GOCE DELCEV UNIVERSITY - STIP
FACULTY OF COMPUTER SCIENCE

The journal is indexed in

EBSCO

ISSN 2545-4803 on line
DOI: 10.46763/BJAMI

BALKAN JOURNAL OF APPLIED MATHEMATICS AND INFORMATICS (BJAMI)

BALKAN JOURNAL
OF APPLIED MATHEMATICS AND INFORMATICS

YEAR 2023

VOLUME VI, Number 1

AIMS AND SCOPE:
BJAMI publishes original research articles in the areas of applied mathematics and informatics.

Topics:
1. Computer science;
2. Computer and software engineering;
3. Information technology;
4. Computer security;
5. Electrical engineering;
6. Telecommunication;
7. Mathematics and its applications;
8. Articles of interdisciplinary of computer and information sciences with education, economics, environmental, health, and engineering.

BALKAN JOURNAL
OF APPLIED MATHEMATICS AND INFORMATICS
(BJAMI), Vol 6

ISSN 2545-4803 on line
Vol. 6, No. 1, Year 2023

EDITORIAL BOARD

CONTENT

Balkan Journal of Applied Mathematics and Informatics Online ISSN 2545-4083
Volume VI Number 1 Year 2023 UDK: 515.178.2:517.44

SOME ABELIAN RESULTS FOR THE DISTRIBUTIONAL STOCKWELL TRANSFORM

JASMINA VETA BURALIEVA

Abstract. We give several Abelian results characterizing the quasiasymptotic behavior of distributions at origin (resp. at infinity) in $\mathcal{S}_0'(\mathbb{R})$, in terms of their Stockwell transform, using known quasiasymptotic behavior for the polynomial multiplication and m-th derivative of a distribution $f \in \mathcal{S}_0'(\mathbb{R})$.

1. **Introduction**

Generalized asymptotic analysis is an interesting research subject that took the attention of different authors. It refers to the asymptotic analysis of generalized functions, i.e. distributions. In the present paper, we consider the quasiasymptotics of distributions. The motivation for its introduction came from the theoretical questions in the quantum field theory. Russian mathematicians Vladimirov, Drozinov and Zavialov introduced and analyzed the quasiasymptotics of distributions [19], and apply it in order to analyze the asymptotic behavior of some generalized integral transform, such as the Laplace transform of tempered distributions. The great contribution in this field is also of Pilipović and his coworkers (see, e.g. [8, 10, 11]). In [1, 9, 12, 14, 15, 20] and references therein, one can find the generalized asymptotic analysis of distributions with respect to the asymptotic analysis of the Fourier transform, short-time Fourier transform, directional short-time Fourier transform, wavelet transform, Stockwell transform and other transforms.

One can consider the asymptotic behavior of some integral transform through the quasiasymptotic behavior of distribution, if the corresponding integral transform is defined on distribution space. So, generalized integral transforms are also an interesting research subject that has been elaborated in the last 54 years. The book of Zemanin [21] from 1968 was the first systematic monograph in which different integral transforms of generalized functions are collected. Twenty one years later, Brychov and Prudnikov [2] gave the most important integral transforms of generalized functions.

The Stockwell transform is defined and analyzed by Stockwell [17].The authors of [15] generalized the Stockwell transform to the distribution spaces and done some asymptotic analysis. They give several Tauberian type results relating the quasiasymptotic behavior of Lizorkin distributions to the asymptotic analysis of the generalized Stockwell transform. In the present paper we provide several Abelian type results for the

Date: 18 May 2023.
Keywords. Stockwell transform, distributions, quasiasymptotic behavior, Abelian results.

distributional Stockwell transform, using known quasiasymptotic results for the polynomial multiplication and m-th derivative of a distribution $f \in \mathcal{S}_0'(\mathbb{R})$. An Abelian type result is a result in which the asymptotic behavior of the integral transform of some function (distribution) is obtained, knowing the asymptotic behavior of that function (distribution) see, for example [6, 11, 12, 13, 14, 19, 20]).

2. **Preliminaries**

In this section we give the basic notations and definition of spaces in which we work, and also the basic facts from the quasiasymptotic theory.

2.1. **Notations and spaces.** The operators of modulation, translation and dilatation for the measurable function f on $\mathbb{R}$ are given by $M_a f(\cdot) = e^{ia\cdot} f(\cdot)$, $T_b f(\cdot) = f(\cdot - b)$ and $D_{\frac{1}{c}} f(\cdot) = |c| f(c\,\cdot)$, $a, b \in \mathbb{R}$, $c \in \mathbb{R}\backslash\{0\}$, respectively. The notation $\langle f, \varphi \rangle$ means dual pairing between a distribution f and a test function φ, such that $\langle f, g \rangle = (f, g)_{L^2(\mathbb{R})}$ if $f, g \in L^2(\mathbb{R})$. All dual spaces in the paper are equipped with the strong dual topology, [18].

Through the paper, $\mathcal{S}(\mathbb{R})$ stands for the well known Schwartz space of rapidly decreasing smooth functions, i.e., of functions $\varphi \in \mathbb{C}^\infty(\mathbb{R})$ for which all the norms

$$\rho_{k,n}(\varphi) = \sup_{x \in \mathbb{R}} (1 + |x|^2)^{k/2} |\varphi^{(n)}(x)|, \quad k, n \in \mathbb{N}_0 \tag{2.1}$$

are finite. The dual space of the Schwartz space of rapidly decreasing smooth functions is the space of tempered distributions $\mathcal{S}'(\mathbb{R})$, [16]. The Stockwell transform is defined on the space of highly time-frequency localized test functions over the real line, $\mathcal{S}_0(\mathbb{R})$, and its dual space $\mathcal{S}_0'(\mathbb{R})$, the space of Lizorkin distributions, [15]. We say that one element from $\mathcal{S}(\mathbb{R})$ is in $\mathcal{S}_0(\mathbb{R})$ if all its moments are equal to 0, namely $\varphi \in \mathcal{S}_0(\mathbb{R})$ if $\varphi \in \mathcal{S}(\mathbb{R})$ and $\int_{\mathbb{R}} x^m \varphi(x) dx = 0$ for all $m \in \mathbb{N}_0$, [5].

