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CONTENT

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GENERALIZATION OF THE APPLICATION OF A FUNDAMENTAL LEMMA OF VARIATIONAL CALCULUS TO REVOLUTIONIZE TRANSPORTATION BY USING THE SOLUTION OF BRACHISTOCHRONE

Aleksandra Risteska-Kamcheski

Abstract Variational calculus studied methods for finding maximum and minimum values of functional. It has its inception in 1696 year by Johan Bernoulli with its glorious problem of the brachistochrone: to find a curve connecting two points *A and B*, which does not lie in a vertical, so that the heavy point descends along this curve from position A to reach position B in the shortest time. In functional analysis, variational calculus takes the same space, as well as the theory of maximum and minimum intensity in the classic analysis. We will prove a theorem for functional where we prove that the necessary condition for the extreme of the functional is the variation of functional to be equal to zero. We describe the solution of the equation of Euler with an example of application, such as the problem of brachistochrone, and its generalization that has the potential to completely revolutionize transportation.

Key words: extreme, functional, variation, condition, transportation.

1. Introduction

Many problems in mathematics are naturally formulated in terms of identifying a function that minimizes some quantity of interest. A natural example from geometry is the seemingly simple question: which is the shortest length path between two points in R^n? While everyone knows that such a path is the straight line segment connecting the two points, proving that such is the case is more subtle than such a simple question would suggest. A more complex example of a question in the same vein is to ask: given some open set Ω and some boundary conditions, can we identify a surface defined on this set, satisfying the boundary conditions, that has the minimum possible area?

The setting of the calculus of variations is over functionals on general normed vector spaces, specifically vector spaces of functions, the methods of results of the calculus of variations are remarkably simple and powerful and bear a great deal of resemblance to the machinery of a finite-dimensional real analysis.

The Euler-Lagrange equations are a very useful result in variational analysis, since many naturally occurring problems in mathematics, physics and other domains of application can be formulated in terms of minimizing or maximizing an integral on a given domain.

2. Results and discussion

We will explore for the extreme of the functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \ , \qquad (2.1)$$

with the limit points of the allowable set of curves: $y(x_0) = y_0$ and $y(x_1) = y_1$. We will consider that the function $F(x, y, y')$ is three times differentiable. We know that the necessary condition for the extreme is the variation of the functional to be equal to zero ([1]). We will now show how the main theorem is applied to the given functional (2.1).

Let us assume that the extreme reached on the two times differentiable curve $y = y(x)$ (required only the existence of a derived from the first line of residue curves, otherwise it may be that of the curve on which the extreme is reached, there is a second derived). We are taking some close to $y = y(x)$ limit curves $y = \overline{y}(x)$ and include curves $y = y(x)$ and $y = \overline{y}(x)$ to the family curves with one parameter

$$y(x, \alpha) = y(x) + \alpha(\overline{y}(x) - y(x)) \ .$$

When $\alpha = 0$, we receive the curve $y = y(x)$, when $\alpha = 1$, we receive $y = \overline{y}(x)$.

As we already know, the difference $\overline{y}(x) - y(x)$ is called the variation of the function $y(x)$ and means with the δy.

The variation δy in variational problems plays a role analogous to the role of the increase Δx of an independent variable x in problems for the study of the extreme of the function $f(x)$. The variation of function $\delta y = \overline{y}(x) - y(x)$ is a function of the x ([2]).

This function can be differentiated once or several times, as $(\delta y)' = \overline{y}'(x) - y'(x) = \delta y'$ it is generated of the variance to be equal to the variance of the generated, and similarly

$$(\delta y)'' = \overline{y}''(x) - y''(x) = \delta y'',$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(\delta y)^{(k)} = \overline{y}^{(k)}(x) - y^{(k)}(x) = \delta y^{(k)}.$$

And so, we analyze the family $y = y(x, \alpha)$, where $y(x, \alpha) = y(x) + \alpha \delta y$, containing the $\alpha = 0$ curves, which reaches an extreme, and in some $\alpha = 1$ close tolerances and curves that are called curves of comparison ([3]).

If we look at the values of the functional (0.1), only of the family curves $y = y(x, \alpha)$, the functional passes into a functional of α:

$$v[y(x, \alpha)] = \varphi(\alpha),$$

As in the case that we consider $v[y(x,\alpha)]$ is functional depending on parameter, the value of the parameter α determines the curve of the family $y = y(x,\alpha)$ and so determines and the value of functional $v[y(x,\alpha)]$.

