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CONTENT

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SOLUTION AND STABILITY OF A NEW RECIPROCAL TYPE FUNCTIONAL EQUATION

SADANI IDIR

Abstract. The aim of this paper is to obtain the general solution of the following new reciprocal type functional equation

$$\frac{1}{f(x+u,y+v)}=\frac{1}{2f(2x+y,2u+v)}+\frac{1}{2f(y,v)}$$

and investigate its generalized Hyers-Ulam stability in Banach spaces using the direct method. We also show Hyers-Ulam-Rassias stability, Ulam-Găvruta-Rassias stability and J. M. Rassias stability controlled by the mixed product-sum of powers of norms for the same equation.

1. Introduction

The mathematical theory of stability of functional equations gained its significance in the nineties when an interesting talk exhibited by S. M. Ulam [1] in 1940, set off the study of stability problems for various functional equations. The main motivation was from the study of the stability of homomorphisms. Pioneering work in this timeframe was done by Hyers [2] for the Cauchy's functional equation in Banach spaces. Furthermore, the result of Hyers has been generalized by Rassias [3]. Since then, the subject has been extensively explored by many authors, who have discovered and studied many functional equations and the results obtained are very interesting ([10, 7, 5, 6, 14, 8, 9, 11, 13, 12]).

In 2010, K. Ravi and B. V. Senthil Kumar [4] studying the generalized Hyers-Ulam stability for the following reciprocal functional equation

$$g(x+y)=\frac{g(x)g(y)}{g(x)+g(y)}\ (x,y\in X). \tag{1.1}$$

where $g:X\to]0,\infty[$ is a mapping on the space of non-zero real numbers and X is a real normed space.

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Later, Ravi et al., [16] introduced the reciprocal difference functional equation

$$f\left(\frac{x+y}{2}\right) - f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)},$$

and and the reciprocal adjoint functional equation

$$f\left(\frac{x+y}{2}\right) + f(x+y) = \frac{3f(x)f(y)}{f(x)+f(y)}.$$

and investigated its Hyers-Ulam stability, where $f : X \to]0,\infty[$ is a mapping on the space of non-zero real numbers and for all $x, y \in X$.

After that, diverse forms of the reciprocal functional equations were explored which can be found (as examples) in [15, 17, 18, 19, 20, 21].

In this paper, we investigate the Hyers-Ulam stability of the following new reciprocal functional equation

$$\frac{1}{f(x+u,y+v)} = \frac{1}{2f(2x+y,2u+v)} + \frac{1}{2f(y,v)}.$$

Throughout this paper, we denote by X the set $\mathbb{R}^2 \backslash \{(x,y) | x+y=c\}$ such that $c \neq 0$ is the constant appearing in the general solution of considered functional equation.

2. General solution

Theorem 2.1. *Let $f : X \to \mathbb{R}$ be a continuous real-valued function of two real variables satisfying the reciprocal type functional equation*

$$\frac{1}{f(x+u,y+v)} = \frac{1}{2f(2x+y,2u+v)} + \frac{1}{2f(y,v)} \tag{2.1}$$

for all $x, y \in X$. If $f(x,y) \neq 0$ for all $(x,y) \in X$, then f is of the form

$$f(x,y) = \frac{a}{x+y+c},$$

for all $(x,y) \in X$, where $a, c \neq 0$ are constants.

Proof. Let $f : X \to \mathbb{R}$ satisfies (2.1) with $f(x,y) \neq 0$ for all $(x,y) \in X$. First, we show that

$$f(x+y,0) = f(x,y) \tag{2.2}$$

Indeed, equation below is obtained by replacing (x,y,u,v) with $(x,0,y,0)$:

$$\frac{1}{f(x+y,0)} = \frac{1}{2f(2x,2y)} + \frac{1}{2f(0,0)} \tag{2.3}$$

Now, substituting (x,u,y,v) by $(x+y,0,0,0)$ in (2.1), we obtain

$$\frac{1}{f(x+y,0)} = \frac{1}{2f(2x+2y,0)} + \frac{1}{2f(0,0)} \tag{2.4}$$

Comparing equations (2.5) and (2.4), we arrive to

$$\frac{1}{f(2x,2y)}=\frac{1}{f(2x+2y,0)},$$

which implies that $f(2x,2y)=f(2x+2y,0)$. Thus $f(x,y)=f(x+y,0)$.