In the present paper $\mathbb{Y}$ stands for $\mathbb{R} \times (\mathbb{R}\backslash\{0\})$, and $\mathcal{S}(\mathbb{Y})$ is the space of highly localized test functions over $\mathbb{Y}$ as the space of smooth functions Φ on $\mathbb{Y}$ for which

$$\rho_{s,r}^{l,m}(\Phi) = \sup_{(b,a) \in \mathbb{Y}} \left(|a|^s + \frac{1}{|a|^s} \right) (1 + b^2)^{r/2} \left| \frac{\partial^l}{\partial a^l} \frac{\partial^m}{\partial b^m} \Phi(b, a) \right| < \infty \tag{2.2}$$

for all $l, m, s, r \in \mathbb{N}_0$, [5]. The topology of this space is defined by means of the seminorms (2.2). The dual space of $\mathcal{S}(\mathbb{Y})$ is the space $\mathcal{S}'(\mathbb{Y})$ which contains the range of the Stockwell transform [15].

2.2. **Quasiasymptotics of distributions.** The quasiasymptotics of distribution is defined with respect to a slowly varying function. So, a function L is called slowly varying at the origin (resp. at infinity) if it is a measurable real-valued function, defined and positive on the interval $(0, A]$ (resp. $[A, \infty)$), $A > 0$ and

$$\lim_{\varepsilon \to 0^+} \frac{L(a\varepsilon)}{L(\varepsilon)} = 1 \qquad \left(\text{resp.} \lim_{\lambda \to \infty} \frac{L(a\lambda)}{L(\lambda)} = 1 \right) \quad \text{for each} \quad a > 0.$$

More about slowly varying functions one can find in [11].

Let L be a slowly varying function at the origin (resp. at infinity). We say that the distribution $f \in \mathcal{S}_0'(\mathbb{R})$ has quasiasymptotic behavior of degree $\alpha \in \mathbb{R}$ at the origin (resp.

at infinity) with respect to L if there exists $h \in \mathcal{S}_0'(\mathbb{R})$ such that for each $\varphi \in \mathcal{S}_0(\mathbb{R})$ hold

$$\lim_{\varepsilon\to 0^+}\left\langle \frac{f(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(x)\right\rangle = \langle h(x), \varphi(x)\rangle \qquad \left(\text{resp.} \lim_{\lambda\to\infty}\right). \tag{2.3}$$

Through the paper, we use the following notation for the quasiasymptotic behavior at the origin (resp. at infinity)

$$f(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) h(x) \text{ as } \varepsilon \to 0^+ \qquad (\text{resp.}\lambda \to \infty) \text{ in } \mathcal{S}_0'(\mathbb{R}),$$

which should always be interpreted in the week topology of $\mathcal{S}_0'(\mathbb{R})$. The distribution h does not have an arbitrary form, it must be homogeneous with the degree of homogeneity α, [4, 11, 19]. A distribution h is homogeneous with the degree of homogeneity α if it holds $h(ax) = a^\alpha h(x)$, for each $a > 0$, .

In Theorem 2.1 useful properties for the quasiasymptotic behavior of distribution at origin (resp. at infinity) are given, that are important for our results in Section 4.

Theorem 2.1. [11, Proposition 2.8] *Let* $f \in \mathcal{S}_0'(\mathbb{R})$ *and let*

$$f(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) h(x) \quad \text{as} \quad \varepsilon \to 0^+ \quad (\text{resp.} \quad \lambda \to \infty) \quad \text{in} \quad \mathcal{S}_0'(\mathbb{R}),$$

then

(i) $x^m f(\varepsilon x) \sim \varepsilon^{\alpha+m} L(\varepsilon) x^m h(x)$ as $\varepsilon \to 0^+$ (resp. $\lambda \to \infty$) in $\mathcal{S}_0'(\mathbb{R}), \mathrm{m} \in \mathbb{N}$;
(ii) $f^{(m)}(\varepsilon x) \sim \varepsilon^{\alpha-m} L(\varepsilon) h^{(m)}(x)$ as $\varepsilon \to 0^+$ (resp. $\lambda \to \infty$) in $\mathcal{S}_0'(\mathbb{R}), \mathrm{m} \in \mathbb{N}$.

3. **Stockwell transform**

Let $f \in L^2(\mathbb{R})$. The well known Fourier transform

$$\widehat{f}(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\omega} f(x)dx, \ \omega \in \mathbb{R},$$

gives information of the signal f just in a frequency domain, but not in time. So, Gabor in 1946 proposed a new transform which is a modification of the Fourier transform. The so called Gabor transform or short-time Fourier transform (STFT) gives information for the signal at time just in a small section, and is defined as

$$V_g f(b, a) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x)\overline{g}(x-b)e^{-iax}dx, \ a.b \in \mathbb{R}.$$

for $f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Because the window g is fixed, the STFT has some disadvantages, such as disability to detect and analyze low frequencies and incorrect time resolution of high-frequency events. The introduction of the wavelet transform overcomes these disadvantages.