Theorem. If the functional $v(y) = \int_{x_0}^{x_1} F(x, y, y')dx$ has a local extreme in y , the necessary condition for the extreme of the functional is

$$\int_{x_0}^{x_1} [F_y - \frac{d}{dx} F_{y'}]\delta y \, dx = 0, \qquad (2.2)$$

Proof. We analyze the function $\varphi(\alpha)$. It reaches its extreme at $\alpha = 0$, and when $\alpha = 0$, we receive $y = y(x)$, and the functional, in assumption, reaches the extreme compared with any permissible curve, and in particular, in terms of the near families curves $y = y(x,\alpha)$.

The necessary condition for the extreme of the function $\varphi(\alpha)$ at $\alpha = 0$, as is known, is the derivative to be equal to zero at $\alpha = 0$, i.e.,

$$\varphi'(0) = 0 .$$

Since

$$\varphi(\alpha) = \int_{x_0}^{x_1} F(x, y(x,\alpha), y_x'(x,\alpha)) \, dx,$$

It

$$\varphi'(\alpha) = \int_{x_0}^{x_1} \left[F_y' \frac{\partial}{\partial\alpha} y(x,\alpha) + F_{y'}' \frac{\partial}{\partial\alpha} y'(x,\alpha)) \right] dx,$$

Where

$$F_y' = \frac{\partial}{\partial y} F(x, y(x,\alpha), y'(x,\alpha)),$$

$$F_{y'}' = \frac{\partial}{\partial y'} F(x, y(x,\alpha), y'(x,\alpha)),$$

$$\frac{\partial}{\partial\alpha} y(x,\alpha) = \frac{\partial}{\partial\alpha}[y(x) + \alpha\delta y] = \delta y$$

$$\frac{\partial}{\partial\alpha} y'(x,\alpha) = \frac{\partial}{\partial\alpha}[y'(x) + \alpha\delta y'] = \delta y',$$

And we get

$$\varphi'(\alpha) = \int_{x_0}^{x_1} \left[F_y(x, y(x,\alpha), y'(x,\alpha))\delta y + F_{y'}(x, y(x,\alpha), y'(x,\alpha))\delta y' \right] dx,$$

$$\varphi'(0) = \int_{x_0}^{x_1} \left[F_y(x, y(x), y'(x))\delta y + F_{y'}(x, y(x), y'(x))\delta y' \right] dx \quad (npu\ \alpha = 0).$$

As we know, $\varphi'(0)$ is called a variation of the functional and means δv ([4]).
The necessary condition for the extreme of the functional is its variation to be equal to zero

$$\delta v = 0 \ .$$

For the functional (0.1), this condition has a type of

$$\int_{x_0}^{x_1} [F_y{}'\delta y + F'_{y'}\delta y']\, dx = 0 \tag{2.3}$$

Integrate the equation (2.3) in parts, whereas $\delta y' = (\delta y)'$, we get

$$\delta v = [F'_{y'}\delta y']\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} [F_y{}' - \frac{d}{dx}F'_{y'}]\delta y\, dx =$$

$$= \int_{x_0}^{x_1} F'_y \delta y\, dx + F'_{y'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))\delta y(x_1) - F'_{y'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))\delta y(x_0) =$$

$$= \int_{x_0}^{x_1} F'_y \delta y\, dx + F'_{y'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))(\bar{y}(x_1) - y(x_1))$$

$$-F'_{y'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))(\bar{y}(x_0) - y(x_0)) - \int_{x_0}^{x_1} (\delta y) dF'_{y'} =$$

$$= \int_{x_0}^{x_1} F'_y \delta y\, dx + F'_{y'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))(0)$$

$$-F'_{y'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))(0) - \int_{x_0}^{x_1} (\delta y) \frac{d}{dx}F'_{y'}$$

Since, all of the possible (permissible) curves in the given problem pass through fixed limit points, we get

$$\delta v = \int_{x_0}^{x_1} [F_y{}' - \frac{d}{dx}F'_{y'}]\delta y\, dx \ . \qquad \square$$

Note. The first multiplier $F_y{}' - \frac{d}{dx}F'_{y'}$ of the curve $y = y(x)$ reaches the extreme of the continuous function, and the second multiplier δy , random for the choice of the curve in comparison $y = \overline{y}(x)$, is an arbitrary function having passed only certain general conditions, namely: the function δy in the border points $x = x_0$, and $x = x_1$ is equal to zero, continuous and differentiable one or several times, δy or δy and $\delta y'$ are small in absolute value.
To simplify the obtained necessary condition (2.2), we will use the following lemma:

