GOCE DELCEV UNIVERSITY - STIP
FACULTY OF COMPUTER SCIENCE

The journal is indexed in
EBSCO

ISSN 2545-4803 on line
DOI: 10.46763/BJAMI

BALKAN JOURNAL OF APPLIED MATHEMATICS AND INFORMATICS (BJAMI)

YEAR 2023

VOLUME VI, Number 2

AIMS AND SCOPE:
BJAMI publishes original research articles in the areas of applied mathematics and informatics.

Topics:
1. Computer science;
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BALKAN JOURNAL
OF APPLIED MATHEMATICS AND INFORMATICS
(BJAMI), Vol 6

ISSN 2545-4803 on line
Vol. 6, No. 2, Year 2023

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CONTENT

The Appendix

In honor of the first Doctor of Mathematical Sciences Acad. Blagoj Popov, a mathematician dedicated to differential equations, the idea of holding the "Day of DifferentialEquations" was born, prompted by Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski, and Prof. Ph.D. Lazo Dimov. Acad. Blagoj Popov presented his doctoral dissertation on 05.05.1952 in the field of differential equations. This is the main reason for holding the "Day of Differential Equations" at the beginning of May.

This year on May 5th, the "Day of Differential Equations" was held for the seventh time under the auspices of the Faculty of Computer Sciences at "Goce Delcev" University in Stip and Dean Prof. Ph.D. Saso Koceski, organized by Prof. Ph.D. Biljana Zlatanovska, Prof. Ph.D. Marija Miteva and Prof. Ph.D. Limonka Koceva Lazarova.

The participants of this event were:

1. Prof. Ph.D. Aleksa Malcheski from the Faculty of Mechanical engineering at Ss.Cyril and Methodius University in Skopje;
2. Prof. Ph.D. Slagjana Brsakoska from the Faculty of Natural Sciences and Mathematics at Ss.Cyril and Methodius University in Skopje;
3. Prof. Ph.D. Natasa Koceska, Prof. Ph.D. Limonka Koceva Lazarova, Prof. Ph.D. Marija Miteva and Prof. Ph.D. BiljanaZlatanovska from the Faculty of Computer Sciences at Goce Delcev University in Stip;
4. Ass. Prof. Ph.D. Biljana Citkuseva Dimitrovska and Ass. M.Sc. Maja Kukuseva Panova from the Faculty of Electrical Engineering at Goce Delcev University in Stip.

Acknowledgments to Prof. Ph.D. Boro Piperevski, Prof. Ph.D. Borko Ilievski and Prof.Ph.D. Lazo Dimov for the wonderful idea and the successful realization of the event this year and in previous years.

Acknowledgments to the Dean of the Faculty of Computer Sciences, Prof. Ph.D. Saso Koceski for her overall support of the organization and implementation of the "Day of Differential Equations".

The papers that emerged from the "Day of Differential Equations" are in the appendixto this issue of BJAMI.

Balkan Journal of Applied Mathematics and Informatics Online ISSN 2545-4083
Volume VI Number 2 Year 2023 UDC: 517.968.7:517.53

ONE APPROACH TO THE ITERATIONS OF THE VEKUA EQUATION

SLAGJANA BRSAKOSKA AND ALEKSA MALCHESKI

Abstract. In this paper one old and one new approach to the iterations of the Vekua equation in its most simple form are given. Some comparison is made and the conclusions are formulated as theorems.

1. Introduction

The equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F \qquad (1.1)$$

where $A = A(z)$, $B = B(z)$ and $F = F(z)$ are given complex functions from a complex variable $\boldsymbol{z} \in \boldsymbol{D} \subseteq \mathbb{C}$ is the well-known Vekua equation [1] according to the unknown function $W = W(z) = u + iv$. The derivative on the left side of this equation has been introduced by G. V. Kolosov in 1909 [2]. During his work on a problem from the theory of elasticity, he introduced the expressions:

$$\frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] = \frac{\hat{d}W}{dz} \qquad (1.2)$$

and

$$\frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] = \frac{\hat{d}W}{d\bar{z}} \qquad (1.3)$$

known as operator derivatives of a complex function $W = W(z) = u(x, y) + iv(x, y)$ from a complex variable $z = x + iy$ and $\bar{z} = x - iy$ correspondingly. The operating rules for these derivatives are completely given in the monograph of Г. Н. Положий [3] (pages18-31). In the mentioned monograph are defined the so-called operator integrals $\hat{\int} f(z)dz$ and $\hat{\int} f(z)d\bar{z}$ from $z = x + iy$ and $\bar{z} = x - iy$ correspondingly (pages 32-41). As for the complex integration in the same monograph it is emphasized that it is assumed that all operator integrals can be solved in area D.

