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CONTENT

The Appendix

In honor of the first Doctor of Mathematical Sciences Acad. Blagoj Popov, a mathematician dedicated to differential equations, the idea of holding the "Day of DifferentialEquations" was born, prompted by Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski, and Prof. Ph.D. Lazo Dimov. Acad. Blagoj Popov presented his doctoral dissertation on 05.05.1952 in the field of differential equations. This is the main reason for holding the "Day of Differential Equations" at the beginning of May.

This year on May 5th, the "Day of Differential Equations" was held for the seventh time under the auspices of the Faculty of Computer Sciences at "Goce Delcev" University in Stip and Dean Prof. Ph.D. Saso Koceski, organized by Prof. Ph.D. Biljana Zlatanovska, Prof. Ph.D. Marija Miteva and Prof. Ph.D. Limonka Koceva Lazarova.

The participants of this event were:

1. Prof. Ph.D. Aleksa Malcheski from the Faculty of Mechanical engineering at Ss.Cyril and Methodius University in Skopje;
2. Prof. Ph.D. Slagjana Brsakoska from the Faculty of Natural Sciences and Mathematics at Ss.Cyril and Methodius University in Skopje;
3. Prof. Ph.D. Natasa Koceska, Prof. Ph.D. Limonka Koceva Lazarova, Prof. Ph.D. Marija Miteva and Prof. Ph.D. BiljanaZlatanovska from the Faculty of Computer Sciences at Goce Delcev University in Stip;
4. Ass. Prof. Ph.D. Biljana Citkuseva Dimitrovska and Ass. M.Sc. Maja Kukuseva Panova from the Faculty of Electrical Engineering at Goce Delcev University in Stip.

Acknowledgments to Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski and Prof.Ph.D. Lazo Dimov for the wonderful idea and the successful realization of the event this year and in previous years.

Acknowledgments to the Dean of the Faculty of Computer Sciences, Prof. Ph.D. Saso Koceski for her overall support of the organization and implementation of the "Day of Differential Equations".

The papers that emerged from the "Day of Differential Equations" are in the appendixto this issue of BJAMI.

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ABOUT A CLASS OF 2D MATRIX OF DIFFERENTIAL EQUATIONS

BILJANA ZLATANOVSKA AND BORO PIPEREVSKI

Abstract. A class of 2D matrix differential equations and their connection to second-order differential equations with polynomial coefficients are considered. By using the method of transformation, appropriate results for their correlation are obtained. These results enable obtaining appropriate conditions for the integrability of one of the classes and systems of differential equations.The theory is supported by examples.

Dedicated to the Day of Differential Equations in Macedonia 2023.

1. Introduction

In this paper, the class of 2D matrix differential equations of the type

$$PX' + MX = O \tag{1.1}$$

is considered, where

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, X'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}, X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, O = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$a = a_1 t + a_2, b = b_1 t + b_2, c = c_1 t + c_2, d = d_1 t + d_2,$$

$$A, B, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in R,$$

$X(t)$ is matrix function and $x_1(t), x_2(t)$ are real functions of one real variable t by frist derivate $x_1'(t) = \frac{dx_1}{dt}, x_2'(t) = \frac{dx_2}{dt}$.

In [2,3,9] for this class of matrix differential equations (1.1), the matrix polynomial solution

$$X_n = \begin{pmatrix} P_n(x) \\ Q_n(x) \end{pmatrix}$$

is presented, where $P_n(x)$ and $Q_n(x)$ are polynomials of the degree n. For this solution, the following theorem is true.

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Keywords. class of 2D matrix differential equations, second-order differential equations, examples.
2010 Math. Subject Classification: 34A30 .