Now, replacing (x,u,y,v) by $(x,y,0,0)$ in (2.1) we obtain

$$\frac{1}{f(x,y)}=\frac{1}{2f(2x,2y)}+\frac{1}{2f(0,0)} \tag{2.5}$$

which implies that

$$\frac{2}{f(x,y)}-\frac{1}{f(0,0)}=\frac{1}{f(2x,2y)} \tag{2.6}$$

Replacing (x,u,y,v) by $(x+y,0,x+y,0)$ in (2.1) and using equation (2.2), we get

$$\frac{1}{f(x+y,x+y)}=\frac{1}{2f(3x+3y,0)}+\frac{1}{2f(x+y,0)} \tag{2.7}$$

This implies that

$$\frac{1}{f(2x,2y)}=\frac{1}{2f(3x,3y)}+\frac{1}{2f(x,y)} \tag{2.8}$$

Using (2.6) in (2.8), we obtain

$$\frac{3}{f(x,y)}-\frac{2}{f(0,0)}=\frac{1}{f(3x,3y)} \tag{2.9}$$

By induction on $n>0$, we acquire

$$\frac{n}{f(x,y)}-\frac{n-1}{f(0,0)}=\frac{1}{f(nx,ny)}. \tag{2.10}$$

Now, we set $s=f(0,0)$. For $(x,y)=(1,1)$, the equation (2.10) becomes

$$\frac{n}{t}-\frac{n-1}{s}=\frac{1}{f(n,n)} \tag{2.11}$$

for some constant $t=f(1,1)\neq 0$. Then

$$f(n,n)=\frac{st}{(s-t)n+t} \tag{2.12}$$

We can put $a'=\frac{st}{s-t}$ and $c'=\frac{t}{s-t}$ such that $s\neq t$ and $a',c'\neq 0$, we get

$$f(n,n)=f(n+n,0)=f(2n,0)=\frac{a'}{n+c'}=\frac{2a'}{n+2c'} \tag{2.13}$$

Then,

$$f(n,0)=\frac{a}{n+c},$$

with $a=2a'$ and $c=2c'$. Let (x,y) be any fixed element in X, then there is a sequence $\{(x_n,y_n)\}$ of non-zero rational numbers such that $\lim_{n\to\infty}x_n=x$

and $\lim_{n\to\infty} y_n = y$. Then, by the continuity of f, where, we have $f(x,y) = \lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} f(x_n + y_n, 0) = \lim_{n\to\infty} \frac{a}{x_n+y_n+c} = \frac{a}{x+y+c}$. Hence, we conclude that $f(x,y) = \frac{a}{x+y+c}$ for all $(x,y) \in X$, which completes the proof of Theorem. $\square$

3. Stability of equation (2.1)

Theorem 3.1. *Let $\phi : X^2 \to \mathbb{R}$ be a mapping satisfying*

$$\Phi(x,u,y,v) = \sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i u, 2^i y, 2^i v)}{2^i} < \infty \tag{3.1}$$

for all $(x,u,y,v) \in X^2$. Let $f : X \to \mathbb{R}$ be a mapping such that

$$\left\| \frac{1}{f(x+u, y+v)} - \frac{f(2x+y, 2u+v) + f(y,v)}{2f(2x+y, 2u+v)f(y,v)} \right\| \le \phi(x,u,y,v) \tag{3.2}$$

for all $(x,y,u,v) \in X^2$. Then, there exists a unique mapping $h : X \to \mathbb{R}$ which satisfies the functional equation (2.1) and the inequality

$$\left\| \frac{1}{f(x,y)} - \frac{1}{h(x,y)} - \frac{1}{f(0,0)} \right\| \le \Phi(x,0,y,0) \tag{3.3}$$

for all $(x,y) \in X$. The mapping h is defined by

$$h(x,y) = \lim_{n\to\infty} 2^n f(2^n x, 2^n y). \tag{3.4}$$

Proof. Replacing (x,y,u,v) by $(x,0,y,0)$ in (3.2), we get

$$\left\| \frac{1}{f(x+y,0)} - \frac{f(2x,2y) + f(0,0)}{2f(2x,2y)f(0,0)} \right\| = \left\| \frac{1}{f(x,y)} - \frac{1}{2f(2x,2y)} - \frac{1}{2f(0,0)} \right\| \le \phi(x,0,y,0) \tag{3.5}$$

for all $(x,y) \in X$.