In the Vekua equation (1.1) the unknown function $W = W(z)$ is under the sign of a complex conjugation which is equivalent to the fact that $B = B(z)$ is not identically equaled to zero in D. That is why for (1.1) the quadratures that we have for the equations where the unknown function $W = W(z)$ is not under the sign of a complex conjugation, stop existing.

This equation is important not only for the fact that it came from a practical problem, but also because depending on the coefficients A, B and F, the equation (1.1)

2010 *Mathematics Subject Classification.* 34M45, 35Q74.
Keywords. *Vekua equation, areolar differential, areolar integral, iterations*

defines different classes of generalized analytic functions. For example, for $F = F(z) \equiv 0$ in D the equation (1.1) i.e.,

$$\frac{\hat{d}W}{d\overline{z}} = AW + B\overline{W} \qquad (1.4)$$

which is called canonical Vekua equation, defines the so-called generalized analytic functions from the fourth class; and for $A \equiv 0$ and $F \equiv 0$ in D, the equation (1.1) i.e., the equation $\frac{\hat{d}W}{d\overline{z}} = B\overline{W}$ defines the so-called generalized analytic functions from the third class or the (r+is)-analytic functions [3], [4].

Those are the cases when $B \neq 0$. But if we put $B \equiv 0$, we get the following special cases. In the case $A \equiv 0$, $B \equiv 0$ and $F \equiv 0$ in the working area $\boldsymbol{D} \subseteq \mathbb{C}$ the equation (1.1) takes the following expression $\frac{\hat{d}W}{d\overline{z}} = 0$ and this equation, in the class of the functions $W = u(x,y) + iv(x,y)$ whose real and imaginary parts have unbroken partial derivatives u'_x, u'_y, v'_x and v'_y in D, is a complex writing of the Cauchy - Riemann conditions. In other words, it defines the analytic functions in the sense of the classic theory of the analytic functions. In the case $B \equiv 0$ in D, i.e., $\frac{\hat{d}W}{d\overline{z}} = AW + F$ is the so cold areolar linear differential equation [3] (pages 39-40) and it can be solved with quadratures by the formula:

$$W = e^{\hat{\int} A(z)d\overline{z}} \left[\Phi(z) + \hat{\int} F(z) e^{-\hat{\int} A(z)d\overline{z}} d\overline{z} \right].$$

Here $\Phi = \Phi(z)$ is an arbitrary analytic function in the role of an integral constant.

2. *Iterations in the Vekua equation* $\frac{\partial W}{\partial \overline{z}} = A(z,\overline{z})\overline{W}$

Let us write an integral form of the equation

$$\frac{\partial W}{\partial \overline{z}} = A(z,\overline{z})\overline{W} \qquad (2.1)$$

i.e.,

$$W = \hat{\int} A(z,\overline{z})\overline{W} d\overline{z} + C(z) \qquad (2.2)$$

Let us define a sequence

$$W^{[n]}(z,\overline{z}) = \hat{\int} A(z,\overline{z})\overline{W^{[n-1]}} d\overline{z} + C(z) \qquad (2.3)$$

with the condition $\qquad W^{[0]} \overset{def}{=} C(z) \qquad (2.4)$

We have that:

$$W^{[1]} \overset{def}{=} C(z) + \hat{\int} A(z,\bar{z})\overline{W^{[0]}}d\bar{z} =$$

$$= C(z) + \hat{\int} A(z,\bar{z})\overline{C(z)}d\bar{z}$$

and then

$$W^{[2]} \overset{def}{=} C(z) + \hat{\int} A(z,\bar{z})\overline{W^{[1]}}d\bar{z} =$$

