GOCE DELCEV UNIVERSITY - STIP
FACULTY OF COMPUTER SCIENCE

The journal is indexed in

EBSCO

ISSN 2545-4803 on line
DOI: 10.46763/BJAMI

BALKAN JOURNAL OF APPLIED MATHEMATICS AND INFORMATICS (BJAMI)

BALKAN JOURNAL
OF APPLIED MATHEMATICS AND INFORMATICS

YEAR 2023

VOLUME VI, Number 1

AIMS AND SCOPE:
BJAMI publishes original research articles in the areas of applied mathematics and informatics.

Topics:
1. Computer science;
2. Computer and software engineering;
3. Information technology;
4. Computer security;
5. Electrical engineering;
6. Telecommunication;
7. Mathematics and its applications;
8. Articles of interdisciplinary of computer and information sciences with education, economics, environmental, health, and engineering.

BALKAN JOURNAL
OF APPLIED MATHEMATICS AND INFORMATICS
(BJAMI), Vol 6

ISSN 2545-4803 on line
Vol. 6, No. 1, Year 2023

EDITORIAL BOARD

CONTENT

Balkan Journal of Applied Mathematics and Informatics Online ISSN 2545-4083
Volume VI Number 1 Year 2023 UDK: 517.972.8

APPLICATION OF FUNDAMENTAL LEMMA OF VARIATIONAL CALCULUS TO THE PROBLEM OF PLATEAU

Aleksandra Risteska-Kamcheski and Mirjana Kocaleva Vitanova

Abstract. In this paper, we will prove a theorem for a functional where we prove that the necessary condition for the extreme of a functional is the variation of the functional to be equal to zero and we will give an example of its application, the problem of the Plateau.

Key words: extreme, functional, variation, condition.

1. Introduction

Many problems in mathematics are naturally formulated in terms of identifying a function that minimizes some quantity of interest. A natural example from geometry is the seemingly simple question: which is the shortest length path between two points in R^n? While everyone knows that such a path is the straight-line segment connecting the two points, proving that such is the case is more subtle than such a simple question would suggest. A more complex example of a question in the same vein is to ask: given some open set Ω, and some boundary conditions, can we identify a surface defined on this set, satisfying the boundary conditions, that has the minimum possible area?

The setting of the calculus of variations is over functionals on general normed vector spaces, specifically vector spaces of functions, the methods of results of the calculus of variations are remarkably simple and powerful and bear a great deal of resemblance to the machinery of finite-dimensional real analysis.

The Euler-Lagrange equations are a very useful result in variational analysis since many naturally occurring problems in mathematics, physics and other domains of application can be formulated in terms of minimizing or maximizing an integral on a given domain.

2. Results and discussion

We will explore for extreme of the functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \,, \qquad (2.1)$$

With the limit points of the allowable set of curves: $y(x_0) = y_0$ and $y(x_1) = y_1$. We will consider that the function $F(x, y, y')$ is three times differentiable. We know that the necessary condition for the extreme is the variation of the functional to be equal to zero ([1]). We will now show how the main theorem is applied to the given functional (2.1).

Let's assume that the extreme reached on two times differentiable curve $y = y(x)$ (required only the existence of a derived from the first line of residue curves, otherwise it may be that of the curve on which the extreme is reached, there is a second derived). We are taking some close to $y = y(x)$ limit curves $y = \overline{y}(x)$ and include curves $y = y(x)$ and $y = \overline{y}(x)$ to the family of curves with one parameter

$$y(x,\alpha) = y(x) + \alpha(\overline{y}(x) - y(x)) \ .$$

When $\alpha = 0$, we receive the curve $y = y(x)$, when $\alpha = 1$, we receive $y = \overline{y}(x)$.

As we already know, the difference $\overline{y}(x) - y(x)$ is called the variation of the function $y(x)$ and is marked with δy .

