing competitions and preparing students for just several days before competitions. We believe that this approach does not even remotely meet the needs of gifted students, therefore, in this paper we have made an attempt to develop a curriculum for working with mathematically gifted students aged 16-17, that is, for students in the second year of secondary education.
Key Words: gifted students, curriculum, curriculum goals and activities.

1. Introduction

Paper [2] provides the curriculum for working with mathematically gifted students from their first year in secondary education, that is, for students aged 15-16. The development of this curriculum, as well as the curriculum that we will present hereinafter, aims to make up for the weaknesses in working with mathematically gifted students. Namely, we believe that it is not enough to solely declare the need for care for these students, provide 2 classes to work with them and for the competent authorities to accredit citizens' associations for organizing and conducting competitions, without thereby developing a curriculum and appropriate materials for its realization. That is why we will here provide a curriculum for the mathematically gifted students at the age of 16-17, that is, for students from the second year of secondary education. During the preparation of this program, the knowledge about work with gifted students given in the papers [5], [6,], [8] and [9] was used.

This paper is in a way a continuation of the abovementioned papers. In addition, based on the experience of the authors, but also the experience of the countries in the immediate and wider surrounding, an attempt was made for the part of the topic Inequalities to give an example of a system of problems that would determine the level that students should reach at this age.

2. Curriculum for working with mathematically gifted students aged 16-17

In this section, we will present a curriculum for working with mathematically gifted students aged 16-17, that is, for students in the second year of secondary education. The offered curriculum actually builds on the respective teaching curriculum that was previously prepared for students in the first year of secondary education and is presented in paper [2]. During the preparation of the curriculum, the method of concentric circles was used, which means that part of the contents that were adopted in the previous years at a certain level are expanded and extended. This curriculum should be implemented continuously, and not only in periods when students are preparing for certain math competitions. The goals of the curriculum for students aged 16-17 are:

- To develop students' qualities of thinking such as: flexibility, stereotyping, width, rationality, depth and critical thinking.
- To strive for the student to adopt scientific methods: observation, comparison, experiment, analysis, synthesis, classification, systematization and axiomatic method,
- To strive for the student to adopt the types of conclusions: induction, deduction and analogy, whereby it is of particular importance to present suitable examples from which the student will realize that the analogy conclusion is not always correct,
- The student to adopt the prescribed contents in the field of complex numbers and to enable them to apply them when solving appropriate problems,
- The student to adopt the prescribed contents in the field of quadratic equations, inequalities and quadratic functions and to enable them to apply these when solving appropriate problems,
- The student to adopt the prescribed contents in the field of inequalities and to enable them to apply them when solving appropriate problems,
- The student to adopt the prescribed contents in the field of linear differential equations and to enable them to apply the same when solving appropriate problems,
- The student to adopt the prescribed contents in the field of combinatorics and to enable them to apply them when solving appropriate problems,
- The student to adopt the prescribed contents in the field of exponential and logarithmic equations, functions and inequalities and to enable them to apply these when solving appropriate problems,
- The student to adopt the prescribed contents in the field of the Number theory and to enable them to apply them when solving appropriate problems,
- The student to adopt the prescribed contents in the field of trigonometry and to enable them to apply the same when solving appropriate problems,
- The student to adopt the prescribed contents in the field of a plane figure and to enable them to apply the same when solving appropriate problems, and
- The student to adopt the prescribed contents in the field of stereometry and to enable them to apply them when solving appropriate problems.

In order to achieve the aforementioned goals, it is necessary to adopt the following contents:

Algebra (4 classes per week – 144 classes in a school year).

Field of complex numbers: notion of a complex number (complex number as an ordered pair of real numbers), addition, subtraction and multiplication of complex numbers, basic properties, complex conjugate number, modulus of a complex number, distance in C (set of complex numbers) , imaginary unit, algebraic notation of a complex number, triangle inequality and parallelogram equation, division of complex numbers, raising complex numbers to a power, field of complex numbers, square root of a complex number, geometric interpretation of complex numbers, complex plane, partitioning a line-segment in a given ratio.