Theorem 1.1. *Let the class of 2D matrix differential equations (1.1) be given. Let the conditions $a \cdot b \cdot c \cdot d \cdot (a \cdot d - b \cdot c) \cdot A \cdot B \neq 0, b' \neq 0, c' \neq 0$ are satisfied. If there exist a natural number n (the smaller one if there are two) that satisfies the condition*

$$r(M + nP') = 1, r(M + kP') = 2, k < n, k \in N, r - \text{a rang of the matrix}$$

$$A + na' \neq 0, B + nd' \neq 0,$$

i.e. the condition $(a'd' - b'c') \cdot n^2 + (Ad' + Ba') \cdot n + AB = 0$ then the 2D matrix differential equations (1.1) has the matrix pilynomial solution of a degree n and no other polynomial solution of degree less than n, given by the Rodrigues matrix formula

$$X_n = T \cdot \frac{d^{n-1}}{dt^{n-1}}[T_1 \cdot U_1],$$

where

$$T = \begin{pmatrix} f(t) & 0 \\ 0 & f(t) \end{pmatrix}, T_1 = \begin{pmatrix} B(k \cdot b + a) & 0 \\ 0 & A(k \cdot d + c) \end{pmatrix}, U_1 = \frac{(a \cdot d - b \cdot c)^{n-1}}{f(t)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$k = -\frac{nc'}{B + nd'} = -\frac{A + na'}{nb'}, f(t) = e^{-\int \frac{aB + Ad}{ad - bc} dt}.$$

i.e.

$$X_n = f(t) \cdot \begin{pmatrix} \frac{d^{n-1}}{dt^{n-1}}[f^{-1}(t) \cdot B(kb + a) \cdot (ad - bc)^{n-1}] \\ \frac{d^{n-1}}{dt^{n-1}}[f^{-1}(t) \cdot A(kd + c) \cdot (ad - bc)^{n-1}] \end{pmatrix}$$

In [1,2,4] for this class 2D matrix differential equations (1.1), the matrix polynomial solution $X_n = \begin{pmatrix} P_{n-1}(t) \\ Q_n(t) \end{pmatrix}$ is presented, where $P_{n-1}(t)$ and $Q_n(t)$ are polynomials of degree $n - 1$ i.e. n.

For this matrix polynomial solution, the following theorem is true.

Theorem 1.2. *The subclass of matrix differential equations (1.1) with condition*

$$a \cdot b \cdot c \cdot d \cdot (a \cdot d - b \cdot c) \cdot A \cdot B \neq 0, b' = 0, c' \neq 0, a'd' = 1$$

has a matrix polynomial solution of degree n and no other polynomial solution of degree $k < n, k \in N$ if and only if there exist a positive integer n such that the conditions

$$nd' + B = 0, ka' + A \neq 0, k < n, k \in N$$

are satisfied. By the Rodrigues matrix formula, the matrix polynomial solution

$$X_n = f(t) \cdot \begin{pmatrix} \frac{d^{n-1}}{dt^{n-1}}[(ad - bc)^{n-1} \cdot f^{-1}(t)] \\ \frac{A}{Bb} \cdot \frac{d^{n-1}}{dt^{n-1}}[d \cdot (ad - bc)^{n-1} \cdot f^{-1}(t)] \end{pmatrix}$$

is given, where $f(t) = e^{-\int \frac{aB + Ad}{ad - bc} dt}$.

The class second-order differential equation with polynomial coefficients of the type

$$(St+T)(t^2+Qt+R)x''+(\beta_2t^2+\beta_1t+\beta_0)x'+(\gamma_1t+\gamma_0)x=0 \tag{1.2}$$
$$S,T,Q,R,\beta_2,\beta_1,\beta_0,\gamma_1,\gamma_0\in R$$

has a fundamental role in the theory of differential equations related to numerical mathematics, special functions, the Sturm-Liouville problem, as well as calculus of variations and applications.

In [1,2,3], the following theorem was proved.