The wavelet transform of $f \in L^2(\mathbb{R})$ and a wavelet $g \in L^2(\mathbb{R})$ (a function for which hold $\int_{-\infty}^{+\infty} \frac{|\widehat{g}(\omega)|^2}{|\omega|} d\omega < \infty$), is defined as

$$\mathcal{W}_g f(b, a) = \int_{\mathbb{R}} f(x)|a|^{-1/2}\overline{g}\left(\frac{x-b}{a}\right)dx, \ b \in \mathbb{R}, \, a \in \mathbb{R}\backslash\{0\}.$$

The wavelet transform produces time-scale plots that are very complicated, and one of its disadvantages is that it does not retain the absolute phase information. In [3, 7] one can find a detailed theory for the wavelet transform.

So, the integral transform introduced and named by Stockwell [17] is a hybrid between the STFT and wavelet transform. The Stockwell transform is considered as a frequency dependent STFT or a phase corrected wavelet transform, because it overcomes the disadvantages of these two transforms. Let $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that $\int_{\mathbb{R}} g(x)dx = 1$. The Stockwell transform, of a signal $f \in L^2(\mathbb{R})$ with respect to the window g is defined by

$$S_g f(b,a) = \frac{|a|}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixa} f(x) \overline{g(a(x-b))} dx = \frac{1}{\sqrt{2\pi}} (f, M_a T_b D_{\frac{1}{a}} g)_{L^2(\mathbb{R})}. \tag{3.1}$$

for all $b \in \mathbb{R}$ and $a \in \mathbb{R}\backslash\{0\}$. The first definition of the Stockwell transform is given for the window $g(x) = \frac{1}{\sqrt{(2\pi)}} e^{-\frac{x^2}{2}}$, [17]. The idea of authors of [15] was to define the Stockwell transform for a larger class of functions, i.e. distributions. So, first they prove a useful relation between Stockwell and Fourier transform

$$S_g f(b,a) = \frac{|a|}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ib(\omega-a)} \widehat{f}(\omega) \overline{\widehat{g}}(\frac{\omega-a}{a}) d\omega$$

for $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $f \in L^2(\mathbb{R})$, [15, Lemma 2.1.]. Then, if $g \in \mathcal{S}(\mathbb{R})$ is a non-trivial window and $\psi \in \mathcal{S}(\mathbb{R})$ is a reconstruction window for it, and if $f \in L^1(\mathbb{R})$ is a function such that $\hat{f} \in L^1(\mathbb{R})$, the following reconstruction formula [15, Prop. 3.1],

$$f(x) = \frac{1}{\sqrt{2\pi} C_{g,\psi}} \int_{\mathbb{R}} \int_{\mathbb{R}} S_g f(b,a) M_a T_b D_{\frac{1}{a}} \psi(x) db \frac{da}{|a|}, \tag{3.2}$$

holds pointwisely, where $C_{g,\psi} := \int_{\mathbb{R}} \widehat{\psi}(\omega-1)\overline{\widehat{g}}(\omega-1)\frac{d\omega}{|\omega|} < \infty$. The inversion formula (3.2) allows one to define an operator that maps functions on $\mathbb{Y}$ to functions on $\mathbb{R}$. This operator named the Stockwell synthesis operator is defined as

$$S_g^* F(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} F(b,a) M_a T_b D_{\frac{1}{a}} g(x) db \frac{da}{|a|}, \tag{3.3}$$

for $g \in \mathcal{S}(\mathbb{R})$, [15]. Let us note that the integral (3.3) is absolutely convergent, if $F \in \mathcal{S}(\mathbb{Y})$, and that the inversion formula (3.2) can also be rewritten as $(S_\psi^* \circ S_g) f = C_{g,\psi} f$. In order to be defined the Stockwell transform and the Stockwell synthesis operator to a distribution spaces, first it is proven that the mappings $S_g : \mathcal{S}_0(\mathbb{R}) \to \mathcal{S}(\mathbb{Y})$ and $S_g^* : \mathcal{S}(\mathbb{Y}) \to \mathcal{S}_0(\mathbb{R})$ are continuous, for $g \in \mathcal{S}_0(\mathbb{R})$, [15, Thrm. 4.1 and 4.2]. Then, for $g \in \mathcal{S}_0(\mathbb{R})$, the definition of the Stockwell transform of f with respect to g via the transposed mapping $< S_g f, \Phi > := < f, S_{\overline{g}}^* \Phi >, \Phi \in \mathcal{S}(\mathbb{Y})$, and its direct definition

$$S_g f(b,a) = \frac{1}{\sqrt{2\pi}} \langle f, \overline{M_a T_b D_{\frac{1}{a}} g} \rangle, \tag{3.4}$$

are equivalent, while the Stockwell synthesis operator $S_g^* : \mathcal{S}'(\mathbb{Y}) \to \mathcal{S}_0'(\mathbb{R})$ with respect to g is defined as $< S_g^* F, \varphi > := < F, S_{\overline{g}} \varphi >, F \in \mathcal{S}'(\mathbb{Y}), \varphi \in \mathcal{S}_0(\mathbb{R})$, [15, Sec. 5]. The mappings $S_g : \mathcal{S}_0'(\mathbb{R}) \to \mathcal{S}'(\mathbb{Y})$ and $S_g^* : \mathcal{S}'(\mathbb{Y}) \to \mathcal{S}_0'(\mathbb{R})$ are also continuous, and the reconstruction formula (3.2) is generalized to the Lizorkin distributions for $g, \psi \in \mathcal{S}_0(\mathbb{R})$, [15, Prop. 5.1 and Thrm.5.1], respectively. In the same paper [15], the authors provide some generalized asymptotic analysis for the Stockwell transform. They prove several Abelian and Tauberian type results characterizing the quasiasimptotic behavior of Lizorkin distributions in terms of their Stockwell transform, using that for $g \in \mathcal{S}_0(\mathbb{R})$ and $K \subset \mathbb{Y}$ compact, the set

$$\{M_a T_b D_{\frac{1}{a}} g : (b,a) \in K\}$$

is bounded in $\mathcal{S}(\mathbb{R})$, [15, Lemma 6.1].

