GOCE DELCEV UNIVERSITY - STIP
FACULTY OF COMPUTER SCIENCE

The journal is indexed in
EBSCO

ISSN 2545-4803 on line
DOI: 10.46763/BJAMI

BALKAN JOURNAL OF APPLIED MATHEMATICS AND INFORMATICS (BJAMI)

YEAR 2024

VOLUME 7, Number 2

AIMS AND SCOPE:
BJAMI publishes original research articles in the areas of applied mathematics and informatics.

Topics:
1. Computer science;
2. Computer and software engineering;
3. Information technology;
4. Computer security;
5. Electrical engineering;
6. Telecommunication;
7. Mathematics and its applications;
8. Articles of interdisciplinary of computer and information sciences with education, economics, environmental, health, and engineering.

BALKAN JOURNAL
OF APPLIED MATHEMATICS AND INFORMATICS
(BJAMI), Vol 7

ISSN 2545-4803 on line
Vol. 7, No. 2, Year 2024

EDITORIAL BOARD

CONTENT

Balkan Journal of Applied Mathematics and Informatics Online ISSN 2545-4803
Volume 7 Number 2 Year 2024 UDC: 517.925.4:519.612

ON THE INTERGRABILITY OF A SUBCLASS OF 2D MATRIX DIFFERENTIAL EQUATIONS

BILJANA ZLATANOVSKA AND BORO M. PIPEREVSKI

Abstract. In this paper, the 2D matrix differential equations are considered. Under certain conditions, using the Rodrigues' formula for these 2D matrix differential equations, a particular solution is obtained. Finally, this theory is supported by examples.

Dedicated to the Day of Differential Equations in Macedonia 2024

1. Introduction

The subclass a 2D matrix differential equation of the form

$$P \cdot X' + M \cdot X = O \tag{1.1}$$

is considered, where

$$P = \begin{bmatrix} t-a & 0 \\ 0 & t-b \end{bmatrix}, M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, X' = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

$X(t)$ is matrix function, $A, B, C, D, a, b \in \mathbb{R}$ and $x_1(t)$, $x_2(t)$ are real functions of one real variable t by first derivate $x'_1(t)$, $x'_2(t)$.

In [1,2,3], the following theorem is proved:

Theorem 1.1. The 2D matrix differential equation (1.1) with the condition $A \cdot B \cdot C \cdot D \cdot (A \cdot D - B \cdot C) \neq 0$ has the polynomial solution of a degree n and no other polynomial solution of degree less that n if and only if there exists a natural number n such that

$$r(M + nE) = 1, r(M + mE) = 2, E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, m < n, m - \text{natural number},$$

i.e., n is the root of the quadratic equation

$$(A + k)(D + k) - BC = 0 \tag{1.2}$$

In the case where two natural numbers are the roots of the equation (1.2), then the number n is the smaller one. The solution is given by the formula

Keywords. 2D matrix differential equations, Rodrigues' formula, integrability .

$$X = \begin{bmatrix} (t-a)^{-A}(t-b)^{1-D}[(t-a)^{A+n}(t-b)^{D+n-1}]^{(n)} \\ -\dfrac{A+n}{B}(t-a)^{1-A}(t-b)^{-D}[(t-a)^{A+n-1}(t-b)^{D+n}]^{(n)} \end{bmatrix}$$

Let the differential equation of the form

$$(a_2t^2 + a_1t + a_0)\, x'' + (b_1t + b_0)\, x' + c_0\, x = 0,$$

be given and let a and b be different real roots of the quadratic equation $a_2t^2 + a_1t + a_0 = 0,\ a_2 \neq 0$. Let this differential equation be written in the form

$$(t-a)(t-b)\, x'' + [(p+q)t - (aq+bp)]\, x' + r\, x = 0$$

i.e.,

$$x'' + \left(\frac{p}{t-a} + \frac{q}{t-b}\right) x' + \frac{r}{(t-a)(t-b)}\, x = 0, \tag{1.3}$$