Theorem 1.3. *The second-order differential equation (1.2) has one polynomial solution of degree n and no other polynomial solution of degree $k<n, k\in N$ if there exist a positive integer n (the smaller one if there are two), which is the root of the characteristical equation $Sx^2+(\beta_2-S)x+\gamma_1=0$ and if the conditions*

$$S^2(\beta_0+SR-QT)+T^2(S+\beta_2)-T\beta_1S=0$$
$$S^2(\gamma_0\beta_1+\gamma_0^2-\gamma_1\beta_0)+T(\gamma_1+\beta_2)(T\gamma_1-2S\gamma_0)=0 \tag{1.3}$$

are satisfied. In this case, the polynomial solution is given by the formula

$$x(t)=f(t)\cdot\frac{d^{n-1}}{dt^{n-1}}[(t+K)(t^2+Qt+R)^{n-1}\cdot f^{-1}(t)],$$

where $f(t)=e^{-\int\frac{Mt+N}{t^2+Qt+R}dt}, M=\frac{\beta_2-S}{S}, N=\frac{S\beta_1-T\beta_2-TS}{S^2}$, $K=\frac{1}{S\gamma_1}[T\gamma_1+n(2T\beta_2-S\beta_1+T\gamma_1-S\gamma_0)]$.

For a second-order linear homogeneous differential equation of the type

$$P(x)y''+Q(x)y'+\lambda R(x)y=0,$$

Brenke [17] shows that it will satisfy the condition for the existence of polynomial solutions of degree n for each $n\in N$, for a suitable value of parameter λ and if $P(x),Q(x),R(x)$ are polynomials of second, first and zero degrees respectively. He also gives the general formula for the sequence of polynomial solutions of that equation and under certain conditions shows their orthogonality with appropriate weight. Let us note that in that case, when all members of the sequence $(\lambda_n), n=0,1,2,...$ are different, then λ_n are called eigenvalues, and the polynomials $y_n(x)$ are eigenfunctions. In [11, 14], this second-order linear homogeneous differential equation is also the subject of studying. For this important class of second-order differential equations, it is important to mention the classical results regarding polynomial solutions of the very important hypergeometric differential equation, as an equation with polynomial coefficients of the type (1.2). Legendre, Jacobi, Tschebyscheff, Hermite, Laguerre, etc. polynomials appear as special cases of such known orthogonal polynomials as solutions of differential equations of type (1.2). which finds great application in numerical mathematics.

In the theory of partial differential equations and the calculus of variations, the classic result for solving a Dirichlet interior problem for a contour problem for Laplace's partial differential equation on a sphere is known. By transforming into spherical coordinates and using the Fourier method of separation of variables, differential equations of the type (1.2) are obtained whose solutions are the classic orthogonal Legendre's polynomials which are also eigenfunctions for the corresponding Sturm-Liouville problem. In doing so, the solutions of Laplace's partial differential equation are obtained in the form of homogeneous polynomials of the appropriate degree and they are called spherical harmonic functions. By introducing elliptic coordinates in Laplace's partial differential equation, this approach to solving the same problem but for an ellipsoid was used by Lame. By using elliptic functions and the Fourier method of separation of variables, the second-order differential equation of the type (1.2) was obtained whose polynomial solutions are the Lame polynomials [8, 10, 16]. The Lame equation is a special case of the Schrödinger equation with a periodic potential and a spectrum that has exactly n layers on the semiaxis. This equation is a special case of Hill's equation.

2. **Main results**

We consider the connection between a 2D matrix differential equation of the form (1.1) and a second-order differential equation of the form (1.2).

Theorem 2.1. *Let the 2D matrix differential equation (1.1) be given. Let the conditions $a\cdot b\cdot c\cdot d\cdot(a\cdot d-b\cdot c)\cdot A\cdot B\neq 0, b'\neq 0, c'\neq 0$ be satisfied. If the conditions (1.3) are satisfied then the 2D matrix differential equation (1.1) corresponds to the second-order differential equation (1.2).*

Proof. Let the 2D matrix differential equation (1.1) be given. The 2D matrix equation (1.1) corresponds to the second-order differential equations. In relation to the first component function $x_1(t)$, the second-order differential equation is

$$b(ad-bc)x_1''+(ad'b-c'b^2+a'db-adb'+Abd+abB)x_1'+A(bd'-b'd+Bb)x_1=0 \quad (2.1)$$

In terms of the second component function $x_2(t)$, the equation is

$$c(ad-bc)x_2''+(ad'c-b'c^2+a'dc-adc'+Acd+acB)x_2'+B(ca'-c'a+Ac)x_2=0 \quad (2.2)$$