4. **Abelian type results**

In this section we provide several Abelian type results relating the quasiasymptotic behavior of Lizorkin distributions to the asymptotics of their Stockwell transform, using the quasiasymptotic behavior of the polynomial multiplication of Lizorkin distributions and quasiasymptotic behavior of the $m-$th derivative of Lizorkin distributions.

Proposition 4.1. *Let L be a slowly varying function at the origin, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}_0'(\mathbb{R})$. Suppose that $f(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) h(x)$ as $\varepsilon \to 0^+$ in $\mathcal{S}_0'(\mathbb{R})$. Then for its Stockwell transform with respect to a window $g \in \mathcal{S}_0(\mathbb{R}) \backslash \{0\}$ we have*

$$S_g x^m f(\varepsilon b, \frac{a}{\varepsilon}) \sim \varepsilon^{\alpha+2m} L(\varepsilon) S_g x^m h(b,a) \quad as \quad \varepsilon \to 0^+,\ m \in \mathbb{N} \tag{4.1}$$

uniformly on compact subsets of $\mathbb{Y}$.

Proof. By definition of the Stockwell transform (3.4), substitution $\varepsilon t = x$, the quasi-asymptotic behavior of f near the origin and Theorem 2.1 (i) we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{S_g x^m f(\varepsilon b, \frac{a}{\varepsilon})}{\varepsilon^{\alpha+2m} L(\varepsilon)} &= \lim_{\varepsilon\to 0^+} \frac{(2\pi)^{-\frac{1}{2}}}{\varepsilon^{\alpha+2m} L(\varepsilon)} \langle x^m f(x), \overline{M_{\frac{a}{\varepsilon}} T_{\varepsilon b} D_{\frac{1}{\frac{a}{\varepsilon}}} g(x)} \rangle \\
&= \lim_{\varepsilon\to 0^+} \frac{(2\pi)^{-\frac{1}{2}}}{\varepsilon^{\alpha+m} L(\varepsilon)} \langle \Big(\frac{x}{\varepsilon}\Big)^m f(x), e^{-ix\frac{a}{\varepsilon}} \frac{|a|}{\varepsilon} \overline{g}(\frac{a}{\varepsilon}(x-\varepsilon b)) \rangle \\
&= \lim_{\varepsilon\to 0^+} \frac{(2\pi)^{-\frac{1}{2}}}{\varepsilon^{\alpha+m} L(\varepsilon)} \langle t^m f(\varepsilon t), e^{-ita} |a| \overline{g}(a(t-b)) \rangle \\
&= \lim_{\varepsilon\to 0^+} (2\pi)^{-\frac{1}{2}} \langle \frac{t^m f(\varepsilon t)}{\varepsilon^{\alpha+m} L(\varepsilon)}, \overline{M_a T_b D_{\frac{1}{a}} g(t)} \rangle \\
&= (2\pi)^{-\frac{1}{2}} \langle t^m h(t), \overline{M_a T_b D_{\frac{1}{a}} g(t)} \rangle = S_g t^m h(b,a).
\end{aligned}$$

By boundedness of the set (3) and the fact that the weak and the strong topology on $\mathcal{S}_0'(\mathbb{R})$ are equivalent, we obtain the uniform convergence on compact subsets of $\mathbb{Y}$. □

In Proposition 4.2 the asymptotic result relating the quasiasymptotic behavior of Lizorkin distributions at infinity to their Stockwell transform is given. The proof goes on in a similar way as the proof of the Proposition 4.1 .

Proposition 4.2. *Let L be a slowly varying function at infinity, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}_0'(\mathbb{R})$. Suppose that $f(\lambda x) \sim \lambda^\alpha L(\lambda) h(x)$ as $\lambda \to \infty$ in $\mathcal{S}_0'(\mathbb{R})$. Then for its Stockwell transform with respect to a window $g \in \mathcal{S}_0(\mathbb{R})\backslash\{0\}$ we have*

$$S_g x^m f(\lambda b, \frac{a}{\lambda}) \sim \lambda^{\alpha+2m} L(\lambda) S_g x^m h(b,a) \quad as \quad \lambda \to \infty,\ m \in \mathbb{N} \tag{4.2}$$

uniformly on compact subsets of $\mathbb{Y}$.

In Proposition 4.3 and 4.4 an other asymptotic result for the Stockwell transform is given with respect to both variables and one variable, using the quasiasymptotic behavior at origin and infinity of Lizorkin distributions, respectively.

Proposition 4.3. *Let L be a slowly varying function at the origin, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}_0'(\mathbb{R})$. Suppose that $f(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) h(x)$ as $\varepsilon \to 0^+$ in $\mathcal{S}_0'(\mathbb{R})$. Then for its Stockwell transform with respect to a window $g \in \mathcal{S}_0(\mathbb{R})\backslash\{0\}$ we have*

$$S_g x^m f(\varepsilon^2 b, \frac{a}{\varepsilon}) \sim \varepsilon^{\alpha+2m} L(\varepsilon) S_g x^m h(0,a) \quad as \quad \varepsilon \to 0^+,\ m \in \mathbb{N} \tag{4.3}$$

uniformly on compact subsets of $\mathbb{Y}$.