where

$$p = \frac{b_1\, a + b_0}{a-b},\ q = \frac{b_1\, b + b_0}{b-a},\ r = c_0.$$

Using the substitutions

$$\begin{aligned} x &= (t-a)^{1-p}\, z_1, \\ x &= (t-b)^{1-q}\, z_2, \\ x &= (t-a)^{1-p}\, (t-b)^{1-q}\, z_3, \end{aligned} \tag{1.4}$$

the equation (1.3) transforms into at most three other equations of the same type [4, 5, 6], which are given with the differential equations

$$(t-a)(t-b)\, z_1'' + [(2-p+q)t - (aq + b(2-p))]\, z_1' + (r - pq + q)\, z_1 = 0,$$

$$(t-a)(t-b)\, z_2'' + [(2+p-q)t - (a(2-q) + b(2-p))]\, z_2' + (r - pq + p)\, z_2 = 0,$$

$$(t-a)(t-b)\, z_3'' + [(4-p-q)t - (a(2-q) + b(2-p))]\, z_3' + (r - p - q + 2)\, z_3 = 0,$$

i.e., with the differential equations

$$\begin{aligned} z_1'' + \left(\frac{2-p}{t-a} + \frac{q}{t-b}\right) z_1' + \frac{r - pq + q}{(t-a)(t-b)}\, z_1 &= 0, \\ z_2'' + \left(\frac{p}{t-a} + \frac{2-q}{t-b}\right) z_2' + \frac{r - pq + p}{(t-a)(t-b)}\, z_2 &= 0, \\ z_3'' + \left(\frac{2-p}{t-a} + \frac{2-q}{t-b}\right) z_3' + \frac{r - p - q + 2}{(t-a)(t-b)}\, z_3 &= 0. \end{aligned} \tag{1.5}$$

The condition for the characteristic equation

$$k^2 + (p+q-1)\, k + r = 0, \tag{1.6}$$

of the differential equation (1.3) to have a natural root n, where n is the smaller if there are two natural number roots, is a necessary and sufficient condition for the differential equation (1.3) to have a particular polynomial solution of degree n.

This condition for a polynomial solution applied to the differential equation (1.3) and the differential equations (1.5) will be given accordingly by the relations

$$\begin{aligned} &n^2+(p+q-1)n+r=0,\\ &n^2+(q-p+1)n+r-pq+q=0,\\ &n^2+(p-q+1)n+r-pq+p=0,\\ &n^2+(3-p-q)n+r-p-q+2=0, \end{aligned} \tag{1.7}$$

where in each of the relations, n is a natural number (the smaller one if the relation is satisfied for two natural numbers). From these relations, the conditions of the following theorem, proved in [6] are obtained.

Theorem 1.2. Let the differential equation (1.3) be given and let $(p+q-1)^2-4r\geq 0$. Let k be the root of the characteristic equation (1.6). If the root k satisfies one of the following conditions

1^0 $k\in\mathbb{N}$, the smaller if there are two natural numbers,

2^0 $k+p-1\in\mathbb{N}$ or $-(k+q)\in\mathbb{N}$, the smaller if there are two natural numbers,

3^0 $k+q-1\in\mathbb{N}$ or $-(k+p)\in\mathbb{N}$, the smaller if there are two natural numbers,

4^0 $k+p+q-2\in\mathbb{N}$ or $-(k+1)\in\mathbb{N}$, the smaller if there are two natural numbers,

then the differential equation (1.3) is integrable into closed form.

