GOCE DELCEV UNIVERSITY - STIP
FACULTY OF COMPUTER SCIENCE

The journal is indexed in
EBSCO

ISSN 2545-4803 on line
DOI: 10.46763/BJAMI

BALKAN JOURNAL OF APPLIED MATHEMATICS AND INFORMATICS (BJAMI)

YEAR 2023

VOLUME VI, Number 2

AIMS AND SCOPE:
BJAMI publishes original research articles in the areas of applied mathematics and informatics.

Topics:
1. Computer science;
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BALKAN JOURNAL
OF APPLIED MATHEMATICS AND INFORMATICS
(BJAMI), Vol 6

ISSN 2545-4803 on line
Vol. 6, No. 2, Year 2023

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CONTENT

The Appendix

In honor of the first Doctor of Mathematical Sciences Acad. Blagoj Popov, a mathematician dedicated to differential equations, the idea of holding the "Day of DifferentialEquations" was born, prompted by Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski, and Prof. Ph.D. Lazo Dimov. Acad. Blagoj Popov presented his doctoral dissertation on 05.05.1952 in the field of differential equations. This is the main reason for holding the "Day of Differential Equations" at the beginning of May.

This year on May 5th, the "Day of Differential Equations" was held for the seventh time under the auspices of the Faculty of Computer Sciences at "Goce Delcev" University in Stip and Dean Prof. Ph.D. Saso Koceski, organized by Prof. Ph.D. Biljana Zlatanovska, Prof. Ph.D. Marija Miteva and Prof. Ph.D. Limonka Koceva Lazarova.

The participants of this event were:

1. Prof. Ph.D. Aleksa Malcheski from the Faculty of Mechanical engineering at Ss.Cyril and Methodius University in Skopje;
2. Prof. Ph.D. Slagjana Brsakoska from the Faculty of Natural Sciences and Mathematics at Ss.Cyril and Methodius University in Skopje;
3. Prof. Ph.D. Natasa Koceska, Prof. Ph.D. Limonka Koceva Lazarova, Prof. Ph.D. Marija Miteva and Prof. Ph.D. BiljanaZlatanovska from the Faculty of Computer Sciences at Goce Delcev University in Stip;
4. Ass. Prof. Ph.D. Biljana Citkuseva Dimitrovska and Ass. M.Sc. Maja Kukuseva Panova from the Faculty of Electrical Engineering at Goce Delcev University in Stip.

Acknowledgments to Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski and Prof.Ph.D. Lazo Dimov for the wonderful idea and the successful realization of the event this year and in previous years.

Acknowledgments to the Dean of the Faculty of Computer Sciences, Prof. Ph.D. Saso Koceski for her overall support of the organization and implementation of the "Day of Differential Equations".

The papers that emerged from the "Day of Differential Equations" are in the appendixto this issue of BJAMI.

Balkan Journal of Applied Mathematics and Informatics Online ISSN 2545-4083
Volume VI Number 2 Year 2023 UDC: 517.442/.443:621.31.011.7

APPLICATION OF THE LAPLACE TRANSFORM IN ELECTRICAL CIRCUITS

JASMINA VETA BURALIEVA AND MAJA KUKUSHEVA PANEVA

Abstract. In the present paper the application of the Laplace transform and its inverse transform for solving ordinary differential equations with constant coefficients and initial condition is considered. The solution of an ordinary differential equation that describes a serial RLC circuit is provided. Some results of the simulations are also given.

1. Introduction

The Laplace transform is an essential tool in modern approaches for analyzing and designing engineering systems. In fact, the theory of Laplace and its inverse transform was developed as a result of a necessity to find a mathematical model through which practical problems in engineering could be solved. English electrical engineer Oliver Heaviside developed a method for systematic solution of ordinary differential equations with constant coefficients, [11]. His method gave accurate results, especially in practical problems, beside the fact that it is based on intuition and mathematically not verified. Even so, it gives a solution to some complicated problems, what classical methods cannot do. So, this method led to developments in different areas, including results for the transmission of current and voltage through transmission wires.