Proof. Let $(b,a) \in \mathbb{Y}$ be fixed, $m \in \mathbb{N}$ then by the substitution $x = \varepsilon t$ we have

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{S_g x^m f(\varepsilon^2 b, \frac{a}{\varepsilon})}{\varepsilon^{\alpha+2m} L(\varepsilon)} &= \lim_{\varepsilon\to 0^+} \frac{(2\pi)^{-\frac{1}{2}}}{\varepsilon^{\alpha+2m} L(\varepsilon)} \langle x^m f(x), \overline{M_{\frac{a}{\varepsilon}} T_{\varepsilon^2 b} D_{\frac{1}{\frac{a}{\varepsilon}}} g(x)} \rangle \\
&= \lim_{\varepsilon\to 0^+} \frac{(2\pi)^{-\frac{1}{2}}}{\varepsilon^{\alpha+m} L(\varepsilon)} \langle \left(\frac{x}{\varepsilon}\right)^m f(x), e^{-ix\frac{a}{\varepsilon}} \frac{|a|}{\varepsilon} \overline{g}(\frac{a}{\varepsilon}(x-\varepsilon^2 b)) \rangle \\
&= \lim_{\varepsilon\to 0^+} \frac{(2\pi)^{-\frac{1}{2}}}{\varepsilon^{\alpha+m} L(\varepsilon)} \langle t^m f(\varepsilon t), e^{-ita} |a| \overline{g}(a(t-\varepsilon b)) \rangle \\
&= \lim_{(\varepsilon_1,\varepsilon_2)\to(0^+,0^+)} \frac{1}{\sqrt{2\pi}\varepsilon_1^{\alpha+m} L(\varepsilon_1)} \langle t^m f(\varepsilon_1 t), \overline{M_a T_{\varepsilon_2 b} D_{\frac{1}{a}} g(t)} \rangle
\end{aligned}$$

if this last limit exist. Since the weak and the strong topology on $\mathcal{S}_0'(\mathbb{R})$ are equivalent, the boundedness of the set (3), the quasiasymptotic behavior of f near the origin and Theorem 2.1 (i) imply that

$$\lim_{\varepsilon_1\to 0^+} \frac{1}{\sqrt{2\pi}\varepsilon_1^{\alpha+m} L(\varepsilon_1)} \langle t^m f(\varepsilon_1 t), \overline{M_a T_{\varepsilon_2 b} D_{\frac{1}{a}} g(t)} \rangle = \frac{1}{\sqrt{2\pi}} \langle t^m h(t), \overline{M_a T_{\varepsilon_2 b} D_{\frac{1}{a}} g(t)} \rangle,$$

uniformly for $\varepsilon_2 \in [0,1]$. Furthermore, for each $0 < \varepsilon_1 \leqslant 1$, we have

$$\lim_{\varepsilon_2\to 0^+} \frac{1}{\sqrt{2\pi}\varepsilon_1^{\alpha+m} L(\varepsilon_1)} \langle t^m f(\varepsilon_1 t), \overline{M_a T_{\varepsilon_2 b} D_{\frac{1}{a}} g(t)} \rangle = \frac{1}{\sqrt{2\pi}\varepsilon_1^{\alpha} L(\varepsilon_1)} \langle t^m f(\varepsilon_1 t), \overline{M_a D_{\frac{1}{a}} g(t)} \rangle,$$

because $\overline{M_a T_{\varepsilon_2 b} D_{\frac{1}{a}} g(t)} \to \overline{M_a D_{\frac{1}{a}} g(t)}$ as $\varepsilon_2 \to 0^+$ in $\mathcal{S}_0(\mathbb{R})$. Hence

$$\begin{aligned}
&\lim_{(\varepsilon_1,\varepsilon_2)\to(0^+,0^+)} \frac{1}{\sqrt{2\pi}\varepsilon_1^{\alpha+m} L(\varepsilon_1)} \langle t^m f(\varepsilon_1 t), \overline{M_a T_{\varepsilon_2 b} D_{\frac{1}{a}} g(t)} \rangle \\
&= \lim_{\varepsilon_1\to 0^+} \lim_{\varepsilon_2\to 0^+} \frac{1}{\sqrt{2\pi}\varepsilon_1^{\alpha+m} L(\varepsilon_1)} \langle t^m f(\varepsilon_1 t), \overline{M_a T_{\varepsilon_2 b} D_{\frac{1}{a}} g(t)} \rangle \\
&= \lim_{\varepsilon_1\to 0^+} \frac{1}{\sqrt{2\pi}\varepsilon_1^{\alpha+m} L(\varepsilon_1)} \langle t^m f(\varepsilon_1 t), \overline{M_a D_{\frac{1}{a}} g(t)} \rangle
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}}\langle t^m h(t), \overline{M_a T_0 D_{\frac{1}{a}} g(t)}\rangle = S_g t^m h(0,a).$$

□

Proposition 4.4. *Let L be a slowly varying function at infinity, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}_0'(\mathbb{R})$. Suppose that $f(\lambda x) \sim \lambda^\alpha L(\lambda) h(x)$ as $\lambda \to \infty$ in $\mathcal{S}_0'(\mathbb{R})$. Then for its Stockwell transform with respect to a window $g \in \mathcal{S}_0(\mathbb{R})\backslash\{0\}$ we have*

$$S_g x^m f(b, \frac{a}{\lambda}) \sim \lambda^{\alpha+2m} L(\lambda) S_g x^m h(0,a) \quad as \quad \lambda \to \infty,\ m \in \mathbb{N}$$

uniformly on compact subsets of $\mathbb{Y}$.