The second-order differential equation (2.1) is equivalent to the second-order differential equation of the form (1.2), if the following relations

$$\begin{aligned} b_1=S, b_2=T, a_1d_1-b_1c_1=1, a_2d_2-b_2c_2=R,\\ a_1d_2+a_2d_1-b_1c_2-b_2c_1=Q,\\ a_1d_1b_1-c_1b_1^2+Ab_1d_1+Ba_1b_1=\beta_2,\\ 2d_1a_1b_2-2b_1c_1b_2+Ab_1d_2+Ab_2d_1+Ba_1b_2+Ba_2b_1=\beta_1, \end{aligned} \quad (2.3)$$

$$\begin{aligned} d_1a_2b_2 - c_1b_2^2 + a_1d_2b_2 - b_1a_2d_2 + Ab_2d_2 + Ba_2b_2 = \beta_0, \\ ABb_1 = \gamma_1, A(Bb_2 - b_1d_2 + d_1b_2) = \gamma_0 \end{aligned}$$

are satisfied. From the equations of the system relations (2.3), the first condition of the conditions (1.3) is obtained. From the equations of the system relations (2.3) and the first condition from the conditions (1.3), the second condition of the conditions (1.3) is obtained.

The second-order differential equation (2.2) is equivalent to the second-order differential equation of the form (1.2) if the following relations

$$\begin{aligned} c_1 = S, c_2 = T, a_1d_1 - b_1c_1 = 1, a_2d_2 - b_2c_2 = R, \\ a_1d_2 + a_2d_1 - b_1c_2 - b_2c_1 = Q, \\ a_1d_1c_1 - b_1c_1^2 + Ac_1d_1 + Ba_1c_1 = \beta_2, \\ 2d_1a_1c_2 - 2b_1c_1c_2 + Ac_1d_2 + Ac_2d_1 + Ba_1c_2 + Ba_2c_1 = \beta_1, \\ d_1a_2c_2 - b_1c_2^2 + a_1d_2c_2 - c_1a_2d_2 + Ac_2d_2 + Ba_2c_2 = \beta_0, \\ ABc_1 = \gamma_1, B(Ac_2 - c_1a_2 + a_1c_2) = \gamma_0 \end{aligned} \tag{2.4}$$

are satisfied. From the equations of the system relations (2.4), the first condition of the conditions (1.3) is obtained. From the equations of the system relations (2.4) and the first condition from the conditions (1.3), the second condition of the conditions (1.3) is obtained. According to given a 2D matrix differential equation (1.1) corresponds to two second-order differential equations (1.2) that satisfy condition (1.3). □

Theorem 2.2. *Let the second-order differential equation (1.2) be given. Let the conditions (1.3) be satisfied. Then the second-order differential equation (1.2) corresponds to the 2D matrix differential equation (1.1). The coefficients of the the appropriate 2D matrix differential equation with the form (1.1) are given by the formulas*

$$\begin{aligned} b_1 = S, b_2 = T, Sd_1A^2 - (\beta_2 - S)A + a_1\gamma_1 = 0, \\ B = \frac{\gamma_1}{AS}, c_1 = \frac{1}{S}(a_1d_1 - 1), d_2 = \frac{1}{SA}(\frac{\gamma_1 T}{S} + Ad_1T - \gamma_0), \\ a_2 = \frac{1}{TB}(\beta_0 + SR - TQ - TAd_2), c_2 = \frac{1}{T}(a_2d_2 - R) \end{aligned} \tag{2.5}$$

where a_1 and d_1 satisfy the conditions $a_1 \cdot d_1 \neq 1, (\beta_2 - S)^2 - 4S \cdot d_1 \cdot a_1 \cdot \gamma_1 = k^2, k \in R$ i.e., by the formulas

$$\begin{aligned} c_1 = S, c_2 = T, Sd_1A^2 - (\beta_2 - S)A + a_1\gamma_1 = 0, \\ B = \frac{\gamma_1}{AS}, b_1 = \frac{1}{S}(a_1d_1 - 1), d_2 = \frac{1}{TA}(\beta_0 + SR - TQ - BTa_2), \end{aligned} \tag{2.6}$$

$$a_2 = \frac{T}{S}a_1 + \frac{A}{\gamma_1}(\frac{T}{S}\gamma_1 - \gamma_0), b_2 = \frac{1}{T}(a_2 d_2 - R)$$

where a_1 and d_1 satisfy the conditions $a_1 \cdot d_1 \neq 1, (\beta_2 - S)^2 - 4S \cdot d_1 \cdot a_1 \cdot \gamma_1 = k^2, k \in R$.