Remark 1.1. The polynomial solution of the differential equation (1.3) is given by the Rodrigues' formula

$$x=(t-a)^{1-p}(t-b)^{1-q}[(t-a)^{n+p-1}(t-b)^{n+q-1}]^{(n)}$$

2. Main results

Let the 2D matrix differential equation (1.1) be given. Let the conditions $A\cdot B\cdot C\cdot D\cdot(A\cdot D-B\cdot C)\neq 0, C=1$ be satisfied. The corresponding differential equations are

$$(t-a)(t-b)x_1''+[(A+D+1)t-(A+1)b-Da]x_1'+(AD-B)x_1=0, \tag{2.1}$$

$$(t-a)(t-b)x_2''+[(A+D+1)t-(D+1)a-Ab]x_2'+(AD-B)x_2=0. \tag{2.2}$$

The differential equation (2.1) is equivalent to a differential equation of the form

$$x_1''+\left(\frac{p}{t-a}+\frac{q}{t-b}\right)x_1'+\frac{r}{(t-a)(t-b)}x_1=0,$$

if the following relations

$$D = q, A = p-1, B = (p-1)q - r,$$
$$p = A+1, q = D, r = AD - B$$

are satisfied. Therefore,

$$x_1'' + (\frac{A+1}{t-a} + \frac{D}{t-b})x_1' + \frac{AD-B}{(t-a)(t-b)}x_1 = 0. \qquad (2.1^*)$$

The differential equation (2.2) is equivalent to a differential equation of the form

$$x_2'' + (\frac{p^*}{t-a} + \frac{q^*}{t-b})x_2' + \frac{r^*}{(t-a)(t-b)}x_2 = 0,$$

if the following relations

$$p^* = A, q^* = D+1, r^* = AD - B$$

are satisfied. Therefore,

$$x_2'' + (\frac{A}{t-a} + \frac{D+1}{t-b})x_2' + \frac{AD-B}{(t-a)(t-b)}x_2 = 0. \qquad (2.2^*)$$

Theorem 2.1. Let the 2D matrix differential equation (1.1) be given and let $A \cdot B \cdot C \cdot D \cdot (A \cdot D - B \cdot C) \neq 0, C = 1, (A-D)^2 + 4B \geq 0$. Let $k \in \mathbb{R}$ be a root of the characteristic equation (1.2). If the root k satisfies one of the following conditions

1^0 $k \in \mathbb{N}$, the smaller if there are two natural numbers,

2^0 $k + A \in \mathbb{N}$ or $-(k+D) \in \mathbb{N}$, the smaller if there are two natural numbers,

3^0 $k + D - 1 \in \mathbb{N}$ or $-(k + A + 1) \in \mathbb{N}$, the smaller if there are two natural numbers,

4^0 $k + A + D - 1 \in \mathbb{N}$ or $-(k+1) \in \mathbb{N}$, the smaller if there are two natural numbers,

then the 2D matrix differential equation (1.1) is integrable into closed form.

Proof. The conditions for integrability of the 2D matrix differential equation (1.1) are derived from the conditions for integrability of the differential equation (1.3) given by Theorem 1.2 from the introduction, through the differential equation (2.1). ■

Theorem 2.2. Let the 2D matrix differential equation (1.1) be given. Let the conditions of Theorem 2.1 be satisfied.

- Let n_0 denote the natural number if condition 1^0 is satisfied;
- Let n_1 denote the natural number if condition 2^0 is satisfied;
- Let n_2 denote the natural number if condition 3^0 is satisfied;
- Let n_3 denote the natural number if condition 4^0 is satisfied.

Then the particular solution of the differential equation (2.1), as well as the first component function $x_1(t)$ of the 2D matrix differential equation (1.1) is given by the formulas

$$x_1 = (t-b)^{1-D}[(t-a)^{n_1-A}(t-b)^{n_1+D-1}]^{(n_1)},$$
$$x_1 = (t-a)^{-A}[(t-a)^{n_2+A}(t-b)^{n_2+1-D}]^{(n_2)}, \tag{2.3}$$
$$x_1 = [(t-a)^{n_3-A}(t-b)^{n_3+1-D}]^{(n_3)}.$$

In the first condition 1^0, the differential equation (2.1) has a polynomial solution, which is given by Rodrigues' formula

$$x_1 = (t-a)^{-A}(t-b)^{1-D}[(t-a)^{n_0+A}(t-b)^{n_0+D-1}]^{(n_0)}.$$

This polynomial solution of the 2D matrix differential equation (1.1) is given in Theorem 1.1 from the introduction.

