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REICH-TYPE CONTRACTIVE MAPPING INTO A COMPLETE METRIC SPACE AND CONTINUOUS, INJECTIVE AND SUBSEQUENTIALLY CONVERGENT MAPPING

SAMOIL MALCHESKI

Abstract. In this paper, a generalization of the fixed point theorem of the Reich-type mapping on a complete metric space (X,d) is given. Continuous, injective and sub-sequentially convergent mapping T was used, as well as it is taken that function f is from the class Θ continuous monotonically nondecreasing functions $f:[0,+\infty)\to[0,+\infty)$ such that $f^{-1}(0)=\{0\}$, where it is additionally assumed that it is subadditive, i.e. $f(x+y)\le f(x)+f(y)$, for each $x,y\in[0,+\infty)$.

1. Introduction

Let (X,d) be a metric space. The mapping $S:X\to X$ is called a contraction if it exists $\lambda\in(0,1)$ such that for each $x,y\in X$ holds true

$$d(Sx,Sy)\le\lambda d(x,y). \quad (1.1)$$

If the metric space (X,d) is complete, then the mapping T for which condition (1.1) is satisfied has a unique fixed point. This result is known in the literature as Banach's fixed point principle ([4]). Later, in 1968, it was generalized by R. Kannan ([3]), and in 1972 its generalization was given by S. K. Chatterjea ([5]).

In 1971, S. Reich ([10]) gave a new generalization of the Banach's fixed point principle as follows:

Theorem 1.1. *If the mapping $S:X\to X$ where (X,d) is a complete metric space, it satisfies the inequality*

$$d(Sx,Sy)\le ad(x,Sx)+bd(y,Sy)+cd(x,y), \quad (1.2)$$

where $a>0,b>0$ and $c>0$ are such that $a+b+c<1$ and $x,y\in X$, then S has a unique fixed point.

If S satisfies the condition (1.2), then for S we say that it is a Reich-type mapping.

Furthermore, in [9] S. Moradi and D. Alimohammadi generalize R. Kannan's result, using the sequentially convergent mappings, and in [1] several generalizations of Kannan and Chatterjea's theorems are proved, using the sequentially convergent mappings, which are defined as follows.

Keywords. Reich-type contractive mapping, complete metric space, sequentially convergent, subsequentially convergent mapping.

Definition 1.1. ([8]). *Let* (X,d) *be a metric space. A mapping* $T: X \to X$ *is said to be sequentially convergent if we have, for every sequence* $\{y_n\}$*, if* $\{Ty_n\}$ *is convergence then* $\{y_n\}$ *is also convergence. A mapping* T *is said to be sub-sequentially convergent if we have, for every sequence* $\{y_n\}$*, if* $\{Ty_n\}$ *is convergence then* $\{y_n\}$ *has a convergent subsequence.*

In [8] S. Moradi and A. Beiranvand introduce the concept of T_f contractive mapping, whereby they use the class Θ of continuous monotonically nondecreasing functions $f:[0,+\infty) \to [0,+\infty)$ such that $f^{-1}(0)=\{0\}$, which is defined as follows.

Definition 2 ([8]). *Let* (X,d) *be a metric space,* $S,T: X \to X$ *and* $f \in \Theta$*. A mapping* S *is said to be* T_f *– contraction if there exist* $\lambda \in (0,1)$ *such that*

$$f(d(TSx,TSy)) \le \lambda f(d(Tx,Ty)), \qquad (1.3)$$

for all $x,y \in X$.

Let us note here that, if $f \in \Theta$, then from $f^{-1}(0)=\{0\}$ it follows that $f(t)>0$, for each $t>0$. S. Moradi and A. Beiranvand prove that if S is T_f contractive mapping, then S has a unique fixed point. Then, in [2], M. Kir and H. Kiziltunc generalize the result of S. Moradi and A. Beiranvand for Kannan and Chatterjea type mappings. In [7] the results of Kir and Kiziltunc are generalized and their application is given, while in [6] a generalization of the Reich-type mapping using sequentially convergent mappings is given. In the following considerations we will give a generalization of the Reich-type mapping using sub-sequentially convergent mappings, which were also introduced by S. Moradi and A. Beiranvand ([8]).