Proof. By the system relations (2.3), the formulas (2.5) are obtained. By the system relations (2.4), the formulas (2.6) are obtained. □

Remark 2.1: Due to the quadratic equation for coefficient A, two 2D matrix equations (1.1) are obtained. According to the given second-order differential equation (1.2) for which the conditions (1.3) are satisfied, corresponds to two 2D matrix differential equations (1.1).

So one 2D matrix differential equation (1.1) corresponds to two second-order differential equations (1.2) for which the conditions (1.3) are satisfied. One second-order differential equation (1.2) for which the conditions (1.3) are satisfied, corresponds to two 2D matrix equations differential equations (1.1).

Example 2.1. *Let the 2D matrix differential equation*

$$\begin{pmatrix} 3t+1 & 3t+1 \\ t+1 & 2t+2 \end{pmatrix} \cdot X' + \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix} \cdot X = O, X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{2.7}$$

is given by a matrix polynomial solution $X = \begin{pmatrix} 28t^2 + \frac{40}{3}t + \frac{4}{3} \\ -\frac{28}{3}t^2 - 8t + \frac{4}{3} \end{pmatrix}$. The 2D matrix differential equation (2.7) coresponds to the following two second-order differential eqautions,

$$(3t+1)(3t^2+4t+1)x_1'' + (3t^2 - 14t - 5)x_1' + (-24t+8)x_1 = 0 \tag{2.8}$$

$$(t+1)(3t^2+4t+1)x_2'' + (t^2 - 2t - 3)x_2' + (-8t-4)x_2 = 0 \tag{2.9}$$

which satisfy the condition (1.3) by polynomial solutions

$$x_1(t) = t^2 + \frac{10}{21}t + \frac{1}{21}, x_2(t) = t^2 + \frac{6}{7}t - \frac{1}{7}.$$

Let the second-order differential equation (2.8), i.e., the equation

$$(t+\frac{1}{3})(t^2 + \frac{4}{3}t + \frac{1}{3})x_1'' + (\frac{1}{3}t^2 - \frac{14}{9}t - \frac{5}{9})x_1' + (-\frac{8}{9}t + \frac{8}{9})x_1 = 0$$

is given. By the formulas (2.5) and $a_1 = 1, d_1 = 2$, for the quadratic equation per A, two solutions are obtained. For the solution $A = -\frac{4}{3}, B = 2$, the 2D matrix differential equation (2.7) is obtained. But, for the solution $A^\star = 1, B^\star = -\frac{8}{3}$, the 2D matrix differential equation

$$\begin{pmatrix} t+\frac{1}{3} & t+\frac{1}{3} \\ t-\frac{19}{9} & t-\frac{10}{9} \end{pmatrix} \cdot Y' + \begin{pmatrix} 1 & 0 \\ 0 & -\frac{8}{3} \end{pmatrix} \cdot Y = O, Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

is obtained by a matrix polynomial solution $Y = \begin{pmatrix} 28t^2 + \frac{40}{3}t + \frac{4}{3} \\ -42t^2 - \frac{52}{3}t - \frac{10}{3} \end{pmatrix}$. *For the new 2D matrix differential equation by finding the corresponding second-order differential equations for* $y_1(t)$ *same differential equation (2.8) will be obtained as for* $x_1(t)$. *While for* $y_2(t)$ *a new second-order differential equation*

$$(t - \frac{19}{9})(t^2 + \frac{4}{3}t + \frac{1}{3})y_2'' + (\frac{1}{3}t^2 - \frac{130}{27} + \frac{29}{27})y_2' + (-\frac{8}{3}t + \frac{328}{27})y_2 = 0$$

is obtained by a polynomial solution $y_2(t) = t^2 + \frac{26}{63}t + \frac{205}{2583}$.