Proof. Using the substitutions

$$x_1 = (t-a)^{-A} z_1,$$
$$x_1 = (t-b)^{1-D} z_2, \tag{2.4}$$
$$x_1 = (t-a)^{-A}(t-b)^{1-D} z_3$$

the differential equation (2.1) transforms into the differential equations

$$(t-a)(t-b)z_1'' + [(-A+D+1)t-(-A+1)b-Da]z_1' - Bz_1 = 0,$$
$$(t-a)(t-b)z_2'' + [(A-D+3)t-(A+1)b+Da-2a]z_2' + (A-D-B+1)z_2 = 0,$$
$$(t-a)(t-b)z_3'' + [(-A-D+3)t+(A-1)b+Da-2a]z_3' + (AD-A-B-D+1)z_3 = 0,$$

i.e., into to the differential equations

$$z_1'' + \left(\frac{1-A}{t-a}+\frac{D}{t-b}\right)z_1' - \frac{B}{(t-a)(t-b)}z_1 = 0,$$
$$z_2'' + \left(\frac{1+A}{t-a}+\frac{2-D}{t-b}\right)z_2' + \frac{A-B-D+1}{(t-a)(t-b)}z_2 = 0,$$
$$z_3'' + \left(\frac{1-A}{t-a}+\frac{2-D}{t-b}\right)z_3' + \frac{AD-A-B-D+1}{(t-a)(t-b)}z_3 = 0$$

where

$$p_1 = 1-A, q_1 = D, p_2 = 1+A, q_2 = 2-D, p_3 = 1-A, q_3 = 2-D,$$
$$r_1 = -B, r_2 = A-B-D+1, r_3 = AD-A-B-D+1.$$

Using Rodrigues' formula for each of the differential equations, the corresponding formulas

$$z_1 = (t-a)^{A}(t-b)^{1-D}[(t-a)^{n_1-A}(t-b)^{n_1+D-1}]^{(n_1)},$$
$$z_2 = (t-a)^{-A}(t-b)^{D-1}[(t-a)^{n_2+A}(t-b)^{n_2+1-D}]^{(n_2)},$$
$$z_3 = (t-a)^{A}(t-b)^{D-1}[(t-a)^{n_3-A}(t-b)^{n_3+1-D}]^{(n_3)}$$

are obtained.

Using the substitutions (2.4), we get the formulas

$$x_1 = (t-a)^{-A} z_1,\ \ x_1 = (t-b)^{1-D} z_2,\ x_1 = (t-a)^{-A}(t-b)^{1-D} z_3$$

which correspond to the formulas (2.3).
In the first condition 1^0, the differential equation (2.1) has a polynomial solution

$$x_1 = (t-a)^{-A}(t-b)^{1-D}[(t-a)^{n_0+A}(t-b)^{n_0+D-1}]^{(n_0)}.$$

which is given by using Rodrigues' formula. This polynomial solution of the 2D matrix differential equation is given by the formula in Theorem 1.1 from the introduction. ■

The formula for the second component function of the 2D matrix differential equation (1.1) can be obtained from the differential equation

$$(t-a)\,x_1' + Ax_1 + Bx_2 = 0,\ x_2 = -\frac{1}{B}[(t-a)\,x_1' + Ax_1].$$

using the differential equation (2.2) and the same procedure, the following theorem is obtained:

Theorem 2.3. Let the 2D matrix differential equation (1.1) be given and let $A\cdot B\cdot C\cdot D\cdot(A\cdot D - B\cdot C)\neq 0, C=1, (A-D)^2+4B\geq 0$. Let $k\in\mathbb{R}$ be the root of the characteristic equation (1.2). If the root k satisfies one of the following conditions

1^0 $k\in\mathbb{N}$, the smaller if there are two natural numbers,

2^0 $k+A-1\in\mathbb{N}$ or $-(k+D+1)\in\mathbb{N}$, the smaller if there are two natural numbers,

3^0 $k+D\in\mathbb{N}$ or $-(k+A)\in\mathbb{N}$, the smaller if there are two natural numbers,

4^0 $k+A+D-1\in\mathbb{N}$ or $-(k+1)\in\mathbb{N}$, the smaller if there are two natural numbers,

then the 2D matrix differential equation (1.1) is integrable into closed form.