Engineers widely accepted and used the Heaviside's method because of its practicality, such that took the attention of mathematicians, whose main goal was mathematical verification of his method. So, this contributes to rapid developments of many branches in mathematics such as improper integrals, asymptotic series, the theory of transforms etc. The research of this problem continued until it was recognized that the integral transform was developed a century earlier by the French mathematician Pierre-Simon Laplace, who proved the theoretical basis of Heaviside's work. Even more, it was observed that the application of this integral transform, i.e., the Laplace transform, gave better results than Heaviside's method for the theory of differential equations.

The Laplace transform is an ideal tool for solving initial value problems such as those encountered in electrical circuits and mechanical vibration (see e.g. [2]-[4], [6]-[9]). The advantage of the Laplace transform over the classical method for solving differential equations with initial conditions is that the initial conditions are automatically included in the solution. When applying the Laplace transform on a certain differential equation, it converts the differential equation into an algebraic equation, such that the solution of the algebraic equation could be found easier and faster. So, the solution of the corresponding

Keywords. Laplace and inverse Laplace transform, ordinary differential equation, electrical circuit, simulation.

differential equation is obtained when the inverse Laplace transform is applied on the obtained algebraic equation, ([1]-[4], [10]). In the present paper, the solution of ordinary differential equation with constant coefficients and initial conditions is considered, that describes a serial RLC circuit. Also, the results for the current from the simulations are presented.

2. The Laplace and the inverse Laplace transform

Let us suppose that $f(t)$ is a real valued function of the variable $t>0$, and s is a complex parameter. In engineering, t is usually the variable for time and s for frequency. For the real and imaginary part of the complex parameter s, we will use the notations $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$, respectively.

The Laplace transform of $f(t)$ is defined as

$$\mathscr{L}\left(f(t)\right)(s)=F(s)=\int_0^\infty f(t)e^{-st}dt\,, \tag{2.1}$$

whenever the integral absolutely converges. If $F(s)$, $s=\mathrm{Re}(s)+i\,\mathrm{Im}(s)$, is the Laplace transform of the function $f(t)$, i.e. $f(t)H(t)$, then

$$f(t)H(t)=\mathscr{L}^{-1}\left(F(s)\right)(t)=\frac{1}{2\pi i}\int_{\mathrm{Re}s+i\infty}^{\mathrm{Re}s+i\infty}F(s)e^{st}ds\,,$$

is called the inverse Laplace transform, the integral is Bromwich integral, and $H(t)$ stands for the Heaviside function. We will use the notation $\mathscr{L}^{-1}$ for the inverse Laplace transform. One can find more about the inverse Laplace transform in [7].

Table 1. *Table of pairs for the Laplace and the inverse Laplace transform of some basic functions*

$f(t)$	$F(s)=\mathscr{L}\left(f(t)\right)$
c, $c=const.$	$\frac{c}{s}$, $\mathrm{Re}(s)>0$
t	$\frac{1}{s^2}$, $\mathrm{Re}(s)>0$
e^{kt}, $k\in\mathbb{C}$	$\frac{1}{s-k}$, $\mathrm{Re}(s)>\mathrm{Re}(k)$
$\sin kt$, $k\in\mathbb{R}$	$\frac{k}{s^2+k^2}$, $\mathrm{Re}(s)>0$
$\cos kt$, $k\in\mathbb{R}$	$\frac{s}{s^2+k^2}$, $\mathrm{Re}(s)>0$

In Table 1 the pairs of functions and their Laplace transform and reverse, i.e., the Laplace transform and its inverse Laplace transform are given [7, Chap.1].

A function $f(t)$ has exponential order when $t \to \infty$ if there exist $\alpha \in \mathbb{R}$ and $M > 0$ such that for some $T \geq 0$ holds $\left|f(t)\right| \leq Me^{\alpha t}$, for all $t \geq T$.

We say that the $f(t)$ has exponential order if it does not grow faster than any exponential function. The infimum of set of α is called abscise of convergence, and we will use the notation α_c.