Quadratic equation and quadratic function: notion of a quadratic equation, types of quadratic equations, solving incomplete quadratic equations, solving complete quadratic equations, discriminant of quadratic equation (nature of the solutions of a quadratic equation), Vieta's formulas, application, factoring quadratic trinomial to linear factors, equations reducible to quadratic (fractional-rational and biquadratic), systems of quadratic equations, application of quadratic equations and systems of quadratic equations, notion and definition of parabola, quadratic function, graph of quadratic function: $f(x)=ax^2, f(x)=ax^2+c,\ f(x)=ax^2+bx+c$, properties of the quadratic function and the graph, the sign of the quadratic function, extreme values of a quadratic function, the position of a given number in relation to the zeros of the quadratic function.

Inequalities: Cauchy–Bunyakovsky-Schwarz inequality, Engel's principle of minimum, Abel equations, rearrangement inequality, Chebyshev's inequality, geometric inequalities.

Quadratic and irrational inequalities: quadratic inequalities, system quadratic inequalities, irrational equations, irrational inequalities, application of quadratic inequalities and their systems.

First-order and second-order linear differential equations: first-order linear differential equation, second-order linear differential equation, second-order linear homogeneous differential equation with constant coefficients, system differential equations of type

$$\begin{cases} x_{n+1} = px_n + qy_n \\ y_{n+1} = rx_n + sy_n \end{cases}$$

solving some second-order linear nonhomogeneous differential equations with constant coefficients, triangular numbers, Fibonacci numbers, Lucas numbers, solving some nonlinear differential equations.

Basic combinatorial configurations: mathematical induction (revision), principle of sum, product, inclusion and exclusion (revision), permutations without repetitions and permutations with repetitions, variations without repetition and variations with repetition (variations of a given type), combinations without repetition and combinations with repetition, triangulation of n-gon and problem of parenthesis, Catalan numbers, binomial formula, polynomial formula.

Exponential function, logarithmic function, exponential and logarithmic equations and inequalities: exponential function, properties and graph, exponential equations, inverse function (revision) and notion of logarithm, basic properties of logarithm, logarithmic function, properties and graph, logarithmic equations, system of exponential and logarithmic equations, exponential and logarithmic inequalities, powers with irrational exponent, exponential, and logarithmic inequalities.

Number theory: divisibility - general and special divisibility criteria, GCD (The greatest common divisor) , Euclidean algorithm and LCD (lowest common denominator), prime and complex numbers, fundamental theorem of arithmetic and functions $y=[x]$, $y=\{x\}$, multiplicative functions, perfect numbers, convolution product, congruence, properties of congruences and their application , residue systems (complete and reduced) and their properties, Euler function, Euler theorem, Fermat's little theorem, Carmichael's theorem and Wilson's theorem, linear congruent equation, systems of linear congruent equations and Chinese remainder theorem, non-linear congruence equations, linear Diophantine equation, Euler method for Diophantine equations, Pythagorean triples, Fermat's method, sum of four squares, sum of two squares, general quadratic Diophantine equations, exponential Diophantine equations.

Geometry (3 classes per week – 108 classes in a school year).

Trigonometry: trigonometry of a right triangle , extension of the term angle, directed angle, measurement of angles and arcs, trigonometric circle, trigonometric functions of arbitrary angle, signs of trigonometric functions in separate quadrants, basic relations between trigonometric functions of the same angle, reduction of trigonometric functions of an arbitrary angle to trigonometric functions of acute angle, periodicity, even-odd properties of trigonometric functions, increase and decrease intervals, transformations of trigonometric functions, graphs of basic trigonometric functions, graph and basic properties of the function $y=a\sin(bx+c)$ and similar functions, trigonometric functions of sum and difference of two angles, trigonometric functions of double-angle and half-angle expressed by the function of that angle, transformation of algebraic sum of trigonometric functions into a product and vice versa, inverse trigonometric functions, transformations of inverse trigonometric functions, basic trigonometric equations, homogeneous trigonometric equations, equations of type $a\sin x+b\cos x=c$, where *a,b* and *c* are real numbers, basic trigonometric inequalities, systems of trigonometric equations and trigonometric inequalities, law of sines, application, law of cosines, application, solving a triangle, solving a quadrilateral.