Fundamental lemma of the variational calculus. If for any continuous function $\eta(x)$ it is true that

$$\int_{x_0}^{x_1} \Phi(x)\eta(x)\,dx = 0, [6]$$

Where the function $\Phi(x)$ is continuous in the interval $[x_0, x_1]$, it

$$\Phi(x) \equiv 0$$

in this interval ([5]).
Proof. We accept that, in the point $x = \overline{x}$, resting in the interval (x_0, x_1) , $\Phi(x) \neq 0$, is a contradiction.
Indeed, the continuity of the function $\Phi(x)$, it follows that if $\Phi(\overline{x}) \neq 0$ it $\Phi(x)$ keeps characters in vicinity of $\overline{x}$ ($x_0 \le x \le x_1$). We choose the function $\eta(x)$ which also retains the mark in that vicinity and is equal to zero outside of this vicinity. We receive

$$\int_{x_0}^{x_1} \Phi(x)\eta(x)\,dx = \int_{\overline{x}_0}^{\overline{x}_1} \Phi(x)\eta(x)\,dx \neq 0,$$

Since the product $\Phi(x)\eta(x)$ retains its mark in the interval $x_0 \le x \le x_1$ and is equal to zero in the same interval.
And so, we come to a contradiction, therefore $\Phi(x) \equiv 0$. □
Note. The adoption of the lemma and its proof remain unchanged if the function $\eta(x)$ requires the following restrictions:

$$\eta(x_0) = \eta(x_1) = 0,$$

$\eta(x)$ There is a continuous derived to line n ,

$$\left|\eta^{(s)}(x)\right| < \varepsilon, \quad (s = 0,1,\ldots,q; q \le n) .$$

The function $\eta(x)$ can be selected, e.g.:

$$\eta(x)=\begin{cases}k(x-\overline{x}_0)^{2n}(x-\overline{x}_1)^{2n}, & x\in[\overline{x}_0,\overline{x}_1]\\ 0 & x\in[x_0,x_1]\backslash[\overline{x}_0,\overline{x}_1]\end{cases},$$

where n is a positive number, k is a constant.

Apparently, the function $\eta(x)$ satisfies the above conditions: it is a continuous, there is a continuous derived to line $2n-1$, in the points x_0 and x_1 is equal to zero and by reducing the factor by k we can do $\left|\eta^{(s)}(x)\right|<\varepsilon$ for the $\forall x\in[x_0,x_1]$.

Now we will apply the fundamental lemma of variational calculus to simplify the above necessary condition for the extreme (2) of the functional (2.1).

Consequence. If the functional $v(y)=\int_{x_0}^{x_1}F(x,y,y')dx$ reaches the extreme of the curve $y=y(x)$, and $F_y^{'}$ are $\frac{d}{dx}F_{y'}^{'}$ continuous, then $y=y(x)$ is a solution to the differential equation (equation of Euler)([1])

$$F_y-\frac{d}{dx}F_{y'}=0,$$

Or in an expanded form

$$F_y-F_{xy'}-F_{yy'}y'-F_{y'y'}y''=0 \; .$$

Proof. The proof of consequence 1.1 follows immediately from the fundamental lemma of the variational calculus. □

This equation is called the equation of Euler (1744 year) ([2]). The integral curve $y=y(x,C_1,C_2)$ the equation of Euler is called extreme.

To find a curve, on which the extreme of the functional is reached (2.1) we integrate the equation of Euler and spell out random constants, satisfying the general solution of this equation, of the conditions of borders $y(x_0)=y_0, y(x_1)=y_1$.

Only if they are satisfied with these conditions, the extreme of the functional can be reached.

However, in order to determine whether they are really extreme (maximum or minimum), sufficient conditions for the extreme must also be studied.

To recall, that border problem

$$F_y'-\frac{d}{dx}F_{y'}^{'}=0,\quad y(x_0)=y_0,\quad y(x_1)=y_1 \; ,$$

not always has a solution, and if there is a solution, then this may not be the only one. It should be taken into account that in many variational problems the existence of solutions is evident, from physical or geometrical sense of the problem, and in the solution of the equations of Euler satisfying the border conditions, only a single extreme may be the solution of the given problem.