2. Mains results

Theorem 2.1. *Let* (X,d) *be a complete metric space* $S: X \to X$*,* $f \in \Theta$ *is such that* $f(p+q) \le f(p)+f(q)$*, for each* $p,q \in [0,+\infty)$ *and mapping* $T: X \to X$ *is continuous, injective and sub-sequentially convergent. If exist* $a>0, b>0$ *and* $c>0$ *such that* $a+b+c<1$ *and*

$$f(d(TSx,TSy)) \le af(d(Tx,TSx)) + bf(d(Ty,TSy)) + cf(d(Tx,Ty)) \qquad (2.4)$$

for each $x,y \in X$*, then* S *has a single fixed point and for each* $x_0 \in X$ *the sequence* $\{S^n x_0\}$ *converges to the fixed point.*

Proof. Let x_0 be an arbitrary point from X and let the sequence $\{x_n\}$ be determined by $x_{n+1}=Sx_n$, $n=0,1,2,3,...$. From inequality (4) it follows that

$$\begin{aligned} f(d(Tx_{n+1},Tx_n)) &= f(d(TSx_n,TSx_{n-1})) \\ &\le af(d(Tx_n,TSx_n)) + bf(d(Tx_{n-1},TSx_{n-1})) + cf(d(Tx_n,Tx_{n-1})) \\ &= af(d(Tx_n,Tx_{n+1})) + (b+c) f(d(Tx_{n-1},Tx_n)), \end{aligned}$$

i.e.

$$f(d(Tx_{n+1},Tx_n)) \le \tfrac{b+c}{1-a} f(d(Tx_n,Tx_{n-1})) .$$

According to that, for $\lambda = \tfrac{b+c}{1-a} < 1$ holds true

$$f(d(Tx_{n+1},Tx_n)) \le \lambda f(d(Tx_n,Tx_{n-1})) , \quad (2.5)$$

for each $n = 1,2,3,\dots$. From inequality (2.5) it follows that

$$f(d(Tx_{n+1},Tx_n)) \le \lambda^n f(d(Tx_1,Tx_0)) , \quad (2.6)$$

for each $n = 1,2,3,\dots$. Now, from inequality (2.6), the properties of the metric and the monotonicity and subadditivity of the function f, it follows that for each $m,n \in R$ $n > m$ holds true

$$\begin{aligned} f(d(Tx_n,Tx_m)) &\le f(\sum_{k=m}^{n-1} d(Tx_{k+1},Tx_k)) \\ &\le \sum_{k=m}^{n-1} f(d(Tx_{k+1},Tx_k)) \\ &\le \sum_{k=m}^{n-1} \lambda^k f(d(Tx_1,Tx_0)) \\ &< \tfrac{\lambda^m}{1-\lambda} f(d(Tx_1,Tx_0)). \end{aligned}$$

From the last inequality, it follows

$$\lim_{m,n\to\infty} f(d(Tx_n,Tx_m)) = 0 ,$$

and because $f \in \Theta$ we have $\lim_{m,n\to\infty} d(Tx_n,Tx_m) = 0$. According to that, $\{Tx_n\}$ is a Cauchy sequence. But X is a complete metric space, so therefore the sequence $\{Tx_n\}$ is convergent. Further, the mapping $T : X \to X$ is sub-sequentially convergent, therefore the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n(k)}\}$, i.e., there exists $u \in X$ such that $\lim_{n\to\infty} x_{n(k)} = u$. From the continuity of T follows $\lim_{n\to\infty} Tx_{n(k)} = Tu$. Now, the sequence $\{Tx_{n(k)}\}$ is a subsequence of the convergent sequence $\{Tx_n\}$, so the equation holds true