Area of a plane figure: notion of area, area of a parallelogram, area of a triangle, Heron's and other formulas for area of a triangle, areas of similar triangles, area of a trapezoid and trapezium, perimeter and area of a regular polygon, perimeter of a circle, area of a circle, arc length, and areas of parts of a circle

Stereometry: perpendicularity of lines and planes, dihedron, trihedron and edge, notion of polyhedron, regular polyhedra, tetrahedron, prism, intersections of a prism with a plane, area and volume of a prism, Cavalieri's principle, pyramid, intersections of a pyramid with a plane, area and volume of a pyramid, truncated pyramid, notion of rotational, cylindrical and conical surface, cylinder, area and volume, cone, area and volume, truncated cone, area and volume, sphere and ball, intersections of a ball, area and volume of a ball and its parts, application of trigonometry in stereometry.

3. **Example of a system of problems from the section on Inequalities**

In order to realize the suggested curricula for working with gifted 16-17 years old students, it is necessary to make appropriate teaching aids, that is to say, textbooks that must be accompanied by appropriate books with collections of problems. Hereinafter, we will present a system of problems for Inequalities which we deem will help students acquire knowledge and skills to achieve the curriculum's goal at the highest possible level. The selection of tasks is made according to books [1], [3], [5], [6], [7], [10] and [11]. At the beginning, we will provide several tasks related to means, in order to ensure revision of the previously adopted contents in the field of Inequalities:

1. Let $a_1, a_2, ..., a_n$ be positive real numbers such that $a_1 a_2 ... a_n = 1$. Prove that
$$(4+a_1)(4+a_2)...(4+a_n) \geq 5^n.$$
2. Prove that for non-negative real numbers a and b the following is valid
$$\frac{(a+b)^2}{2} + \frac{a+b}{4} \geq a\sqrt{b} + b\sqrt{a}$$
3. Let a_1, a_2, a_3, a_4, a_5 be positive real numbers such that
$$a_1 + a_2 + a_3 + a_4 + a_5 = 1.$$
Prove that
$$\left(\frac{1}{a_1}-1\right)\left(\frac{1}{a_2}-1\right)\left(\frac{1}{a_3}-1\right)\left(\frac{1}{a_4}-1\right)\left(\frac{1}{a_5}-1\right) \geq 1024.$$
4. Let $a,b,c \in \mathbf{R}^+$ be such that $(a+b)(b+c)(c+a) = 8$. Prove the inequality
$$\frac{a+b+c}{3} \geq \sqrt[27]{\frac{a^3+b^3+c^3}{3}}.$$
5. Let a,b,c be real numbers such that $a+b+c=1$. Prove that
$$\frac{a^3}{a^2+b^2} + \frac{b^3}{b^2+c^2} + \frac{c^3}{c^2+a^2} \geq \frac{1}{2}.$$
6. Let a,b and c be positive real numbers. Prove that
$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}.$$
7. Let a,b,c be positive real numbers. Prove that
$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}.$$
8. Find the greatest number A so that the inequality

$$\frac{x}{\sqrt{y^2+z^2}}+\frac{y}{\sqrt{z^2+x^2}}+\frac{z}{\sqrt{x^2+y^2}}\geq A$$

is fulfilled for every positive real number x,y,z.

9. Prove the inequality:

$$(n!)^2<[\frac{(n+1)(n+2)}{6}]^n,\quad n>1.$$

10. Let $x_1,x_2,...,x_n$ be non-negative real numbers and $r_1,r_2,...,r_n$ positive rational numbers such that $r_1+r_2+...+r_n=1$. Prove that

$$x_1^{r_1}x_2^{r_2}...x_n^{r_n}\leq r_1^{r_1}r_2^{r_2}...r_n^{r_n}(x_1+x_2+...+x_n)$$

11. Find the smallest natural number M so that the inequality

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)|\leq M(a^2+b^2+c^2)^2$$

is valid for every real number a,b,c.