Replacing (x,y) by $(2x,2y)$ in (3.5), dividing by 2 and summing the resulting inequality with (3.5), we obtain

$$\left\| \frac{1}{f(x,y)} - \frac{1}{4f(4x,4y)} - \frac{1}{2f(0,0)} - \frac{1}{4f(0,0)} \right\| \le \phi(x,0,y,0) + \frac{1}{2}\phi(2x,0,2y,0) \tag{3.6}$$

Proceeding further and using induction on a positive integer n, we obtain

$$\left\| \frac{1}{f(x,y)} - \frac{1}{2^n f(2^n x, 2^n y)} - \left(1 - \frac{1}{2^n}\right) \frac{1}{f(0,0)} \right\| \le \sum_{i=0}^{n-1} \frac{\phi(2^i x, 0, 2^i y, 0)}{2^i} \tag{3.7}$$

Next, replacing (x, y) by $(2^k x, 2^k y), k \in \mathbb{N}$ in (3.7) and multiplying by 2^{-k}, we get

$$\left\| \frac{1}{2^k f(2^k x, 2^k y)} - \frac{1}{2^k}\left(1 - \frac{1}{2^n}\right)\frac{1}{f(0,0)} - \frac{1}{2^{n+k} f(2^{n+k}x, 2^{n+k}y)} \right\| \leq \sum_{i=0}^{n-1} \frac{\phi(2^{i+k}x, 0, 2^{i+k}y, 0)}{2^{i+k}} \tag{3.8}$$

Since the right-hand side of the inequality (3.8) tends to 0 as k tends to infinity, the sequence $\left\{\frac{1}{2^n f(2^n x, 2^n y)}\right\}$ is a Cauchy sequence. Therefore, we may define $h^{-1}(x, y) = \frac{1}{h(x,y)} = \lim_{n\to\infty} \frac{1}{2^n f(2^n x, 2^n y)}$ for all $(x, y) \in X$. Letting $n \to \infty$ in (3.7), we arrive at (3.3). Next, we have to show that h satisfies (2.1). For that, Replacing x, y, u, v by $2^n x, 2^n y, 2^n u, 2^n v$ in (3.2) and dividing 2^n, it then follows that

$$\frac{1}{2^n}\left\| \frac{1}{f(2^n(x+u), 2^n(y+v))} - \frac{f(2^n(2x+y), 2^n(2u+v) + f(2^n y, 2^n v)}{2f(2^n(2x+y), 2^n(2u+v))f(2^n y, 2^n v)} \right\| \leq \frac{1}{2^n}\phi(2^n x, 2^n u, 2^n y, 2^n v). \tag{3.9}$$

Taking the limit as $n \to \infty$, using (3.4) and (3.1), we see that

$$\left\| \frac{1}{h(x+u, y+v)} - \frac{h(2x+y, 2u+v) + h(y, v)}{2h(2x+y, 2u+v)h(y, v)} \right\| \leq 0. \tag{3.10}$$

which gives

$$\frac{1}{h(x+u, y+v)} - \frac{h(2x+y, 2u+v) + h(y, v)}{2h(2x+y, 2u+v)h(y, v)} = 0. \tag{3.11}$$

Therefore, we have that h satisfies (2.1) for all $(x, u, y, v) \in X^2$. To prove the uniqueness of the mapping h, let us assume that there exists a mapping $h' : X \to \mathbb{R}$ which satisfies (2.1) and the inequality (3.3). We have

$$\begin{aligned}\left\| \frac{1}{h(x,y)} - \frac{1}{h'(x,y)} \right\| &= 2^{-n}\left\| \frac{1}{h(2^n x, 2^n y)} - \frac{1}{h'(2^n x, 2^n y)} \right\| \\ &\leq 2^{-n}\left(\left\| \frac{1}{h(2^n x, 2^n y)} - \frac{1}{f(2^n x, 2^n y)} \right\| + \left\| \frac{1}{f(2^n x, 2^n y)} - \frac{1}{h'(2^n x, 2^n y)} \right\| \right) \\ &\leq 2\sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i x, 2^i y, 2^i y)}{2^{i+n}} \end{aligned} \tag{3.12}$$

for all $(x, y) \in X$. Allowing $n \to \infty$ in (3.12) and using (3.1), we obtain that h is unique. □

Corollary 3.1. *Let $f : X \to \mathbb{R}$ be a mapping such that*

$$\left\| \frac{1}{f(x+u, y+v)} - \frac{f(2x+y, 2u+v) + f(y,v)}{2f(2x+y, 2u+v)f(y,v)} \right\| \leq \epsilon \tag{3.13}$$

for all $x, y \in X$, with $\epsilon > 0$. Then there exists a unique mapping $h : X \to \mathbb{R}$ which satisfies the functional equation (2.1) and the inequality

$$\left\| \frac{1}{f(x,y)} - \frac{1}{f(0,0)} - \frac{1}{h(x,y)} \right\| \leq \epsilon.$$