Theorem 2.1. ([7, Thrm.1.11]) *If $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order α_c, then the Laplace transform (2.1) of $f(t)$ exists for $\mathrm{Re}(s) > \alpha_c$ and converges absolutely.*

One of the most basic and useful properties of the Laplace operator $\mathscr{L}$ is its *linearity*, i.e., if the Laplace transform of the functions $f(t)$ and $g(t)$ exists for $\mathrm{Re}(s) > \alpha_f$ and $\mathrm{Re}(s) > \alpha_g$, respectively, then the Laplace transform of the function $\gamma f(t) + \beta g(t)$ exists for $\mathrm{Re}(s) > \alpha$, $\alpha = \max\{\alpha_f, \alpha_g\}$, such that

$$\mathscr{L}\,(\gamma f(t) + \beta g(t)) = \gamma\,\mathscr{L}\,(f(t)) + \beta\,\mathscr{L}\,(g(t)),$$

for arbitrary constants γ and β. Now, if the Laplace transform of $f(t)$ and $g(t)$ is $\mathscr{L}\left(f(t)\right) = F(s)$ and $\mathscr{L}\left(g(t)\right) = G(s)$, respectively, then the inverse Laplace transform $\mathscr{L}^{-1}$ is also linear operator, i.e.,

$$\mathscr{L}^{-1}\,(\gamma F(s) + \beta G(s)) = \gamma\,\mathscr{L}^{-1}\,(F(s)) + \beta\,\mathscr{L}^{-1}\,(G(s)),$$

holds for arbitrary real constants γ and β.

Theorem 2.2. (First Translation Theorem) [7, Thrm.1.27] *If the Laplace transform of the function $f(t)$ exists for $\mathrm{Re}(s) > \alpha_c$, and is $F(s)$, then the Laplace transform of the function $e^{kt} f(t)$, $k \in \mathbb{C}$ exists and is*

$$\mathscr{L}\left(e^{kt} f(t)\right) = F(s-k),\ \mathrm{Re}(s) > \alpha_c + \mathrm{Re}(k).$$

Theorem 2.3. ([7, Thrm.1.34]) *If the Laplace transform of the function $f(t)$ exists for $\mathrm{Re}(s) > \alpha_c$, and is $F(s)$, then the Laplace transform of the functions $t^n f(t)$ $(n = 1, 2, \ldots)$ also exist and are*

$$\mathscr{L}\,(t^n f(t)) = (-1)^n \frac{d^n F(s)}{ds^n},\ \mathrm{Re}(s) > \alpha_c.$$

Proposition 2.1. *The inverse Laplace transform of* $F(s-k)$ *is*

$$\mathscr{L}^{-1}\left(F(s-k)\right)=e^{kt}f(t)H(t),$$

where $F(s)$ *is the Laplace transform of the function* $f(t)$.

2.1. The Laplace transform as a tool for solving ordinary differential equations

In this section we will consider the application of the Laplace and the inverse Laplace transform for solving ordinary differential equations with constant coefficients. In Theorem 2.4. the Laplace transform of n^{th} derivative of continuous function is given $f(t)$.

Theorem 2.4. ([7, Thrm. 2.12]) *Let us suppose that* $f(t)$, $f^{'}(t)$, ..., $f^{(n-1)}(t)$ *are continuous on* $(0,\infty)$ *and of exponential order, while* $f^{(n)}(t)$ *is piecewise continuous on* $(0,\infty)$. *Then*

$$\mathscr{L}\left(f^{(n)}(t)\right)=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0). \tag{2.2}$$

The Laplace transform converts the ordinary differential equation into an algebraic equation by the variable s. Applying the inverse Laplace transform to the corresponding algebraic equation, using its linearity, the table of pair or some other properties, the solution of the ordinary differential equation is obtained.

Let us consider the ordinary differential equation from second order with constant coefficients, i.e.,

$$a\frac{d^2f(t)}{dt^2}+b\frac{df(t)}{dt}+cf(t)=g(t),\ \ \mathrm{t}\geq 0, \tag{2.3}$$

and with initial conditions $f(0)=m_0,\ f'(0)=n_0,\ m_0,\ n_0\in\mathbb{R}$.

Applying the Laplace transform on (2.3) and using Theorem 2.4.