The variation δy in variational problems plays a role analogous to the role of the increase Δx of an independent variable x in problems for the study of extreme of function $f(x)$. The variation of function $\delta y = \overline{y}(x) - y(x)$ is a function of the x ([2])·

This function can be differentiated once or several times, as $(\delta y)' = \overline{y}'(x) - y'(x) = \delta y'$ the derive of the variance is equal to the variance of the generated, and similarly

$$(\delta y)'' = \overline{y}''(x) - y''(x) = \delta y'',$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(\delta y)^{(k)} = \overline{y}^{(k)}(x) - y^{(k)}(x) = \delta y^{(k)}.$$

And so, we analyze the family $y = y(x,\alpha)$, where $y(x,\alpha) = y(x) + \alpha \delta y$, containing the $\alpha = 0$ curves, which reaches an extreme, and in some $\alpha = 1$ close tolerances and curves that are called curves of comparison ([3]).

If we look at the values of the functional (2.1), only of the family curves $y = y(x,\alpha)$, their functional turns into a function of α :

$$v[y(x,\alpha)] = \varphi(\alpha),$$

As in the case that we consider $v[y(x,\alpha)]$ is functional depending on parameter, the value of the parameter α determines the curve of the family $y = y(x,\alpha)$ and so determines also the value of functional $v[y(x,\alpha)]$.

Theorem. If functional $v(y) = \int_{x_0}^{x_1} F(x, y, y')dx$ has a local extreme in y , the necessary condition for extreme of functional is

$$\int_{x_0}^{x_1} [F_y - \frac{d}{dx} F_{y'}] \delta y \, dx = 0, \qquad (2.2)$$

Proof. We analyze the function $\varphi(\alpha)$. It reaches its extreme at $\alpha = 0$, and when $\alpha = 0$ we receive $y = y(x)$, and the functional, in assumption, reaches the extreme compared with any permissible curve, and in particular, in terms of the near families curves $y = y(x,\alpha)$.

The necessary condition for the extreme of the function $\varphi(\alpha)$ at $\alpha = 0$, as is known, is its derivative to be equal to zero at $\alpha = 0$, i.e.

$$\varphi'(0) = 0 .$$

Since

$$\varphi(\alpha) = \int_{x_0}^{x_1} F(x, y(x,\alpha), y_x'(x,\alpha))\, dx,$$

It

$$\varphi'(\alpha) = \int_{x_0}^{x_1} \left[F_y' \frac{\partial}{\partial\alpha} y(x,\alpha) + F_{y'}' \frac{\partial}{\partial\alpha} y'(x,\alpha)) \right] dx,$$

Where

$$F_y' = \frac{\partial}{\partial y} F(x, y(x,\alpha), y'(x,\alpha)),$$

$$F_{y'}' = \frac{\partial}{\partial y'} F(x, y(x,\alpha), y'(x,\alpha)),$$

$$\frac{\partial}{\partial\alpha} y(x,\alpha) = \frac{\partial}{\partial\alpha}[y(x) + \alpha\delta y] = \delta y$$

$$\frac{\partial}{\partial\alpha} y'(x,\alpha) = \frac{\partial}{\partial\alpha}[y'(x) + \alpha\delta y'] = \delta y',$$

And we get

$$\varphi'(\alpha) = \int_{x_0}^{x_1} \left[F_y(x, y(x,\alpha), y'(x,\alpha))\delta y + F_{y'}(x, y(x,\alpha), y'(x,\alpha))\delta y' \right] dx,$$

$$\varphi'(0) = \int_{x_0}^{x_1} \left[F_y(x, y(x), y'(x))\delta y + F_{y'}(x, y(x), y'(x))\delta y' \right] dx \quad (npu\ \alpha = 0).$$

As we know, $\varphi'(0)$ is called the variation of functional and means δv ([4]).