Proof. Taking $\phi(x, y) = \epsilon$ for all $(x, y) \in X$ in Theorem 3.1, we obtain the required result. □

Corollary 3.2. *Let $f : X \to \mathbb{R}$ be a mapping such that*

$$\left\| \frac{1}{f(x+u, y+v)} - \frac{f(2x+y, 2u+v) + f(y,v)}{2f(2x+y, 2u+v)f(y,v)} \right\| \leq \theta(\|x\|^p + \|u\|^p + \|y\|^p + \|v\|^p) \tag{3.14}$$

for all $x, y \in X$, where, $\theta \geq 0$ and $p > -1$.

Then there exists a unique mapping $h : X \to \mathbb{R}$ which satisfies the functional equation (2.1) and the inequality

$$\left\| \frac{1}{f(x,y)} - \frac{1}{f(0,0)} - \frac{1}{h(x,y)} \right\| \leq \frac{2^{p+2}\theta(\|x\|^p + \|y\|^p)}{2^{p+1} - 1}.$$

Proof. Taking $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $(x, y) \in X$ in Theorem 3.1, thus, the desired result follows. □

Corollary 3.3. *Let $f : X \to \mathbb{R}$ be a mapping such that*

$$\left\| \frac{1}{f(x+u, y+v)} - \frac{f(2x+y, 2u+v) + f(y,v)}{2f(2x+y, 2u+v)f(y,v)} \right\| \leq \theta(\|x\|^p + \|u\|^p)(\|y\|^q + \|v\|^q) \tag{3.15}$$

for all $x, y \in X$, where, $\theta \geq 0$ and $p + q < 1$.

Then there exists a unique mapping $h : X \to \mathbb{R}$ which satisfies the functional equation (2.1) and the inequality

$$\left\| \frac{1}{f(x,y)} - \frac{1}{f(0,0)} - \frac{1}{h(x,y)} \right\| \leq \theta \left(\frac{\|x\|^p \, \|y\|^q}{1 - 2^{p+q-1}} \right).$$

Proof. Taking $\phi(x, y) = \theta(\|x\|^p + \|u\|^p)(\|y\|^q + \|v\|^q)$ for all $(x, y) \in X$ in Theorem 3.1, we get the required result. □

Corollary 3.4. *Let $f : X \to \mathbb{R}$ be a mapping such that*

$$\left\| \frac{1}{f(x+u, y+v)} - \frac{f(2x+y, 2u+v) + f(y,v)}{2f(2x+y, 2u+v)f(y,v)} \right\| \leq \theta(\|x\|^p + \|u\|^p)^q(\|y\|^q + \|v\|^q)^p \tag{3.16}$$

for all $x, y \in X$, where, $\theta \geq 0$ and $pq < 1$. Then, there exists a unique mapping $h : X \to \mathbb{R}$ which satisfies the functional equation (2.1) and the inequality

$$\left\| \frac{1}{f(x,y)} - \frac{1}{f(0,0)} - \frac{1}{h(x,y)} \right\| \leq \theta \left(\frac{\|x\|^{pq} \|y\|^{pq}}{1 - 2^{pq-1}} \right).$$

Proof. Replacing $\phi(x,y) = \theta(\|x\|^p + \|u\|^p)^q(\|y\|^q + \|v\|^q)^p$ for all $(x,y) \in X$ in Theorem 3.1, we get the desired result. □

4. Conclusion and open problem

- In this work, we have obtained the general solutions of the functional equation (2.1). Also, by using direct method, we have studied its Ulam-Hyers stability in which the exact solution of the functional equation (2.1) is explicitly constructed as a limit of a sequence $\{2^n f(2^n x, 2^n y)\}_n$. This kind of problems, in particular, the study of reciprocal type functional equations and its stability is a very interesting subject of mathematics and find many applications in other scientific domains, we refer the interested reader to reference [20].
- As an open problem is to prove that
  (1) The general solution of the following generalized reciprocal type functional equation of the functional equation (2.1)

$$\frac{1}{g(x_1, a_1, \ldots, x_n, a_n)} = \frac{1}{2g\left(\sum_{k=0}^{n-1} x_k + x_n\right)} + \frac{1}{2g(x_n, a_2, a_3, \ldots, a_n)} \tag{4.1}$$

  is of the form

$$g(x_1, \cdots, x_n) = \frac{a}{x_1 + x_2 + \ldots + x_n + b}.$$

  (2) Hyers-Ulam-Rassias stability of (4.1).

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