12. If $a\geq 1,\ b\geq 1$, then

$$3(\frac{a^2-b^2}{8})^2+\frac{ab}{a+b}\geq\sqrt{\frac{a^2+b^2}{8}}.$$

Prove it!

13. The real numbers $x_1,x_2,...,x_6$ fulfil the inequalities

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2=6$$
$$x_1+x_2+x_3+x_4+x_5+x_6=0.$$

Prove that

$$x_1x_2x_3x_4x_5x_6\leq\frac{1}{2}.$$

14. If for the positive real numbers a,b,c,d, $a^2+b^2+c^2+d^2=1$ is valid, then prove that

$$a^2b^2cd+ab^2c^2d+abc^2d^2+a^2bcd^2+a^2bc^2d+ab^2cd^2\leq\frac{3}{32}.$$

15. Let a,b,c be positive real numbers such that

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\geq 1.$$

Prove that

$$a+b+c\geq ab+bc+ca.$$

16. The real numbers a,b,c satisfy the condition $a^2+b^2+c^2=3$. Prove that

$$|a|+|b|+|c|-abc\leq 4.$$

17. Prove that for the positive real numbers a,b,c such that $a+b+c=3$,

$$\frac{a^2+3b^2}{ab^2(4-ab)}+\frac{b^2+3c^2}{bc^2(4-bc)}+\frac{c^2+3a^2}{ca^2(4-ca)}\geq 4$$

is valid.

18. Let x, y, z be positive numbers such that $xyz \geq 1$. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2} \geq 0.$$

19. Let $a_i, i = 1,2,...,n$ be non-negative numbers. Prove that the inequality

$$(\sum_{i=1}^{n} a_i x_i^2)^2 \leq \sum_{i=1}^{n} a_i x_i^4$$

is true for arbitrary real numbers $x_i, i = 1,2,...,n$ if and only if $\sum_{i=1}^{n} a_i \leq 1$.

20. The positive real numbers $x_1, x_2, ..., x_{1997}$ satisfy the condition

$$x_{n+1}^2 \geq \frac{x_1^2}{1^3}+\frac{x_2^2}{2^3}+...+\frac{x_n^2}{n^3}, \text{ for } n = 1,2,...,1996.$$

Prove that there is a natural number N such that

$$\sum_{n=1}^{N} \frac{x_{n+1}}{x_1+x_2+...+x_n} > 1{,}997.$$

21. Let $a,b,c,d \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and let

$$\sin a + \sin b + \sin c + \sin d = 1, \ \cos 2a + \cos 2b + \cos 2c + \cos 2d \geq \frac{10}{3}.$$

Prove that $a,b,c,d \in [0, \frac{\pi}{6}]$.

22. Let x, y, z be positive real numbers. Prove that

$$\frac{x^3}{x^3+2y^3}+\frac{y^3}{y^3+2z^3}+\frac{z^3}{z^3+2x^3} \geq 1.$$

23. Let x, y, z be real numbers such that $x + y + z = 0$. Prove that

$$\frac{x(x+2)}{2x^2+1}+\frac{y(y+2)}{2y^2+1}+\frac{z(z+2)}{2z^2+1} \geq 0.$$

When is the sign of equality valid?

24. Let x, y, z be positive real numbers and $x + y + z = 1$. Prove that

$$\frac{x}{y^2+z}+\frac{y}{z^2+x}+\frac{z}{x^2+y} \geq \frac{9}{4}.$$

25. Let x, y, z be positive real numbers such that $x + y + z = 1$. Prove that $A \geq B^2$, where

$$A = \frac{(1+xy+yz+zx)(1+3x^2+3y^2+3z^2)}{9(x+y)(y+z)(z+x)} \text{ and } B = \frac{x\sqrt{x+1}}{\sqrt[4]{3+9x^2}}+\frac{y\sqrt{y+1}}{\sqrt[4]{3+9y^2}}+\frac{z\sqrt{z+1}}{\sqrt[4]{3+9z^2}}.$$