Proof. Let $(b,a) \in \mathbb{Y}$ be fixed, using the substitution $x = \lambda t$ and similar techniques as in the proof of Proposition 4.3 we obtain

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{S_g x^m f(b,\frac{a}{\lambda})}{\lambda^{\alpha+2m} L(\lambda)} &= \lim_{\lambda\to\infty} \frac{(2\pi)^{-\frac{1}{2}}}{\lambda^{\alpha+2m} L(\lambda)} \langle x^m f(x), \overline{M_{\frac{a}{\lambda}} T_b D_{\frac{1}{\frac{a}{\lambda}}} g(x)}\rangle \\
&= \lim_{\lambda\to\infty} \frac{1}{\sqrt{2\pi}\lambda^{\alpha+m} L(\lambda)} \langle t^m f(\lambda t), \overline{M_a T_{\frac{b}{\lambda}} D_{\frac{1}{a}} g(t)}\rangle \\
&= \frac{1}{\sqrt{2\pi}}\langle t^m h(t), \overline{M_a T_0 D_{\frac{1}{a}} g(t)}\rangle = S_g t^m h(0,a).
\end{aligned}$$

□

Let $f_\varepsilon(x) = f(\varepsilon x)$, and let us denote $[f_\varepsilon(x)]^{(m)} = f_\varepsilon^{(m)}$ such that $[f_\varepsilon(x)]^{(m)} = [f(\varepsilon x)]^{(m)} = \varepsilon^m f^{(m)}(\varepsilon x)$. The same notation is used when $\varepsilon \leftrightarrow \lambda$.

Proposition 4.5. *Let L be a slowly varying function at the origin, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}_0'(\mathbb{R})$. Suppose that $f(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) h(x)$ as $\varepsilon \to 0^+$ in $\mathcal{S}_0'(\mathbb{R})$. Then for its Stockwell transform with respect to a window $g \in \mathcal{S}_0(\mathbb{R})\backslash\{0\}$ we have*

$$S_g f^{(m)}(\varepsilon b, \frac{a}{\varepsilon}) \sim \varepsilon^{\alpha-m} L(\varepsilon) S_g h^{(m)}(b,a) \quad as \quad \varepsilon \to 0^+,\ m \in \mathbb{N} \tag{4.4}$$

uniformly on compact subsets of $\mathbb{Y}$.

Proof. First we prove the relation

$$S_g f_\varepsilon^{(m)}(b,a) = \varepsilon^m S_g f^{(m)}(\varepsilon b, \frac{a}{\varepsilon}). \tag{4.5}$$

Indeed, using the definition of the Stockwell transform and the substitution $\varepsilon x = t$ we have

$$\begin{aligned}
S_g f_\varepsilon^{(m)}(b,a) &= (2\pi)^{-\frac{1}{2}} \langle \varepsilon^m f^{(m)}(\varepsilon x), e^{-ixa}|a|\overline{g}(a(x-b))\rangle \\
&= (2\pi)^{-\frac{1}{2}} \langle \varepsilon^m f^{(m)}(t), e^{-it\frac{a}{\varepsilon}} \frac{|a|}{\varepsilon} \overline{g}(\frac{a}{\varepsilon}(t-\varepsilon b))\rangle \\
&= (2\pi)^{-\frac{1}{2}} \langle \varepsilon^m f^{(m)}(t), \overline{M_{\frac{a}{\varepsilon}} T_{\varepsilon b} D_{\frac{1}{\frac{a}{\varepsilon}}} g(t)}\rangle = \varepsilon^m S_g f^{(m)}(\varepsilon b, \frac{a}{\varepsilon}).
\end{aligned}$$

Now, by relation (4.5), the quasiasymptotics of f near the origin and Theorem 2.1 (ii) we obtain

$$\lim_{\varepsilon\to 0^+} \frac{S_g f^{(m)}(\varepsilon b, \frac{a}{\varepsilon})}{\varepsilon^{\alpha-m} L(\varepsilon)} = \lim_{\varepsilon\to 0^+} \frac{S_g f_\varepsilon^m(b,a)}{\varepsilon^\alpha L(\varepsilon)} = \lim_{\varepsilon\to 0^+} (2\pi)^{-\frac{1}{2}} \langle \frac{f^{(m)}(\varepsilon t)}{\varepsilon^{\alpha-m} L(\varepsilon)}, \overline{M_a T_b D_{\frac{1}{a}} g(t)}\rangle$$

$$= \quad (2\pi)^{-\frac{1}{2}}\langle h^{(m)}(t), \overline{M_a T_b D_{\frac{1}{a}} g(t)}\rangle = S_g h^{(m)}(b,a).$$

The uniform convergence on compact subsets of $\mathbb{Y}$ follows from the boundedness of the set (3) and the fact that the weak and the strong topology on $\mathcal{S}_0'(\mathbb{R})$ are equivalent. □

In Proposition 4.6 the Abelian type result is given for the asymptotic behavior of the Stockwell transform with respect to the quasiasymptotic behavior of Lizorkin distributions at infinity.

Proposition 4.6. *Let L be a slowly varying function at infinity, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}_0'(\mathbb{R})$. Suppose that $f(\lambda x) \sim \lambda^\alpha L(\lambda) h(x)$ as $\lambda \to \infty$ in $\mathcal{S}_0'(\mathbb{R})$. Then for its Stockwell transform with respect to a window $g \in \mathcal{S}_0(\mathbb{R})\backslash\{0\}$ we have*

$$S_g f^{(m)}(\lambda b, \frac{a}{\lambda}) \sim \lambda^{\alpha - m} L(\lambda) S_g h^{(m)}(b,a) \quad as \quad \lambda \to \infty,\ m \in \mathbb{N} \tag{4.6}$$

uniformly on compact subsets of $\mathbb{Y}$.

In Proposition 4.7 and 4.8 the asymptotic result for the Stockwell transform with respect to the one variable and both variables is given, using the quasiasymptotic behavior of Lizorkin distributions at infinity and origin, respectively.

Proposition 4.7. *Let L be a slowly varying function at infinity, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}_0'(\mathbb{R})$. Suppose that $f(\lambda x) \sim \lambda^\alpha L(\lambda) h(x)$ as $\lambda \to \infty$ in $\mathcal{S}_0'(\mathbb{R})$. Then for its Stockwell transform with respect to a window $g \in \mathcal{S}_0(\mathbb{R})\backslash\{0\}$ we have*

$$S_g f^{(m)}(b, \frac{a}{\lambda}) \sim \lambda^{\alpha - m} L(\lambda) S_g h^{(m)}(0,a) \quad as \quad \lambda \to \infty,\ m \in \mathbb{N}$$

uniformly on compact subsets of $\mathbb{Y}$.