$$= C(z) + \hat{\int} A(z,\bar{z})\left[\overline{C(z)} + \overline{\hat{\int} A(z,\bar{z})\overline{C(z)}d\bar{z}}\right]d\bar{z} =$$

$$= C(z) + \hat{\int} A(z,\bar{z})\left[\overline{C(z)} + \hat{\int} \bar{A}(z,\bar{z})\overline{\overline{C(z)}}d\bar{\bar{z}}\right]d\bar{z} =$$

$$= C(z) + \hat{\int} A\bar{C}d\bar{z} + \hat{\int} A\left(\hat{\int} \bar{A}Cdz\right)d\bar{z}$$

$$W^{[3]} \overset{def}{=} C(z) + \hat{\int} A(z,\bar{z})\overline{W^{[2]}}d\bar{z} =$$

$$= C(z) + \hat{\int} A\overline{\left(C(z) + \hat{\int} A\bar{C}d\bar{z} + \hat{\int} A\left(\hat{\int} \bar{A}Cdz\right)d\bar{z}\right)}d\bar{z} =$$

$$= C(z) + \hat{\int} A\left(\bar{C} + \overline{\hat{\int} A\bar{C}d\bar{z}} + \overline{\hat{\int} A\left(\hat{\int} \bar{A}Cdz\right)d\bar{z}}\right)d\bar{z} =$$

$$= C(z) + \hat{\int} A\left(\bar{C} + \hat{\int} \bar{A}\bar{\bar{C}}d\bar{\bar{z}} + \right)d\bar{z} + \hat{\int} A\left(\overline{\hat{\int} A\left(\hat{\int} \bar{A}Cdz\right)d\bar{z}}\right)d\bar{z}$$

$$W^{[3]} = C(z) + \hat{\int} A\left(\bar{C} + \hat{\int} \bar{A}\bar{\bar{C}}d\bar{\bar{z}}\right)d\bar{z} + \hat{\int} A\left(\overline{\hat{\int} A\left(\hat{\int} \bar{A}Cdz\right)d\bar{z}}\right)d\bar{z} =$$

$$= C(z) + \hat{\int} A\left(\bar{C} + \hat{\int} \bar{A}Cdz\right)d\bar{z} + \hat{\int} A\left(\hat{\int} \bar{A}\left(\hat{\int} \bar{\bar{A}}\bar{C}d\bar{z}\right)d\bar{\bar{z}}\right)d\bar{z} =$$

$$= C(z) + \hat{\int} A\bar{C}d\bar{z} + \hat{\int} A\left(\hat{\int} \bar{A}Cdz\right)d\bar{z} + \hat{\int} A\left(\hat{\int} \bar{A}\left(\hat{\int} A\bar{C}d\bar{z}\right)dz\right)d\bar{z}$$

From the previous, we can see that:

$$W^{[1]} = W^{[0]} + \hat{\int} A(z,\bar{z})\overline{C(z)}d\bar{z}$$

$$W^{[2]} = W^{[1]} + \hat{\int} A\left(\hat{\int} \bar{A}Cdz\right)d\bar{z}$$

$$W^{[3]} = W^{[2]} + \hat{\int} A\left(\hat{\int} \bar{A}\left(\hat{\int} A\bar{C}d\bar{z}\right)dz\right)d\bar{z}$$

where from with induction we can approximate the difference:

$$W^{[n+1]} - W^{[n]} = \underbrace{\hat{\int} A \left(\hat{\int} \overline{A} \left(\hat{\int} A \ldots \left(\hat{\int} A\overline{C} dz \right) d\overline{z} \right) dz \right) d\overline{z}}_{n+1 \quad \text{интеграли}}$$

where the last integral is either $\hat{\int} A\overline{C}dz$ or $\hat{\int} \overline{A}Cd\overline{z}$ depending on whether n is even or odd, but that is not relevant if we want to take a module of it. So:

$$\left| W^{[n+1]} - W^{[n]} \right| < \hat{\int} |A| \hat{\int} |A| \hat{\int} |A| \ldots \hat{\int} |A| |C(z)| |dz|^n$$

since $|d\overline{z}| = |dz|, \left|\overline{C}(z)\right| = |C(z)|$.