Proof. The conditions for the integrability of the 2D matrix differential equation (1.1) are transferred from the conditions for integrability of the differential equation (1.3) given by Theorem 1.2 of the introduction, through the differential equation (2.2). ■

Theorem 2.4. Let the 2D matrix differential equation (1.1) be given. Let the conditions of Theorem 2.3 be satisfied.

- Let n_0^* denote the natural number if condition 1^0 is satisfied.
- Let n_1^* denote the natural number if condition 2^0 is satisfied.
- Let n_2^* denote the natural number if condition 3^0 is satisfied.
- Let n_3^* denote the natural number if condition 4^0 is satisfied.

Then the particular solution of the differential equation (2.2), as well as the second component function $x_2(t)$ of the 2D matrix differential equation (1.1) is given by the formulas

$$x_2 = (t-b)^{-D}[(t-a)^{n_1^*+1-A}(t-b)^{n_1^*+D}]^{(n_1^*)},$$
$$x_2 = (t-a)^{1-A}[(t-a)^{n_2^*+A-1}(t-b)^{n_2^*-D}]^{(n_2^*)}, \qquad (2.5)$$
$$x_2 = [(t-a)^{n_3^*+1-A}(t-b)^{n_3^*-D}]^{(n_3^*)}.$$

In the first condition 1^0, the differential equation (2.2) has a polynomial solution, which is given by Rodrigues' formula

$$x_2 = (t-a)^{1-A}(t-b)^{-D}[(t-a)^{A+n_0^*-1}(t-b)^{D+n_0^*}]^{(n_0^*)}.$$

This polynomial solution of the 2D matrix differential equation (1.1) is given by the formula in Theorem 1.1 from the introduction.

Proof. Using the substitutions

$$x_2 = (t-a)^{1-A} z_1^*,$$
$$x_2 = (t-b)^{-D} z_2^*, \qquad (2.6)$$
$$x_2 = (t-a)^{1-A}(t-b)^{-D} z_3^*$$

the differential equation (2.2) transforms into the differential equations

$$(t-a)(t-b)z_1^{*\prime\prime} + [(-A+D+3)t - (-A+2)b - Da - a]z_1^{*\prime} + (D-A-B+1)z_1^* = 0,$$
$$(t-a)(t-b)z_2^{*\prime\prime} + [(A-D+1)t - Ab + Da - a]z_2^{*\prime} - Bz_2^* = 0,$$
$$(t-a)(t-b)z_3^{*\prime\prime} + [(-A-D+3)t + Ab + Da - a - 2b]z_3^{*\prime} + (AD-A-B-D+1)z_3^* = 0,$$

i.e., into the differential equations

$$z_1^{*\prime\prime} + \left(\frac{2-A}{t-a} + \frac{D+1}{t-b}\right)z_1^{*\prime} + \frac{D-A-B+1}{(t-a)(t-b)}z_1^* = 0,$$
$$z_2^{*\prime\prime} + \left(\frac{A}{t-a} + \frac{1-D}{t-b}\right)z_2^{*\prime} - \frac{B}{(t-a)(t-b)}z_2^* = 0,$$
$$z_3^{*\prime\prime} + \left(\frac{2-A}{t-a} + \frac{1-D}{t-b}\right)z_3^{*\prime} + \frac{AD-A-B-D+1}{(t-a)(t-b)}z_3^* = 0$$

where

$$p_1{*} = 2-A,\, q_1{*} = D+1,\, p_2{*} = A,\, q_2{*} = 1-D,\, p_3{*} = 2-A,\, q_3{*} = 1-D,$$
$$r_1{*} = D-A-B+1,\, r_2{*} = -B,\, r_3{*} = AD-A-B-D+1.$$