By repeating the same procedure for the second-order differential equation (2.9), the second-order differential equation

$$(\frac{1}{2}t + \frac{1}{2})(t^2 + \frac{4}{3}t + \frac{1}{3})x_2'' + (\frac{1}{6}t^2 - \frac{1}{3}t - \frac{1}{2})x_2' + (-\frac{4}{3}t - \frac{2}{3})x_2 = 0$$

is obtained.

By the formulas (2.6) and $a_1 = 2, d_1 = 1$, *for the quadratic equations per* A, *two solutions are obtained. For the solution* $A = -\frac{8}{3}, B = 1$, *the same 2D matrix differential equation (2.7) is obtained. But, for the solution* $A^\star = 2, B^\star = -\frac{4}{3}$, *the new 2D matrix differential equation*

$$\begin{pmatrix} 2t+3 & 2t + \frac{16}{3} \\ \frac{1}{2}t + \frac{1}{2} & t+1 \end{pmatrix} \cdot Z' + \begin{pmatrix} 2 & 0 \\ 0 & -\frac{4}{3} \end{pmatrix} \cdot Z = O, Z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$

is obtained by a polynomial solution $Z = \begin{pmatrix} \frac{56}{9}t^2 + \frac{176}{9}t - 8 \\ -\frac{28}{3}t^2 - 8t + \frac{4}{3} \end{pmatrix}$. *For the new 2D matrix differential equation by finding the corresponding second-order differential equations for* $z_2(t)$ *same differential equation (2.9) will be obtained as for* $x_2(t)$. *While for* $z_1(t)$ *a new second-order differential equation*

$$(2t + \frac{16}{3})(t^2 + \frac{4}{3}t + \frac{1}{3})z_1'' + (\frac{2}{3}t^2 + \frac{28}{9}t - \frac{38}{9})z_1' + (-\frac{16}{3}t - \frac{68}{9})z_1 = 0$$

is obtained by a polynomial solution

$$z_1(t) = t^2 + \frac{22}{7}t - \frac{153}{119}.$$

Remark 2.2: The 2D matrix differential equation (1.1) is better for examining the integrability of second-order differential equations and it can also be used to study systems of differential equations.

In [1, 2, 3, 4], the case of the subclass of 2D matrix equation (1.1) when $b' = 0, c' \neq 0$ connected to a second-order differential equation (1.2) is considered.

Theorem 2.3. *Let the 2D matrix differential equation (1.1) be given. Let the conditions*

$$a \cdot b \cdot c \cdot d \cdot (a \cdot d - b \cdot c) \cdot A \cdot B \neq 0, b' = 0, c' \neq 0, \tag{2.10}$$

be satisfied.

If the conditions

$$S = 0, \gamma_1 = 0, \beta_2 = 0 \tag{2.11}$$

i.e., the conditions (1.3) are satisfied, then the 2D matrix differential equation (1.1) corresponds to the second-order differential equation (1.2)

Proof. Let the 2D matrix differential equation (1.1) be given. Let the conditions (2.10) are satisfied. The 2D matrix equation (1.1) corresponds to the second-order differential equations. In relation to the first component function $x_1(t)$, the equation is

$$(ad - bc)x_1'' + (ad' - c'b + a'd + Ad + aB)x_1' + A(d' + B)x_1 = 0 \tag{2.12}$$

In terms of the second component function $x_2(t)$, the equation is

$$c(ad - bc)x_2'' + (ad'c + a'dc - adc' + Acd + acB)x_2' + B(ca' - c'a + Ac)x_2 = 0 \tag{2.13}$$

The second-order differential equation (2.12) is equivalent to the second-order differential equation of the form (1.2), if the following relations

$$\begin{gathered} b_1 = S = 0, b_2 = T, a_1 d_1 = 1, a_2 d_2 - b_2 c_2 = R, a_1 d_2 + a_2 d_1 - b_2 c_1 = Q, \\ \beta_2 = 0, 2d_1 a_1 b_2 + A b_2 d_1 + B a_1 b_2 = \beta_1, \\ d_1 a_2 b_2 - c_1 b_2^2 + a_1 d_2 b_2 + A b_2 d_2 + B a_2 b_2 = \beta_0, \gamma_1 = 0, A(B b_2 + d_1 b_2) = \gamma_0 \end{gathered} \tag{2.14}$$

are satisfied. The condition (2.11) is obtained by them.