or $\left| W^{[n+1]} - W^{[n]} \right| < \left| A(z,\overline{z}) \right|^{n+1} |C(z)| \dfrac{|z|^{n+1}}{(n+1)!}$

but because of

$$\left| W^{[1]} - W^{[0]} \right| = \left| \hat{\int} A(z,\overline{z}) \overline{C}(z) d\overline{z} \right| < \max \left| A(z,\overline{z}) \right| \cdot \max |C(z)| \cdot |\overline{z}|$$

from

$$\left| W^{[n+1]} - W^{[n]} \right| < \max \left| A(z,\overline{z}) \right| \cdot \max |C(z)| \frac{|z|^{n+1}}{(n+1)!} =$$

$$= \max \left(\left| A(z,\overline{z}) \right| \cdot |C(z)| \right) \cdot |z| \cdot \frac{|z|^n}{n!} \cdot \frac{1}{n+1}$$

we have that

$$\left| W^{[n+1]} - W^{[n]} \right| < \left| W^{[1]} - W^{[0]} \right| \cdot \frac{|z|^n}{n!} \cdot \frac{1}{n+1}$$

Since for every finite $|z| = R$, the function $|z|^n$ is smaller than $n!$, and for large n

$$|z|^n < n! \text{ for } |z| = R \text{ or } \frac{|z|^n}{n!} < q < 1$$

and we will have

$$\left| W^{[n+1]} - W^{[n]} \right| < \left| W^{[1]} - W^{[0]} \right| \cdot \frac{q}{n+1}$$

which means that

$$\left| W^{[n+1]} - W^{[n]} \right| \to 0, \quad n \to \infty$$

i.e., the operator (2.3) is an operator of contraction, and with inequality for connection between $n!$ and a^n, we can write the following inequalities

$$\left| W^{[n+1]} - W^{[n]} \right| < q^n \left| W^{[1]} - W^{[0]} \right|$$

where $q < 1$. So, we have the following:

Theorem 2.1. *The process of iteration in the conjugated Vekua equation (2.1) is convergent, the operator (2.2) is an operator of contraction and the sequence* $W^{[0]}, W^{[1]}, W^{[2]}, W^{[3]}, \ldots, W^{[n]}, \ldots$ *is convergent and its limit is the solution of the equation (2.1).*

As the function does not have a successful quadrature solution to the equation (2.1), mainly due to the absence (or unsuitability) of the integral calculus to the operation $\overline{W(z,\overline{z})}$, or the impossibility of realizing the indefinite integral $\hat{\int}\overline{W(z,\overline{z})}d\overline{z}$ exactly through finite formulas, this theorem is still attractive in the treatment of the Vequa equation to this day, and thus to many tasks of mathematical physics.

3. *Vekua equation* $\frac{\partial W}{\partial \overline{z}} = A(z,\overline{z})\overline{W}$

Attempt for algebraic iterations

Let us avoid the nonlinearity in $\overline{W}$, i.e., to replace it with another nonlinearity. So, we have:

$$\frac{\partial W}{\partial \overline{z}} = A(z,\overline{z})\frac{|W|^2}{W}$$

If we multiply with $2W$, we have that

$$2W\frac{\partial W}{\partial \overline{z}} = 2A(z,\overline{z})|W|^2$$

and the left side is the exact derivative

$$\frac{\partial}{\partial \overline{z}}\left(W^2\right) = 2A(z,\overline{z})|W|^2$$

where with the areolar integral

$$W^2 = 2\hat{\int} A(z,\overline{z})|W|^2\, d\overline{z} + C(z)$$

where $C(z)$ is an analytic function of z and is a "generalized constant". All equation operations require an explicit normal form. Therefore, we must introduce a square root, which means a multivalued solution

$$W(z,\overline{z}) = \sqrt{C(z) + 2\hat{\int} A(z,\overline{z})|W|^2\, d\overline{z}}$$

so the nonlinearity is transferred to the square root and the square of the modulus $|W|^2$.