$a\left(s^2F(s)-sf(0)-f'(0)\right)+b\left(sF(s)-f(0)\right)+cF\left(s\right)=G\left(s\right)$ is obtained

Now, using the initial conditions $f(0)=m_0,\ f'(0)=n_0,\ m_0,\ n_0\in\mathbb{R}$ we have

$$a\left(s^2F(s)-sm_0-n_0\right)+b\left(sF(s)-m_0\right)+cF\left(s\right)=G\left(s\right),$$

such that

$$F(s)=\frac{G(s)+(as+b)m_0+an_0}{as^2+bs+c}. \tag{2.4}$$

The solution of the ordinary differential equation (2.3) is obtained, applying the inverse Laplace transform on (2.4). Using the initial conditions, with this method the complete solution of the differential equation (2.3) is obtained, which is not the case with the classical method. The whole procedure could be also applied to a differential equation of n^{th} order with constant coefficients, i.e.,

$$a_n \frac{d^n f(t)}{dt^n} + a_{n-1} \frac{d^{n-1} f(t)}{dt^{n-1}} + \cdots + a_1 \frac{df(t)}{dt} + a_0 f(t) = g(t), \quad t \geq 0 \qquad (2.5)$$

and initial conditions $f(0) = f_0,\ f'(0) = f_1, \ldots, f^{(n-1)}(0) = f_{n-1}$.

3. Application of the Laplace transform in electrical circuits

In electrical engineering the Laplace transform is used to transform the time domain curcuits into a complex s domain, such that the differential equations are converted to algebraic equations. If the differential equation (2.5) that describes some electrical circuit is considered, than $g(t)$ is known as the excitation or input signal, and $f(t)$ is the output or the response signal. If $g(t)$ is continious and has exponetial order, than the output $f(t)$ also has exponential order and is continuous. This helps to justify the application of the Laplace transform.

The substitution of time domain curcuit elements with Laplace domain equvalents is given in Table 2, such that R, C, L, M are real constants, while in Figure the serial RLC curcuit that is used in the analysis1 is given.

Table 2. *Time domain curcuit elements with their Laplace equivalents*

Description	Time domain (t)	s - domain
Resistor	R	R
Capacitor	C	1/s C
Inductor	L	s L
Positive step change in voltage	M	M/s

With $E(t)$ in Figure 1 and in the rest of the paper we denote the voltage in the time domain, while L, R and C stand for inductance, resistance and capacitance, respectively. For the current in time and complex domain $i(t)$ and are used $I(s)$, respectively.

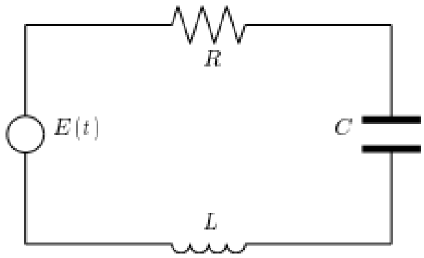

Figure 1. *Serial RLC circuit*

The serial RLC circuit given in Figure 1 consists of a serial connection between the inductor, resistor and capacitor with voltage drops across them, i.e., $L\dfrac{di}{dt}, Ri, \dfrac{1}{C}, \int_0^t i(\tau)d\tau$, respectively. By Kirchoff's voltage law, i.e., the sum of the voltage drops across individual components in closed circuit is equal to the impressed voltage, it follows that

$$L\frac{di}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau=E(t)\,. \tag{3.1}$$

The differential equation (3.1) describes the serial RLC circuit in Figure 1. If (3.1) is differentiated, it follows that

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=\frac{dE}{dt}\,. \tag{3.2}$$

Now, if the Laplace transform is applied on (3.2) and using Theorem 2.4., for the current in the curcuit in complex *s* domain it is obtained

$$I(s)=\frac{M}{s\left(R+sL+\dfrac{1}{sC}\right)}=\frac{M}{L}\cdot\frac{1}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}}, \tag{3.3}$$

using the initial condition $i(0)=0,\ \ i'(0)=0$ and the fact that voltage is a step function given in Table 2. Because M and L are constants only the term

$$\frac{1}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}}, \tag{3.4}$$

is considered.

The Denominator of (3.4) depending on its discriminator could have three posible solutions: real and equal, real and different, and complex conjugate, which are coresppording poles for (3.4). Because of enginering purposes, only the complex conjugate poles are considered.

Let us note that the denominator of (3.4) can be rewritten as

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = s^2 + 2as + (a^2 + b^2)$$

where $2a = R/L$, $(a^2 + b^2) = \dfrac{1}{LC}$ and $b = \sqrt{\dfrac{1}{LC} - \left(\dfrac{R}{2L}\right)^2}$.