The second-order differential equation (2.13) is equivalent to the second-order differential equation of the form (1.2), if the following relations

$$\begin{aligned} c_1 = S, c_2 = T, a_1 d_1 = 1, a_2 d_2 - b_2 c_2 = R, a_1 d_2 + a_2 d_1 - b_2 c_1 = Q, \\ a_1 d_1 c_1 + A c_1 d_1 + B a_1 c_1 = \beta_2, \\ 2d_1 a_1 c_2 + A c_1 d_2 + A c_2 d_1 + B a_1 c_2 + B a_2 c_1 = \beta_1, \\ d_1 a_2 c_2 + a_1 d_2 c_2 - c_1 a_2 d_2 + A c_2 d_2 + B a_2 c_2 = \beta_0, \\ A B c_1 = \gamma_1, B(A c_2 - c_1 a_2 + a_1 c_2) = \gamma_0 \end{aligned} \tag{2.15}$$

are satisfied.

From the equations of the system relations (2.14), the first condition of (1.3) is obtained. From the equations of the system relations (2.15) and the first condition from the conditions (1.3), the second condition of (1.3) is obtained.

According to a 2D matrix differential equation (1.1) correspond to two second-order differential equations (1.2) that satisfy conditions (1.3), i.e., the conditions (2.11). □

Theorem 2.4. *Let the second-order differential equation (1.2) be given. Let the conditions (2.11), i.e., the condition (1.3) be satisfied. Then the second-order differential equation (1.3) corresponds to the 2D matrix differential equation (1.1) for* *which the condition (2.10) is satisfied. The coefficients of the appropriate 2D matrix differential equation with the form (1.1) are given by the formulas*

$$\begin{aligned} b_1 = S = 0, b_2 = T, TA^2 - (\beta_1 - T)A + a_1\gamma_0 = 0,\\ a_1TB = \beta_1 - 2T - d_1TA, a_2 = \frac{1}{B}(\frac{\beta_0}{T} - Q - Ad_2),\\ c_1 = \frac{1}{T}(a_1d_2 + a_2d_1 - Q), c_2 = \frac{1}{T}(a_2d_2 - R), d_1 = \frac{1}{a_1} \end{aligned} \tag{2.16}$$

which satisfy the condition $(\beta_1 - T)^2 - 4 \cdot T \cdot a_1 \cdot \gamma_0 = k^2, k \in R$ *(*a_1 *and* d_2 *- parameters for conditions (2.11)), i.e., by the formulas*

$$\begin{aligned} c_1 = S, c_2 = T, Sd_1A^2 - (\beta_2 - S)A + a_1\gamma_1 = 0, B = \frac{\gamma_1}{AS},\\ a_2 = \frac{T}{S}a_1 + \frac{A}{\gamma_1}(\frac{T}{S}\gamma_1 - \gamma_0), d_2 = \frac{1}{TA}(\beta_0 + SR - TQ - BTa_2),\\ b_2 = \frac{1}{T}(a_2d_2 - R), d_1 = \frac{1}{a_1} \end{aligned} \tag{2.17}$$

which satisfy the condition $(\beta_2 - S)^2 - 4S \cdot \gamma_1 = k^2, k \in R$ *(*a_1 *- parameter for the conditions (1.3)).*

Proof. By relations (2.14), the formulas (2.16) are obtained. By relations (2.15), the formulas (2.17) are obtained. □