Proof. Let us first prove the relation

$$S_g f_\lambda^{(m)}\left(\frac{b}{\lambda}, a\right) = \lambda^m S_g f^{(m)}(b, \frac{a}{\lambda}). \tag{4.7}$$

Indeed, using the substitution $\lambda x = t$ we have

$$\begin{aligned} S_g f_\lambda^{(m)}(\frac{b}{\lambda}, a) &= (2\pi)^{-\frac{1}{2}}\langle \lambda^m f^{(m)}(\lambda x), e^{-ixa}|a|\overline{g}(a(x - \frac{b}{\lambda}))\rangle \\ &= (2\pi)^{-\frac{1}{2}}\langle \lambda^m f^{(m)}(t), e^{-it\frac{a}{\lambda}}\frac{|a|}{\lambda}\overline{g}(\frac{a}{\lambda}(t-b))\rangle \\ &= (2\pi)^{-\frac{1}{2}}\langle \lambda^m f^{(m)}(t), \overline{M_{\frac{a}{\lambda}} T_b D_{\frac{1}{\frac{a}{\lambda}}} g(t)}\rangle = \lambda^m S_g f^{(m)}(b, \frac{a}{\lambda}). \end{aligned}$$

Let $(b,a) \in \mathbb{Y}$ be fixed, then by (4.7) we have

$$\begin{aligned} \lim_{\lambda\to\infty} \frac{S_g f^{(m)}(b, \frac{a}{\lambda})}{\lambda^{\alpha-m} L(\lambda)} &= \lim_{\lambda\to\infty} \frac{1}{\sqrt{2\pi}\lambda^\alpha L(\lambda)}\langle \lambda^m f^{(m)}(\lambda t), \overline{M_a T_{\frac{b}{\lambda}} D_{\frac{1}{a}} g(t)}\rangle \\ &= \lim_{(\lambda_1,\lambda_2)\to(\infty,\infty)} \frac{1}{\sqrt{2\pi}\lambda_1^{\alpha-m} L(\lambda_1)}\langle f^{(m)}(\lambda_1 t), \overline{M_a T_{\frac{b}{\lambda_2}} D_{\frac{1}{a}} g(t)}\rangle \end{aligned}$$

if this last limit exist. Since the weak and the strong topology on $\mathcal{S}_0'(\mathbb{R})$ are equivalent, the boundedness of the set (3), the quasiasymptotic behavior of f near the infinity and Theorem 2.1 (ii) imply that

$$\lim_{\lambda_1\to\infty}\frac{1}{\sqrt{2\pi}\lambda_1^{\alpha-m}L(\lambda_1)}\langle f^{(m)}(\lambda_1 t),\overline{M_aT_{\frac{b}{\lambda_2}}D_{\frac{1}{a}}g(t)}\rangle=\frac{1}{\sqrt{2\pi}}\langle h^{(m)}(t),\overline{M_aT_{\frac{b}{\lambda_2}}D_{\frac{1}{a}}g(t)}\rangle,$$

uniformly for $\lambda_2>1$. Furthermore, for each $\lambda_1>1$, we have

$$\lim_{\lambda_2\to\infty}\frac{1}{\sqrt{2\pi}\lambda_1^{\alpha-m}L(\varepsilon_1)}\langle f^{(m)}(\lambda_1 t),\overline{M_aT_{\frac{b}{\lambda_2}}D_{\frac{1}{a}}g(t)}\rangle=\frac{1}{\sqrt{2\pi}\lambda_1^{\alpha-m}L(\lambda_1)}\langle f^{(m)}(\lambda_1 t),\overline{M_aD_{\frac{1}{a}}g(t)}\rangle,$$

because $\overline{M_aT_{\frac{b}{\lambda_2}}D_{\frac{1}{a}}g(t)}\to\overline{M_aD_{\frac{1}{a}}g(t)}$ as $\lambda_2\to\infty$ in $\mathcal{S}_0(\mathbb{R})$. Hence

$$\begin{aligned}\lim_{\lambda\to\infty}\frac{S_gf^{(m)}(b,\frac{a}{\lambda})}{\lambda^{\alpha-m}L(\lambda)}&=\lim_{(\lambda_1,\lambda_2)\to(\infty,\infty)}\frac{1}{\sqrt{2\pi}\lambda_1^{\alpha-m}L(\lambda_1)}\langle f^{(m)}(\lambda_1 t),\overline{M_aT_{\frac{b}{\lambda_2}}D_{\frac{1}{a}}g(t)}\rangle\\
&=\lim_{\lambda_1\to\infty}\lim_{\lambda_2\to\infty}\frac{1}{\sqrt{2\pi}\lambda_1^{\alpha-m}L(\lambda_1)}\langle f^{(m)}(\lambda_1 t),\overline{M_aT_{\frac{b}{\lambda_2}}D_{\frac{1}{a}}g(t)}\rangle\\
&=\lim_{\lambda_1\to\infty}\frac{1}{\sqrt{2\pi}\lambda_1^{\alpha-m}L(\lambda_1)}\langle f^{(m)}(\lambda_1 t),\overline{M_aD_{\frac{1}{a}}g(t)}\rangle\\
&=\frac{1}{\sqrt{2\pi}}\langle h^{(m)}(t),\overline{M_aT_0D_{\frac{1}{a}}g(t)}\rangle=S_gh^{(m)}(0,a).\end{aligned}$$