The necessary condition for the extreme of the functional is its variation to be equal to zero

$$\delta v = 0 .$$

For the functional (2.1), this condition has a type of

$$\int_{x_0}^{x_1} [F_y{}'\delta y + F'_{y'}\delta y'] \, dx = 0 \qquad (2.3)$$

Integrate the equation (2.3) in parts, whereas $\delta y' = (\delta y)'$, we get

$$\delta v = [F'_{y'}\delta y']\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} [F_y{}' - \frac{d}{dx}F'_{y'}]\delta y \, dx =$$

$$= \int_{x_0}^{x_1} F'_y \delta y \, dx + F'_{y'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))\delta y(x_1) - F'_{y'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))\delta y(x_0) =$$

$$= \int_{x_0}^{x_1} F'_y \delta y \, dx + F'_{y'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))(\overline{y}(x_1) - y(x_1))$$

$$-F'_{y'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))(\overline{y}(x_0) - y(x_0)) - \int_{x_0}^{x_1} (\delta y) dF'_{y'} =$$

$$= \int_{x_0}^{x_1} F'_y \delta y \, dx + F'_{y'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))(0)$$

$$-F'_{y'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))(0) - \int_{x_0}^{x_1} (\delta y) \frac{d}{dx} F'_{y'}$$

Since all of the possible (permissible) curves in the given problem pass through the fixed limit points, we get

$$\delta v = \int_{x_0}^{x_1} [F_y{}' - \frac{d}{dx}F'_{y'}]\delta y \, dx \ . \ \square$$

Note. The first multiplier $F_y{}' - \frac{d}{dx}F'_{y'}$ of the curve $y = y(x)$ reaches the extreme of the continuous function, and the second multiplier δy , random for the choice of the curve in comparison $y = \overline{y}(x)$, is an arbitrary function having passed only certain general conditions, namely: the function δy in the border points $x = x_0$, and $x = x_1$ is equal to zero, continuous and differentiable one or several times, δy or δy and $\delta y'$ are small in absolute value.

To simplify the obtain necessary condition, we will use the following lemma:

Fundamental lemma of variational calculus. If for any continuous function $\eta(x)$ is true

$$\int_{x_0}^{x_1} \Phi(x)\eta(x) \, dx = 0,$$

Where the function $\Phi(x)$ is continuous in the interval $[x_0, x_1]$, it

$$\Phi(x) \equiv 0$$

in this interval ([5]).

Proof. We accept that, in the point $x = \overline{x}$, resting in the interval (x_0, x_1) , $\Phi(x) \neq 0$, is a contradiction.

Indeed, if the continuity of the function $\Phi(x)$, it follows that if $\Phi(\overline{x}) \neq 0$ it $\Phi(x)$ keeps characters in the vicinity of $\overline{x}$ ($x_0 \leq x \leq x_1$). We choose the function $\eta(x)$ which also retains the mark in that vicinity and is equal to zero outside of this vicinity. We receive

$$\int_{x_0}^{x_1} \Phi(x)\eta(x)\, dx = \int_{\overline{x}_0}^{\overline{x}_1} \Phi(x)\eta(x)\, dx \neq 0,$$

since the product $\Phi(x)\eta(x)$ retains its mark in the interval $x_0 \leq x \leq x_1$ and is equal to zero in the same interval.

And so, we come to a contradiction, therefore $\Phi(x) \equiv 0$. □

Note. The adoption of lemma and its proof remain unchanged if the function $\eta(x)$ requires the following restrictions:

$$\eta(x_0) = \eta(x_1) = 0,$$

$\eta(x)$ There is a continuous derived to the line n ,

$$\left|\eta^{(s)}(x)\right| < \varepsilon, \quad (s = 0,1,\ldots,q; q \leq n) \ .$$

The function $\eta(x)$ can be selected, e.g.:

$$\eta(x) = \begin{cases} k(x-\overline{x}_0)^{2n}(x-\overline{x}_1)^{2n}, & x \in [\overline{x}_0, \overline{x}_1] \\ 0 & x \in [x_0, x_1] \setminus [\overline{x}_0, \overline{x}_1] \end{cases} ,$$

where n is a positive number, k is a constant.

Apparently, the function $\eta(x)$ satisfies the above conditions: it is a continuous, there is a continuous derived to line $2n-1$, in the points x_0 and x_1 it is equal to zero and by reducing the factor by k we can do $\left|\eta^{(s)}(x)\right| < \varepsilon$ for the $\forall x \in [x_0, x_1]$.