Example 2.2. *Let the 2D matrix differential equation*

$$\begin{pmatrix} t-1 & 3 \\ t+2 & t-2 \end{pmatrix} \cdot X' + \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \cdot X = O, X_n(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{2.18}$$

is given by a matrix polynomial solution $X = \begin{pmatrix} -(3t-10) \\ 2t^2 - 11t + 8 \end{pmatrix}$. *The 2D matrix differential equation (2.18) coresponds to two second-order differential eqautions,*

$$3(t^2 - 6t - 4)x_1'' + (9t - 30)x_1' - 9x_1 = 0 \tag{2.19}$$

$$(t+2)(t^2 - 6t - 4)x_2'' + (2t^2 + 2t - 16)x_2' - (6t + 18)x_2 = 0 \tag{2.20}$$

which satisfy the condition (1.3), i.e., the condition (2.11) by polynomial solutions

$$x_1(t) = t - \frac{10}{3}, x_2(t) = t^2 - \frac{11}{2}t + 4.$$

Let the second-order differential equation (2.19) is given. By the formulas (2.16), for the quadratic equation per A*, two solutions are obtained. For the solution* $A = 3, B = -2$*, the 2D matrix differential equation (2.18) is obtained. But, for the solution* $A^\star = -1, B^\star = 2$*, the 2D matrix differential equation*

$$\begin{pmatrix} t-3 & 3 \\ \frac{1}{3}t + \frac{10}{3} & t-2 \end{pmatrix} \cdot Y' + \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \cdot Y = O, Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

is obtained by a matrix polynomial solution $Y = \begin{pmatrix} -3(3t-10) \\ t+16 \end{pmatrix}$. *For the new 2D matrix differential equation by finding the corresponding second-order differential equations for* $y_1(t)$ *same differential equation(2.19) will be obtained as for* $x_1(t)$. *While for* $y_2(t)$ *a new second-order differential equation*

$$(\frac{1}{3}t + \frac{10}{3})(t^2 - 6t - 4)y_2'' + (\frac{2}{3}t^2 + \frac{26}{3}t - 32)y_2' + (-\frac{2}{3}t + 2)y_2 = 0$$

i.e., the equation

$$(t + 10)(t^2 - 6t - 4)y_2'' + (2t^2 + 26t - 96)y_2' + (-2t + 6)y_2 = 0$$

is obtained by a polynomial solution

$$y_2(t) = t + 16.$$

The same procedure for the second-order differential equation (2.20) is repeated. By using the formulas (2.17) for the solution of the quadratic equation per A, *two solutions are obtained. For the solution* $A = 3, B = -2$, *the same 2D matrix differential equation (2.18) is obtained. But, for the solution* $A^\star = -2, A^\star = 3$, *a new 2D matrix diferential equation*

$$\begin{pmatrix} t+4 & 18 \\ t+2 & t+8 \end{pmatrix} \cdot Z' + \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \cdot Z = O, Z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$

by a matrix polynomial solution $Z = \begin{pmatrix} -(5t^2 - 32t + 35) \\ 2t^2 - 11t + 8 \end{pmatrix}$. *For the new 2D matrix differential equation by finding the corresponding second-order differential equations for* $z_2(t)$ *same differential equation (2.20) will be obtained as for* $x_2(t)$. *While for* $z_1(t)$ *a new second-order differential equation*

$$18(t^2 - 6t - 4)z_1'' + (54t - 180)z_1' + 144z_1 = 0$$

is obtained by a polynomial solution $z_1(t) = t^2 - \frac{32}{5}t + 7$.

3. **Conclusion**

In general, it can be concluded that the 2D matrix differential equation (1.1) is connected with the second-order differential equation (1.2) and with systems of differential equations, [12, 13, 15]. This connection can be used when studying the properties of any of them, as of example: conditions for the existence of a polynomial solution and its formula, a connection with classical polynomials and polynomials orthogonal to a circular arc, etc., [1, 2, 3, 4, 5, 6]).

In [7] from the same aspect in the form of a system of differential equations, another class of 2D matrix differential equations is considered. In fact, that class of 2D matrix differential equations is a subclass of the class of 2D matrix differential equations (1.1).

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