Using Rodrigues' formula for each of the differential equations, the corresponding formulas

$$z_1^* = (t-a)^{A-1}(t-b)^{-D}[(t-a)^{n_1^*+1-A}(t-b)^{n_1^*+D}]^{(n_1^*)},$$

$$z_2^* = (t-a)^{1-A}(t-b)^{D}[(t-a)^{n_2^*+A-1}(t-b)^{n_2^*-D}]^{(n_2^*)},$$

$$z_3^* = (t-a)^{A-1}(t-b)^{D}[(t-a)^{n_3^*+1-A}(t-b)^{n_3^*-D}]^{(n_3^*)}$$

are obtained.
Using the substitutions (2.6), we get the formulas

$$x_2 = (t-a)^{1-A} z_1^*,\ x_2 = (t-b)^{-D} z_2^*,\quad x_2 = (t-a)^{1-A}(t-b)^{-D} z_3^*,$$

which correspond to the formulas (2.5).
In the first condition 1^0, the differential equation (2.2) has a polynomial solution

$$x_2 = (t-a)^{1-A}(t-b)^{-D}[(t-a)^{A+n_0^*-1}(t-b)^{D+n_0^*}]^{(n_0^*)}$$

which is given by using Rodrigues' formula. This polynomial solution of the 2D matrix differential equation (1.1) is given by the formula of Theorem 1.1 from the introduction.■

Remark 2.1. In accordance with Theorem 2.2 and Theorem 2.4, it can be concluded that there are connections

$$n_0^* = n_0,\ \ n_1^* = n_1 - 1,\ \ n_2^* = n_2 + 1,\ \ n_3^* = n_3.$$

Remark 2.2. The results for the integrability of the 2D matrix differential equation (1.1) can also be transformed to the system of first-order differential equations of the form

$$(t-a)\, x_1' + A\, x_1 + B\, x_2 = 0,$$

$$(t-b)\, x_2' + x_1 + D\, x_2 = 0.$$

3. Examples

The results obtained will be shown via examples.

Example 3.1. Let the 2D matrix differential equation

$$\begin{bmatrix} t+1 & 0 \\ 0 & t-3 \end{bmatrix}\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

be given, where $A = 1, B = -2, C = 1, D = 4, a = -1, b = 3.$
The corresponding differential equations for the component functions are

$$x_1'' + (\frac{2}{t+1} + \frac{4}{t-3})\, x_1' + \frac{6}{(t+1)(t-3)}\, x_1 = 0 \tag{3.1}$$

$$x_2'' + (\frac{1}{t+1} + \frac{5}{t-3})\, x_2' + \frac{6}{(t+1)(t-3)}\, x_2 = 0. \tag{3.2}$$

The conditions of Theorem 2.1 and Theorem 2.3

$$A \cdot B \cdot C \cdot D \cdot (A \cdot D - B \cdot C) = 1 \cdot (-2) \cdot 1 \cdot 4 \cdot (1 \cdot 4 - (-2) \cdot 1) = -48 \neq 0,$$
$$C = 1,$$
$$(A - D)^2 + 4B = (1-4)^2 + 4 \cdot (-2) = 1 \geq 0$$

are satisfied. The roots of the characteristic equation (1.2) are real numbers $k_1 = -2$ and $k_2 = -3$.

According to Theorem 2.1, the conditions for integrability are determined from the table

1^0 $-2, -3 \notin \mathbb{N}$;

2^0 $-2+1 = -1 \notin \mathbb{N}, -3+1 = -2 \notin \mathbb{N}$ or $-(-2+4) = -2 \notin \mathbb{N}, -(-3+4) = -1 \notin \mathbb{N}$;

3^0 $-2+4-1 = 1 \in \mathbb{N}, -3+4-1 = 0 \in \mathbb{N}$ or $-(-2+1+1) = 0 \in \mathbb{N}, -(-3+1+1) = 1 \in \mathbb{N}$;

4^0 $-2+1+4-1 = 2 \in \mathbb{N}, -3+1+4-1 = 1 \in \mathbb{N}$ or $-(-2+1) = 1 \in \mathbb{N}, -(-3+1) = 2 \in \mathbb{N}$.