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Tx_{n(k)} = Tu .$$

We will prove that $u \in X$ is a fixed point for the mapping S. We have

$$\begin{aligned} f(d(TSu,Tx_{n+1})) &= f(d(TSu,TSx_n)) \\ &\le af(d(TSu,Tu)) + bf(d(TSx_n,Tx_n)) + cf(d(Tu,Tx_n)) \\ &= af(d(TSu,Tu)) + bf(d(Tx_{n+1},Tx_n)) + cf(d(Tu,Tx_n)). \end{aligned}$$

If in the last inequality we take $n \to \infty$, then from $\lim_{n\to\infty} Tx_n = Tu$ and the continuity of metric and function f follows the inequality

$$f(d(TSu,Tu)) \le \tfrac{b+c}{1-a} f(0) .$$

But, $0<\frac{b+c}{1-a}<1$ and $f^{-1}(0)=\{0\}$, so from the last inequality it follows that $d(TSu,Tu)=0$, i.e., $TSu=Tu$. Finally, T is injection and therefore $Su=u$, i.e., the mapping S has a fixed point.

Let $u,v\in X$ be two fixed points for S, i.e., $Su=u$ и $Sv=v$. From the inequality, (1.4) it follows that

$$\begin{aligned} f(d(Tu,Tv)) &= f(d(TSu,TSv)) \\ &\leq af(d(Tu,TSu))+bf(d(Tv,TSv))+cf(d(Tu,Tv)) \end{aligned}$$

i.e.

$$f(d(Tu,Tv))\leq \frac{a+b}{1-c}f(0),$$

So, similarly as above, we conclude that $d(Tu,Tv)=0$. Therefore, $Tu=Tv$. But T is an injection, and therefore $u=v$, i.e., S has a unique fixed point.

Finally, from the arbitrariness of the point x_0, it follows that for each $x_0\in X$ the sequence $\{S^n x_0\}$ converges to the fixed point. □

References

[1] A. Malčeski, S. Malcheski, K. Anevska, R. Malčeski, *New extension of Kannan and Chatterjea fixed point theorems on complete metric spaces*. British Journal of Mathematics & Computer Science. Vol. 17 No. 1 (2016),1-10.

[2] M. Kir, H. Kiziltunc, T_F *type contractive conditions for Kannan and Chatterjea fixed point theorems*, Adv. Fixed Point Theory, Vol. 4, No. 1 (2014), pp. 140-148

[3] R. Kannan, *Some results on fixed points,* Bull. Calc. Math. Soc. Vol. 60 No. 1, (1968), 71-77

[4] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations intégrales*, Fund. Math. 2 (1922), 133-181

[5] S. K. Chatterjea, *Fixed point theorems,* C. R. Acad. Bulgare Sci., Vol. 25 No. 6 (1972), 727-730

[6] S. Malcheski, R. Malcheski, *Generalization of a Reich-type contractive mapping in a complete metric space,* Proceedings of the CODEMA2022, Armaganka, Skopje, 2023

[7] S. Malcheski, R. Malcheski, *Three Theorems about Fixed Points for* T_f *Contraction in a Complete Metric Space*, Proceedings of the CODEMA2020, 13-21

[8] S. Moradi, A. Beiranvand, *Fixed Point of* T_F *-contractive Single-valued Mappings*, Iranian Journal of Mathematical Sciences and Informatics, Vol. 5, No. 2 (2010), pp 25-32

[9] S. Moradi, D. Alimohammadi, *New extensions of kannan fixed theorem on complete metric and generalized metric spaces*. Int. Journal of Math. Analysis. 2011;5(47):2313-2320.

[10] S. Reich, *Some remarks concerning contraction mappings*. Canad. Math. Bull. Vol. 14 (1), 1971:121-124.

Samoil Malcheski
International Slavic University, Sv. Nikole, R. North Macedonia
E-mail address: samoil.malcheski@gmail.com