Now we will apply the fundamental lemma of variational calculus to simplify the above necessary condition for the extreme of functional (2.1).

Consequence. If functional $v(y)=\int_{x_0}^{x_1} F(x,y,y')dx$ reaches the extreme of the curve $y=y(x)$, and $F_y^{'}$ and are $\frac{d}{dx}F_{y'}^{'}$ continuous, then $y=y(x)$ is a solution to the differential equation (equation of Euler)([1])

$$F_y-\frac{d}{dx}F_{y'}=0,$$

Or in an expanded form

$$F_y-F_{xy'}-F_{yy'}y'-F_{y'y'}y''=0 \ .$$

Proof. The proof of consequence follows immediately from the fundamental lemma of variational calculus. □

This equation is called the equation of Euler (1744 year) ([2]). The integral curve $y=y(x,C_1,C_2)$ equation of Euler is called the extreme.

To find a curve, which reaches the extreme of the functional (2.1), we integrate the equation of Euler and spell out random constants, satisfying the general solution of this equation, of the conditions of borders $y(x_0)=y_0, y(x_1)=y_1$.

Only if they are satisfied with these conditions, the extreme of functional can be reached. However, in order to determine whether they are really extreme (maximum or minimum), the sufficient conditions for the extreme must also be studied.

To recall, that border problem

$$F_y{}'-\frac{d}{dx}F_{y'}^{'}=0, \quad y(x_0)=y_0, \quad y(x_1)=y_1 \ ,$$

not always has a solution, and if there is a solution, then this may not be the only one.

It should be taken into account that in many variational problems the existence of solutions is evident, from physical or geometrical sense of the problem, and in the solution of the equations of Euler satisfying the border conditions, only a single extreme may be the solution of the given problem.

Plateau's Problem. Let us consider the problem of determining the surface φ over the domain $\Omega\subset\mathbb{R}^n$, satisfying some boundary condition $\varphi(x)=g(x)$ on $\partial\Omega$. The area functional is given by

$$J[\varphi]=\int_{\Omega} dA, \qquad (2.4)$$

where $S=\sqrt{1+\nabla_{\varphi}.\nabla_{\varphi}}$. The Euler-Lagrange equation is given by

$$\frac{\partial S}{\partial\varphi}-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\frac{\partial S}{\partial\varphi_j}=0\,. \qquad (2.5)$$

Expanding this equation, we obtain

$$\sum_{j=1}^{n}\left[\frac{\varphi_{ij}}{S}-\varphi_j\sum_{k=1}^{n}\frac{\varphi_k}{S}\varphi_{kj}\right]=0\,.$$

After some algebraic manipulation, this can be reduced to a simpler form:

$$\nabla.\left(\frac{\nabla_{\varphi}}{S(\varphi)}\right)=0 \qquad (2.6)$$

While the above derivation is quite simple, we have omitted several complexities relating the existence, uniqueness, and smoothness of a minimal area surface to the geometry of $\partial\Omega$ and the nature of the boundary conditions.

3. Conclusion

It should be taken into account that in many variational problems the existence of solutions is evident, from physical or geometrical sense of the problem, and in the solution of the equations of Euler satisfying the border conditions, only a single extreme may be the solution of the given problem.

References

[1] Assoc. Dr. B. Златанов - metric spaces;
[2] L.Э.Эльсгольц - ДИФФЕРЕНЦИАЛЬНЫЕ equations AND ВАРИАЦИОННОЕ ИСЧИСЛЕНИЕ;
[3] Russak I.B. - Calculus of variations (MA4311, 1996);
[4] brunt B. - The calculus of variations (ISBN 0387402470), Springer, 2004);
[5] Byerly W. - introduction to the calculus of variations (Cambridge, 1917);

Aleksandra Risteska-Kamcheski
University of Goce Delcev,
Faculty of Computer Science,
Stip, R. North Macedonia
aleksandra.risteska@ugd.edu.mk

Mirjana Kocaleva Vitanova
University of Goce Delcev,
Faculty of Computer Science,
Stip, R. North Macedonia
mirjana.kocaleva@ugd.edu.mk