26. Prove that, if $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ are non-negative real numbers and $c_k = \prod_{i=1}^{k} b_i^{\frac{1}{k}}, k = 1, 2, ..., n$, then

$$nc_n + \sum_{k=1}^{n} k(a_k - 1)c_k \le \sum_{k=1}^{n} a_k^k b_k.$$

27. Prove the following inequality for the positive numbers x and y:

$$\frac{1}{\sqrt[4]{x}} + \frac{1}{\sqrt[4]{y}} \ge \frac{2^{\frac{5}{4}}}{\sqrt[4]{x+y}}.$$

28. Prove that for every natural number n the following inequality is fulfilled

$$(2n^2 + 3n + 1)^n \ge 6^n (n!)^2.$$

29. Let $x_1, x_2, ..., x_n$ be real numbers such that $x_1 \ge x_2 \ge \ \ge ... \ge x_n \ge 0 = x_{n+1}$. Prove that

$$\sqrt{x_1 + x_2 + ... + x_n} \le \sum_{i=1}^{n} \sqrt{i}(\sqrt{x_i} - \sqrt{x_{i+1}}).$$

30. Let $y_1 \ge y_2 \ge ... \ge y_n > 0$ and $x_1, x_2, ..., x_n$ be positive real numbers such that $x_1 x_2 ... x_k \ge y_1 y_2 ... y_k$, for every $k = 1, 2, ..., n$. Prove that

$$x_1 + x_2 + ... + x_n \ge y_1 + y_2 + ... + y_n.$$

31. Let $a_1, a_2, ..., a_n$ be arbitrary real numbers and $b_1, b_2, ..., b_n$ are real numbers such that $1 \ge b_1 \ge b_2 \ge ... \ge b_n \ge 0$. Prove that there is a natural number $k \le n$ such that

$$|a_1 b_1 + a_2 b_2 + ... + a_n b_n| < |a_1 + a_2 + ... + a_k|.$$

32. a) Let a, b, c be positive real numbers. Prove that

$$a^2 b + b^2 c + c^2 a \le a^3 + b^3 + c^3.$$

b) Let $x_i, y_i \in \boxed{?}$, are real numbers $i = 1, 2, ..., n$ and let $x_1 \ge x_2 \ge ... \ge x_n$, $y_1 \ge y_2 \ge ... \ge y_n$. Prove that for every permutation $z_1, z_2, ..., z_n$ of numbers $y_1, y_2, ..., y_n$ the following is valid:

$$\sum_{i=1}^{n} (x_i - y_i)^2 \le \sum_{i=1}^{n} (x_i - z_i)^2.$$

33. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2+c^2}{b} + \frac{b^2+a^2}{c} + \frac{c^2+b^2}{a} \ge 2(a+b+c).$$

34. For every a, b, c (positive real numbers) the following is valid:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}.$$

Prove it!

35.Let $a_1 \geq a_2 \geq \ldots \geq a_n > 0$ and π be arbitrary permutation of the set $\{1,2,\ldots,n\}$. Then the inequality is valid

$$n \leq \frac{a_1}{a_{\pi(1)}} + \frac{a_2}{a_{\pi(2)}} + \ldots + \frac{a_n}{a_{\pi(n)}}.$$

36.Let $x_1, x_2, \ldots, x_k$ be different real numbers such that

$$\sum_{i=1}^{k} x_i \neq 0.$$

Prove that there are integers $n_1, n_2, \ldots, n_k$ such that

$$\sum_{i=1}^{k} n_i x_i > 0$$

and that for every non-identical permutation π of $\{1,2,\ldots,k\}$ is valid

$$\sum_{i=1}^{k} n_i x_{\pi(i)} < 0.$$

37.Prove that for every positive real number x_i, $i = 1,2,\ldots,n$ the following equality is fulfilled

$$\frac{1}{\frac{1}{1+x_1}+\frac{1}{1+x_2}+\ldots+\frac{1}{1+x_n}} - \frac{1}{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}} \geq \frac{1}{n}.$$

When is the sign of equality valid?

38.Find n real numbers $x_1 \leq x_2 \leq \ldots \leq x_n$ which fulfil the inequality

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leq n \sum_{i=1}^{n} x_i x_{n-i+1}.$$

39.Prove that

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)},$$

where a,b,c,x,y and z are positive real numbers such that $a \geq b \geq c$ and $x \leq y \leq z$.