It will interfere a lot in the iterations, which can be seen from the following. If we introduce a sequence

$$W^{[n]} \overset{def}{=} \sqrt{C(z) + 2\hat{\int} A(z,\overline{z})\left|W^{[n-1]}\right|^2 d\overline{z}} \qquad \text{with} \qquad W^{[0]} \overset{def}{=} \sqrt{C(z)}$$

we have that $W^{[1]} = \sqrt{C(z) + 2C(z)\hat{\int} A(z,\overline{z})d\overline{z}} = \sqrt{C(z)\left(1 + 2\hat{\int} A(z,\overline{z})d\overline{z}\right)}$

but $W^{[2]}$ already gives

$$W^{[2]} \overset{def}{=} \sqrt{C(z) + 2\hat{\int} A(z,\overline{z})\left|W^{[1]}\right|^2 d\overline{z}}$$

where from ($W^{[2]} \overset{def}{=} \sqrt{C(z)+2\hat{\int} A(z,\bar{z})\left(C(z)+2C(z)\hat{\int} A(z,\bar{z})d\bar{z}\right)d\bar{z}}$)

$$W^{[2]} \overset{def}{=} \sqrt{C(z)+2\hat{\int} A\left(C(z)+2C(z)\hat{\int} Ad\bar{z}\right)d\bar{z}} =$$

$$= \sqrt{C(z)\left(1+2\hat{\int} Ad\bar{z}+4\hat{\int} A\left(\hat{\int} Ad\bar{z}\right)d\bar{z}\right)}$$

$$W^{[3]} \overset{def}{=} \sqrt{C(z)+2\hat{\int} A\left|W^{[2]}\right|^2 d\bar{z}} =$$

$$= \sqrt{C(z)+2\hat{\int} A\left(C(z)\left(1+2\hat{\int} Ad\bar{z}+4\hat{\int} A\left(\hat{\int} Ad\bar{z}\right)d\bar{z}\right)\right)d\bar{z}} =$$

$$= \sqrt{C(z)\left[1+2\hat{\int} A\left(1+2\hat{\int} Ad\bar{z}+4\hat{\int} A\left(\hat{\int} Ad\bar{z}\right)d\bar{z}\right)d\bar{z}\right]} =$$

$$= \sqrt{C(z)\left[1+2\hat{\int} Ad\bar{z}+4\hat{\int} A\left(\hat{\int} Ad\bar{z}\right)d\bar{z}+8\hat{\int} A\left(\hat{\int} A\left(\hat{\int} Ad\bar{z}\right)d\bar{z}\right)d\bar{z}\right]}$$

$$W^{[4]} \overset{def}{=} \sqrt{C(z)+2\hat{\int} A\left|W^{[3]}\right|^2 d\bar{z}} =$$

$$= \sqrt{C(z)+2\hat{\int} A\left(C(z)\left[1+2\hat{\int} Ad\bar{z}+4\hat{\int} A\left(\hat{\int} Ad\bar{z}\right)d\bar{z}+8\hat{\int} A\left(\hat{\int} A\left(\hat{\int} Ad\bar{z}\right)d\bar{z}\right)d\bar{z}\right]\right)d\bar{z}} =$$

$$= \sqrt{C(z)\left[1+2\hat{\int} Ad\bar{z}+4\hat{\int} A\left(\hat{\int} Ad\bar{z}\right)d\bar{z}+8\hat{\int} A\left[\hat{\int} A\left(\hat{\int} Ad\bar{z}\right)d\bar{z}\right]d\bar{z}+16\hat{\int} A\left(\hat{\int} A\left(\hat{\int} A\left(\hat{\int} Ad\bar{z}\right)d\bar{z}\right)d\bar{z}\right)d\bar{z}\right]}$$

etc.