Problem of the brachistochrone. To determine the curve, connecting two given points A and B, in whose movement, material item provided for the shortest time from A point to point B (friction and resistance of the environment). We will shift the origin of the coordinate system in the point A, we will put the axis Ox horizontally, and the axis Oy vertically. The speed of movement of the stock point is $\frac{ds}{dt} = \sqrt{2gy}$, where we find the time spent in the movement of the point from position A(0,0) to position $B(x_1, y_1)$

$$t[y(x)] = \frac{1}{\sqrt{2g}} \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}} dx; \quad y(0)=0,\ y(x_1)=y_1 \ .$$

Since this functional is one of the simplest types, and the integrand function does not contain x, so the equation of Euler has a first integral

$$F - y' F'_{y'} = C \ ,$$

or in this case,

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y\left(1+y'^2\right)}} = C \ ,$$

where after a simplification,

$$\frac{1+y'^2-y'^2}{\sqrt{y\left(1+y'^2\right)}} = C \ ,$$

we have $\frac{1}{\sqrt{y\left(1+y'^2\right)}} = C$, or $y(1+y'^2) = C_1$.

We are introducing the parameter t by the application of $y' = ctgt$. Therefore, we have

$$y = \frac{C_1}{1+ctg^2 t} = C_1 \sin^2 t = \frac{C_1}{2}(1-\cos^2 t);$$

$$dx = \frac{dy}{y'} = \frac{2C_1 \sin t \cos t dt}{ctgt} = 2C_1 \sin^2 t dt =$$

$$= 2C_1 \frac{1}{2}(1-\cos 2t)dt = C_1(1-\cos 2t)dt;$$

Integrate, and obtain

$$x = C_1\left(t - \frac{\sin 2t}{2}\right) + C_2 = \frac{C_2}{2}(2t - \sin 2t) + C_2 \cdot$$

The equation of the curve in parametric form has the type

$$x - C_2 = \frac{C_1}{2}(2t - \sin 2t),$$

$$y = \frac{C_1}{2}(1 - \cos 2t)$$

If we replace for parameter $2t = t_1$, and take into account that the $C_2 = 0$, $x = 0$, $y = 0$, we receive the equation of family cycloids in normal form:

$$x = \frac{C_1}{2}(t_1 - \sin t_1),$$

$$y = \frac{C_1}{2}(1 - \cos t_1),$$

where $\frac{C_1}{2}$ is the radius of the rolling circle, which is determined by the conditions of the passing cycloid through the point $B(x_1, y_1)$. And so, the brachistochrone is a cycloid.

We will now discuss a generalization of the brachistochrone that has the potential (in theory) to completely revolutionize transportation. Suppose we could build a tunnel through the Earth's crust connecting any city A to any other city B in the world. If we neglect friction, a train departing from A with zero speed would accelerate as the tunnel gets closer to the center of the Earth and then decelerate as it gets further, finally arriving at B with exactly zero speed. There would be no need for engines, fuel, or brakes. We will push the limits of this fantasy further yet: we will determine the profile of the tunnel that will be traversed in the shortest time.

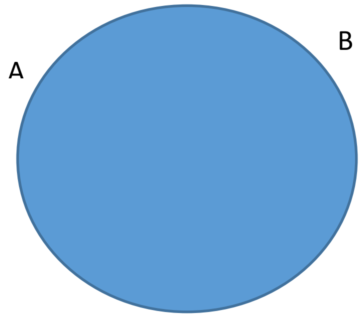

Figure 1. *Distance from point(city) A to point(city) B*

We can model this situation using physics. We model the Earth as a uniform solid sphere of material with constant density, and the two cities A and B as points on its surface. We will draw the tunnel in the plane defined by the two cities and the center of the sphere and parameterize it with the curve $(x, y(x))$. The goal of this is again to find the curve $(x, y(x))$ that will be traversed in the shortest amount of time when powered by gravity alone. What is the difference between this problem and the brachistochrone? The main difference is that the strength and the direction of the force of gravity changes as a function of our position along the path. As with the brachistochrone, the problem is to minimize the integral

$$T = \int \frac{ds}{v},$$

where v designates the speed of the object at point $(x, y(x))$ along its path and ds is an infinitesimally small piece of the trajectory with the length

$$ds = \sqrt{1 + (y')^2}dx.$$

3. Conclusion

It should be taken into account that in many variational problems the existence of solutions is evident, from physical or geometrical sense of the problem, and in the solution of the equations of Euler satisfying the border conditions, only a single extreme may be the solution of the given problem.

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