40.Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \ldots + a_n = n$. Prove that

$$a_1^{n+1} + a_2^{n+1} + \ldots + a_n^{n+1} \geq a_1^n + a_2^n + \ldots + a_n^n.$$

41.Let $a,b,c > 0$ and $a + b + c = 1$. Prove that

$$\frac{9}{10} \leq \frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} < 1.$$

42. Let AD, BE and CF be the medians of triangle ABC. If $m=\overline{AD}+\overline{BE}+\overline{CF}$ and $s=\overline{AB}+\overline{BC}+\overline{CA}$, prove that

$$\frac{3}{4}s<m<\frac{3}{2}s.$$

43. $A_1A_2A_3$ is a triangle with sides $a_1=\overline{A_2A_3}$, $a_2=\overline{A_3A_1}$ and $a_3=\overline{A_1A_2}$. Let s_1,s_2,s_3 be the lengths of the tangents of the inscribed circle in the triangle which begin from A_1,A_2,A_3, respectively. Prove that

$$\frac{s_1}{a_1}+\frac{s_2}{a_2}+\frac{s_3}{a_3}\geq\frac{3}{2}.$$

When is the sign of equality valid?

44. Let P be an inner point in triangle ABC and let D, E and F be its orthogonal projections of lines BC, CA and AB, respectively. Determine all points P for which the sum $\frac{\overline{BC}}{\overline{PD}}+\frac{\overline{CA}}{\overline{PE}}+\frac{\overline{AB}}{\overline{PF}}$ is lowest.

45. Let G and O respectively, be the centroid and the circumcentre of the circumscribed circle around triangle ABC, whereas R and r are the radii of the circumscribed and inscribed circle respectively. Prove that

$$\overline{OG}\leq\sqrt{R(R-2r)}.$$

46. A line intersects the sides AB and BC of the triangle ABC in points M and K, respectively. If the area of triangle ABK is equal to the area of the quadrilateral $AMKC$, prove that

$$\frac{\overline{MB}+\overline{BK}}{\overline{AM}+\overline{CA}+\overline{KC}}\geq\frac{1}{3}.$$

47. Let there be a triangle ABC with sides a,b,c and area S.

a) Prove that there is a triangle $A_1B_1C_1$ with sides $\sqrt{a},\sqrt{b},\sqrt{c}$.

b) If S_1 is the area of the triangle $A_1B_1C_1$, prove that $S_1^2\geq\frac{S\sqrt{3}}{4}$.

48. Points C and D belong to the line segment AB so that $\overline{AC}=\overline{BD}<\frac{1}{2}\overline{AB}$. Prove that for an arbitrary point O which does not belong to the line AB the following is fulfilled

$$\overline{OA}+\overline{OB}>\overline{OC}+\overline{OD}.$$

49. Let d be the sum of the lengths of the diagonals of the convex $n-$gon, ($n>3$), and p be its perimeter. Prove that

$$n-3<\frac{2d}{p}<[\tfrac{n}{2}][\tfrac{n+1}{2}]-2.$$

50. In the tetrahedron $ABCD$, $BD \perp CD$ is valid and the foot of the altitude starting from the vertex D coincides with the orthocentre of the triangle ABC. Prove that

$$(\overline{AB} + \overline{BC} + \overline{AC})^2 \leq 6(\overline{AD}^2 + \overline{BD}^2 + \overline{CD}^2).$$

When is the sign of equality valid?

4. **Conclusion**

Systematic and organized work with gifted students should be a priority because it is a basic precondition for their faster progress. Work should be organized according to the curriculum appropriate for the age of the students, and we have already presented the curriculum for students aged 16-17. We believe that the adoption of theoretical knowledge provided by this curriculum, supported by the collection of problems listed in the presented references, will allow:

- Students to adopt scientific methods and the methods of deduction at a higher level,
- Improving the qualities of students' thinking, especially the depth, width and critical thinking in general,
- Students to systematically prepare for participation in math competitions, including those of higher rank, such as: BMO, APMO, EGMO, IMO, etc.

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