According to the conditions of Theorem 2.1 for the existence of the required natural number (the smallest if there are two), we get $n_2 = 0$ and $n_3 = 1$. For $n_3 = 1$, using the substitution

$$x_1 = (t+1)^{-1}(t-3)^{-3} z_3 \tag{3.3}$$

in the equation (3.1), we obtain the equation

$$z_3'' + \left(\frac{0}{t+1} + \frac{-2}{t-3}\right) z_3' + \frac{2}{(t+1)(t-3)} z_3 = 0.$$

According to the formula from Theorem 2.2, the particular solution is

$$z_3 = -2(t+1).$$

By substitution (3.3), the particular solution of the equation (3.1) is

$$x_1 = -2(t-3)^{-3}.$$

According to Theorem 2.3, the conditions for integrability are determined from the table

1^0 $-2, -3 \notin \mathbb{N}$;

2^0 $-2+1-1 = -2 \notin \mathbb{N}, -3+1-1 = -3 \notin \mathbb{N}$ or $-(-2+5) = -3 \notin \mathbb{N}, -(-3+5) = -2 \notin \mathbb{N}$;

3^0 $-2+4 = 2 \in \mathbb{N}, -3+4 = 1 \in \mathbb{N}$ or $-(-2+1) = 1 \in \mathbb{N}, -(-3+1) = 2 \in \mathbb{N}$;

4^0 $-2+4 = 2 \in \mathbb{N}, -3+4 = 1 \in \mathbb{N}$ or $-(-2+1) = 1 \in \mathbb{N}, -(-3+1) = 2 \in \mathbb{N}$.

According to the conditions of Theorem 2.3 for the existence of the required natural number (the smallest if there are two), we get $n_2^* = 1$ and $n_3^* = 1$. For $n_3^* = 1$, using the substitution

$$x_2 = (t-3)^{-4} z_3^* \tag{3.4}$$

in the equation (3.2), we obtain the equation

$$z_3^{*\prime\prime} + \left(\frac{1}{t+1} + \frac{-3}{t-3}\right) z_3^{*\prime} + \frac{2}{(t+1)(t-3)} z_3^* = 0.$$

According to the formula from Theorem 2.4, the particular solution is

$$z_3^* = -2t - 6.$$

By substitution (3.4), the particular solution of the equation (3.2) is

$$x_2 = -2(t-3)^{-4}(t+3).$$

Finally, the particular solution of the 2D matrix differential equation is

$$X_p = \begin{bmatrix} -2(t-3)^{-3} \\ -2(t-3)^{-4}(t+3) \end{bmatrix}.$$

Example 3.2. Let the 2D matrix differential equation

$$\begin{bmatrix} t-1 & 0 \\ 0 & t-2 \end{bmatrix}\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} + \begin{bmatrix} \frac{5}{2} & -3 \\ 1 & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

be given, where $A = \frac{5}{2}, B = -3, C = 1, D = -\frac{3}{2}, a = 1, b = 2.$

The corresponding differential equations for the component functions are

$$x_1'' + \left(\frac{\frac{7}{2}}{t+1} + \frac{-\frac{3}{2}}{t-3}\right) x_1' + \frac{-\frac{3}{4}}{(t+1)(t-3)} x_1 = 0 \tag{3.5}$$

$$x_2'' + \left(\frac{\frac{5}{2}}{t+1} + \frac{-\frac{1}{2}}{t-3}\right) x_2' + \frac{-\frac{3}{4}}{(t+1)(t-3)} x_2 = 0. \tag{3.6}$$

The conditions of Theorem 2.1 and Theorem 2.3

$$A \cdot B \cdot C \cdot D \cdot (A \cdot D - B \cdot C) = \frac{5}{2} \cdot (-3) \cdot 1 \cdot (-\frac{3}{2}) \cdot (\frac{5}{2} \cdot (-\frac{3}{2}) - (-3) \cdot 1) = -48 \neq 0,$$

$$C = 1,$$

$$(A-D)^2 + 4B = (\frac{5}{2} + \frac{3}{2})^2 + 4 \cdot (-3) = 4 \geq 0$$

are satisfied. The roots of the characteristic equation (1.2) are real numbers $k_1 = \frac{1}{2}$ and $k_2 = -\frac{3}{2}$.