Now, if the inverse Laplace transform is applied on (3.3) , and using Proposition 2.1. and Table 1, the following is obtained

$$i(t) = \frac{M}{L}\mathscr{L}^{-1}\left(\frac{1}{(s+a)^2 + b^2}\right) = \frac{M}{L}\cdot\frac{1}{b}e^{-at}\sin(bt)H(t) . \qquad (3.5)$$

The term $\omega_n = 1/\sqrt{LC}$ is usually referred to as resonant frequency of the circuit so that $b = \sqrt{\omega_n^2 - a^2}$. So, finally, the current (3.5) throug the curcuit in the time domain is

$$i(t) = \frac{M}{L}\cdot\frac{1}{\sqrt{\omega_n^2 - a^2}}e^{-at}\sin(\sqrt{\omega_n^2 - a^2}t)H(t) . \qquad (3.6)$$

In Figure 2 the serial RLC circuit in LTSpice® [10] is given. LTSpice® is a powerful free SPICE simulator software that allows draft, probe and analysis of performance of an electrical circuit design. The simulations are performed using a variable capacitor with the capacitance value range from 10nF to 510nF with step 100nF.

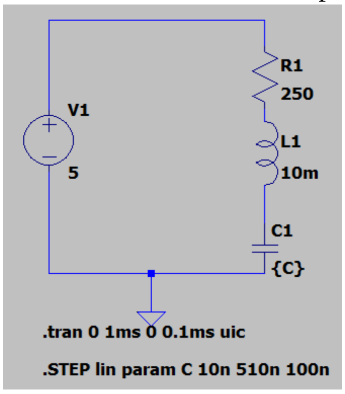

Figure 2. *Serial RLC curcuit in LTSpice*

The simulation results are shown in Figure 3. From the graphs it can be noticed that the current changes as the capacitivy of the capacitor changes. As the capacitance of the capacitor rises, the current through the capacitor rises. Also, from the results it can be noted that the oscilationg behaviour of the circuit is reduced as the capacitance rises.

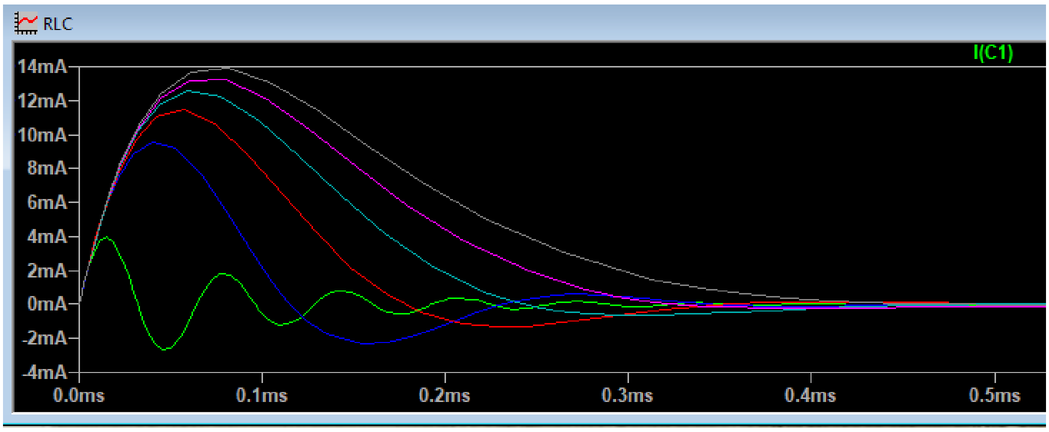

Figure 3. *R=250Ω, E=5 V, L=10 mH, C variable from 10nF to 510nF*

4. Conclusion

In this paper, the application of the Laplace transform of second order differential equation with constant coefficients and initial conditions that describes a serial RLC circuit is analyzed. The results of the current in time and frequency domains are given. In a similar way, ODE from n^{th} order that describes an electrical circuit can be solved using the Laplace and the inverse Laplace transform. The advantage of the application of the Laplace transform is that the final solution is directly obtained, i.e., the initial conditions are directly involved in the final solution, unlike with the classical methods where the general solution is obtained first, and then, using the initial condition, the final solution is obtained (see e.g. [8], [9]). The algebraic equation is easier to solve than the differential equation. This shows that the Laplace transform is a useful and effective tool in solving complex problems in the analysis of electrical circuits.

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