Here we do not have the part with multiple integrals. Is there any contraction? We have

$$\left|W^{[1]}-W^{[0]}\right| = \left|\sqrt{C(z)}\sqrt{1+C(z)\hat{\int} Ad\bar{z}}-\sqrt{C(z)}\right| =$$

$$= \left|\sqrt{C(z)}\right|\left|\sqrt{1+C(z)\hat{\int} Ad\bar{z}}-1\right| \cdot \frac{\left|\sqrt{1+C(z)\hat{\int} Ad\bar{z}}+1\right|}{\left|\sqrt{1+C(z)\hat{\int} Ad\bar{z}}+1\right|} = \frac{\left|\sqrt{C(z)}\right|\left|1+C(z)\hat{\int} Ad\bar{z}-1\right|}{\left|\sqrt{1+C(z)\hat{\int} Ad\bar{z}}+1\right|} =$$

$$= \frac{\left|\sqrt{C(z)}\right|\left|C(z)\right|\left|\hat{\int} Ad\bar{z}\right|}{\left|\sqrt{C(z)}\right|\left|A\right|^{1/2}\left|\bar{z}\right|^{1/2}} < \left(\left|A(z,\bar{z})\right|\left|C(z)\right|\left|\bar{z}\right|\right)^{1/2}$$

$$\left|W^{[n+1]}-W^{[n]}\right|=\left|W^{[1]}-W^{[0]}\right|\cdot\max\left(\left|A\left(z,\overline{z}\right)\right|\left|C\left(z\right)\right|\left|\overline{z}\right|\right)^{1/2}\cdot\frac{|z|^{n}}{n!}\cdot\frac{1}{n+1}<$$
$$<\frac{M^{1/2}R^{n}}{n!\left(n+1\right)}\cdot\left|W^{[1]}-W^{[0]}\right|=q^{n}\left|W^{[1]}-W^{[0]}\right|$$

where $q^{n}=\dfrac{M^{1/2}R^{n}}{n!\left(n+1\right)}<1$

for finite R and large enough. We see that in this case the convergence of the iterations is preserved and the contraction coefficient is almost the same; however, this algebraic method is not suitable for practice, due to the difficulty of calculating the integrals of the square roots in the iterations.

4. *Vekua equation with one continuous coefficient* $A\left(z,\overline{z}\right)$

Let us consider the Vekua equation a bit more generally,

$$\frac{\partial W}{\partial\overline{z}}=A\left(z,\overline{z}\right)$$

If we write an integral form (or an integral equation for the unknown $W\left(z,\overline{z}\right)$):

$$W=\hat{\int}A\left(z,\overline{z}\right)Wd\overline{z}+C\left(z\right)$$

If, in the same way, we introduce a series of functions, which are determined from each other

$$W^{[n]}\left(z,\overline{z}\right)=\hat{\int}A\left(z,\overline{z}\right)W^{[n-1]}d\overline{z}+C\left(z\right)$$

with the initial condition

$$W^{[0]}\overset{def}{=}C\left(z\right)$$

then

$$W^{[1]}\overset{def}{=}\hat{\int}A\left(z,\overline{z}\right)W^{[0]}d\overline{z}+C\left(z\right)=$$
$$=C\left(z\right)\left[1+\hat{\int}A\left(z,\overline{z}\right)d\overline{z}\right]$$

$$W^{[2]}\overset{def}{=}\hat{\int}A\left(z,\overline{z}\right)W^{[1]}d\overline{z}+C\left(z\right)=$$
$$=\hat{\int}A\left(z,\overline{z}\right)\left[C\left(z\right)\left(1+\hat{\int}A\left(z,\overline{z}\right)d\overline{z}\right)\right]d\overline{z}+C\left(z\right)=$$
$$=C\left(z\right)\left[1+\hat{\int}A\left(z,\overline{z}\right)d\overline{z}+\hat{\int}A\left(z,\overline{z}\right)d\overline{z}\hat{\int}A\left(z,\overline{z}\right)d\overline{z}\right]$$

etc. With full induction we have that:

$$W^{[n]} = C(z)\left[1+\hat{\int} A(z,\overline{z})d\overline{z} + \hat{\int} A(z,\overline{z})d\overline{z}\hat{\int} A(z,\overline{z})d\overline{z} + \right.$$

$$\left. +\hat{\int} A(z,\overline{z})d\overline{z}\hat{\int} A(z,\overline{z})d\overline{z}\hat{\int} A(z,\overline{z})d\overline{z} + \cdots\right]$$

The problem now is how to sum these successive multiple integrals

$$I_n = \hat{\int} A(z,\overline{z})d\overline{z}\hat{\int} A(z,\overline{z})d\overline{z}\ldots\hat{\int} A(z,\overline{z})d\overline{z}$$