□

From (4.5) we directly obtain the following relation

$$S_gf_\varepsilon^{(m)}(\varepsilon b,a)=\varepsilon^mS_gf^{(m)}(\varepsilon^2b,\frac{a}{\varepsilon}). \tag{4.8}$$

Proposition 4.8. *Let L be a slowly varying function at the origin, $\alpha\in\mathbb{R}$ and $f\in\mathcal{S}_0'(\mathbb{R})$. Suppose that $f(\varepsilon x)\sim\varepsilon^\alpha L(\varepsilon)h(x)$ as $\varepsilon\to0^+$ in $\mathcal{S}_0'(\mathbb{R})$. Then for its Stockwell transform with respect to a window $g\in\mathcal{S}_0(\mathbb{R})\backslash\{0\}$ we have*

$$S_gf^{(m)}(\varepsilon^2b,\frac{a}{\varepsilon})\sim\varepsilon^{\alpha-m}L(\varepsilon)S_gh^{(m)}(0,a)\quad\text{as}\quad\varepsilon\to0^+,\ m\in\mathbb{N} \tag{4.9}$$

uniformly on compact subsets of $\mathbb{Y}$.

Proof. Let $(b,a)\in\mathbb{Y}$ be fixed, $m\in\mathbb{N}$ then by (4.8), the boundedness of the set (3), Theorem 2.1 (ii), and the fact that the weak and the strong topology on $\mathcal{S}_0'(\mathbb{R})$ are equivalent, we have

$$\begin{aligned}\lim_{\varepsilon\to0^+}\frac{S_gf^{(m)}(\varepsilon^2b,\frac{a}{\varepsilon})}{\varepsilon^{\alpha-m}L(\varepsilon)}&=\lim_{\varepsilon\to0^+}\frac{1}{\sqrt{2\pi}\varepsilon^{\alpha-m}L(\varepsilon)}\langle f^{(m)}(\varepsilon t),\overline{M_aT_{\varepsilon b}D_{\frac{1}{a}}g(t)}\rangle\\
&=\frac{1}{\sqrt{2\pi}}\langle h^{(m)}(t),\overline{M_aT_0D_{\frac{1}{a}}g(t)}\rangle=S_gh^{(m)}(0,a).\end{aligned}$$

□

References

[1] Buralieva J.V, Saneva K, Atanasova S, Directional short-time Fourier transform and quasiasymptotics of distributions. Functional Analysis and Its Applications, 53. pp. 3-10, 2019.
[2] Brychkov Yu.A., Prudnikov A.P., Integral transforms of Generalized Functions, Gordon and Breach Sci. Publ., New York, 1989.
[3] Daubechies I., Ten Lectures on Wavelets, Philadelphia, Pennsylvania: Society for Industrial and applied Mathematics (SIAM), 1992.
[4] Estrada R, Kanwal RP. A Distributional Approach to Asymptotics. Boston: Theory and Applications. Birkhauser; 2002.
[5] Holschneider M. Wavelets. An analysis tool. New York: The Clarendon Press. Oxford University Press; 1995.
[6] Kostadinova S, Pilipović S, Saneva K, Vindas J. The short-time Fourier transform of distributions of exponential type and Tauberian theorems for S-asymptotics. Filomat. 2016;30:3047–3061.
[7] Meyer Y., Wavelets, Philadelphia, Pennsylvania: Society for Industrial and applied Mathematics (SIAM), 1993.
[8] Pilipović S., Quasiasymptotics and S-asymptotics in S' and D', Part 2, Publ. de lInst. Math., 58(72), 1995, 13–20.
[9] Pilipović S., Quasiasymptotic Expansion and the Laplace Transformation, Appl. Anal., 35, 1996, 243-261.
[10] Pilipović S., Stanković B., Structural theorems for the S-asymptotic and quasiasymptotic of distributions, Math. Pannon., 4, 1993, 23-35.
[11] Pilipović S, Stankovic B, Vindas J. Asymptotic behavior of generalized functions. Hackensack(NJ): World Scientific Publishing Co. Pte. Ltd; 2012.
[12] Pilipović S., Stanković B., Takaci A., Asymptotic Behaviour and Stieltjes Transformation of Distribution, Liebzig: Taubner-Texte zur Mathematik-Band, 1990.
[13] Pilipović S, Vindas J. Multidimensional Tauberian theorems for vector-valued distributions. Publ. Inst. Math. (Beograd)(N. S.).2017;95: 1–28.
[14] Saneva K, Aceska R, Kostadinova S. Some Abelian and Tauberian results for the short-time Fourier transform. Novi Sad J. Math. 2013;43(2):81–89.
[15] Saneva H-V.K., Atanasova S., Buralieva V.J., Tauberian theorems for the Stockwell transform of Lizorkin distributions. Applicable Analysis, 99 (4). pp. 596-610, 2020.
[16] Schwartz L. Theorie des distributions a valeurs vectorielles. I. Ann.Inst. Fourier Grenpble. 1957;7:1–141.
[17] Stockwell RG, Mansinha L, Lowe RP. Localization of the complex spectrum: the S transform. IEEE Trans. Signal Process. 1996;44:998–1001.
[18] Treves F. Topological vector spaces, distributions and kernels. New York-London: Academic press; 1967.
[19] Vladimirov VS, Drozinov YuN, Zavialov BI. Tauberian theorems for generalized functions. Dordrecht: Kluwer Academic; 1988.
[20] Vindas J, Pilipović S, Rakić D. Tauberian theorems for the wavelet transform. J. Fourier Anal. Appl. 2011;17:65–95.
[21] Zemanian A. H., Generalized integral transforms, Interscience, John Wiley, New York, 1968.

Jasmina Veta Buralieva
University Goce Delchev,
Faculty of Computer Science,
Krste Misirkov br. 10-A, 2000 Shtip,
North Macedonia.
Email address: jasmina.buralieva@ugd.edu.mk