Let us take the first multiple integral to be equal to I. The integral

$$I_2 = \hat{\int} \underbrace{A(z,\overline{z})d\overline{z}}_{dv} \underbrace{\hat{\int} A(z,\overline{z})d\overline{z}}_{u}$$

If we apply partial integration with

$$u = \hat{\int} A(z,\overline{z})d\overline{z},\ dv = A(z,\overline{z})d\overline{z}$$

$$du = A(z,\overline{z})d\overline{z},\ v = \hat{\int} A(z,\overline{z})d\overline{z}$$

we have that $I_n = \dfrac{\left(\hat{\int} Ad\overline{z}\right)^n}{n!}$.

If in $W^{[n]}(z,\overline{z})$ we substitute $I_2, I_3, \ldots, I_n$, we get

$$W^{[n]}(z,\overline{z}) = C(z)[1+\frac{\left(\hat{\int} Ad\overline{z}\right)^1}{1!}+\frac{\left(\hat{\int} Ad\overline{z}\right)^2}{2!}+\frac{\left(\hat{\int} Ad\overline{z}\right)^3}{3!}+\cdots+\frac{\left(\hat{\int} Ad\overline{z}\right)^n}{n!}]$$

Also

$$e^t = 1+\frac{t^1}{1!}+\frac{t^2}{2!}+\cdots+\frac{t^n}{n!}+\cdots$$

which is valid not only for real t, but also for complex, for $t = \int f(z)dz$, for $t = \hat{\int} A(z,\overline{z})d\overline{z}$ we have

$$e^{\hat{\int} Ad\overline{z}} = \frac{\left(\hat{\int} Ad\overline{z}\right)^1}{1!}+\frac{\left(\hat{\int} Ad\overline{z}\right)^2}{2!}+\frac{\left(\hat{\int} Ad\overline{z}\right)^3}{3!}+\cdots+\frac{\left(\hat{\int} Ad\overline{z}\right)^n}{n!}+\cdots$$

whence it follows that $W^{[n]}(z,\overline{z})$ is n-th partial sum for $e^{\hat{\int} Ad\overline{z}}$. So

$$W^{[n]}(z,\overline{z}) = C(z)\sum_{k=0}^{n}\frac{\left(\hat{\int} Ad\overline{z}\right)^k}{k!}$$

$$\lim_{n\to\infty} W\left(z,\bar{z}\right)=C\left(z\right)\lim_{n\to\infty}\sum_{k=0}^{n}\frac{\left(\hat{\int} A\left(z,\bar{z}\right)d\bar{z}\right)^{k}}{k!}=C\left(z\right)e^{\hat{\int} A\left(z,\bar{z}\right)d\bar{z}} \qquad (4.1).$$

We can formulate the following:

Theorem 4.1. *For the solution of the Vequa equation (1.1) where* $A\left(z,\bar{z}\right)$ *is a continuous coefficient of the two independent variables* z *and* $\bar{z}$ *, the solution is given by (4.1).*

In doing so, the iteration procedure determined by previous formulas converges to (4.1) and represents one contraction, and the contraction coefficient is

$$|q|=\sqrt[n]{\frac{\left|\hat{\int} A\left(z,\bar{z}\right)d\bar{z}\right|^{k}}{\left(n+1\right)\cdot n!}}$$

whereby a contraction condition applies

$$\left|W^{[n+1]}-W^{[n]}\right|<|q|^{n}\left|W^{[1]}-W^{[0]}\right|, |q|<1$$

in the field $|\bar{z}|<R$ and $A\left(z,\bar{z}\right)$ - continuous.

Note: The same can be done for the Vekua equation with generalized derivative $\frac{\partial W}{\partial \bar{z}}=A\left(z,\bar{z}\right)W$ **.**

Important note: Although the derivation is analytic, the solution will not be analytic due to the existence of a non-analytic element $A\left(z,\bar{z}\right)$ **.**

5. **Conclusion**

In the above mentioned iteration methods, all methods have their positive and negative properties, but the algebraic method has a more practical value than the first one.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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