According to Theorem 2.1, the conditions for integrability are determined from the table

1^0 $\frac{1}{2}, -\frac{3}{2} \notin \mathbb{N}$;

2^0 $3, 1 \in \mathbb{N}$ or $1, 3 \in \mathbb{N}$;

3^0 $-2, -4 \notin \mathbb{N}$ or $-4, -2 \notin \mathbb{N}$;

4^0 $\frac{1}{2}, -\frac{3}{2} \notin \mathbb{N}$ or $-\frac{3}{2}, \frac{1}{2} \notin \mathbb{N}$.

According to the conditions of Theorem 2.1 for the existence of the required natural number (the smallest if there are two), we get $n_1 = 1$. For $n_1 = 1$, using the substitution

$$x_1 = (t-1)^{-\frac{5}{2}} z_1 \tag{3.7}$$

in the equation (3.5), we obtain the equation

$$z_1'' + \left(\frac{-\frac{3}{2}}{t-1} + \frac{-\frac{3}{2}}{t-2}\right) z_1' + \frac{3}{(t-1)(t-2)} z_1 = 0.$$

According to the formula from Theorem 2.2, the particular solution is

$$z_1 = -\frac{3}{2}(2t-3).$$

By substitution (3.7), the particular solution of the equation (3.5) is

$$x_1 = -\frac{3}{2}(t-1)^{-\frac{5}{2}}(2t-3).$$

According to Theorem 2.3, the conditions for integrability are determined from the table

1^0 $\frac{1}{2}, -\frac{3}{2} \notin \mathbb{N}$;

2^0 $2, 0 \in \mathbb{N}$ or $0, 2 \in \mathbb{N}$;

3^0 $-1, -3 \notin \mathbb{N}$ or $-3, -1 \notin \mathbb{N}$;

4^0 $\frac{1}{2}, -\frac{3}{2} \notin \mathbb{N}$ or $-\frac{3}{2}, \frac{1}{2} \notin \mathbb{N}$.

According to the conditions of Theorem 2.3 for the existence of the required natural number (the smallest if there are two), we get $n_1^* = 0$. For $n_1^* = 0$, using the substitution

$$x_2 = (t-1)^{-\frac{3}{2}} z_1^* \tag{3.8}$$

in the equation (3.6), we obtain the equation

$$z_1^{*\prime\prime} + \left(\frac{-\frac{1}{2}}{t-1} + \frac{-\frac{1}{2}}{t-2}\right) z_1^{*\prime} + \frac{0}{(t-1)(t-2)} z_1^* = 0.$$

According to the formula from Theorem 2.4, the particular solution is

$$z_1^* = 1.$$

By substitution (3.8), the particular solution of the equation (3.6) is

$$x_2 = (t-1)^{-\frac{3}{2}}.$$

Finally, the particular solution of the 2D matrix differential equation is

$$X_p = \begin{bmatrix} -\frac{3}{2}(t-1)^{-\frac{5}{2}}(2t-3) \\ (t-1)^{-\frac{3}{2}} \end{bmatrix}.$$

References

[1] *Piperevski, M. Boro* (2008) On a class of matrix differential equations whose general solution is polynomial, Bulletin mathematique de la societe des mathematiciens de la Republique de Macedoine, tome 32 (LVIII), (57-64)

[2] *Piperevski, M. Boro* (2001) On existence and construction of a polynomial solution of a class of second order differential equations' systems, Bulletin mathematique de la societe des mathematiciens de la Republique de Macedoine, tome 25, (37-42)

[3] *Piperevski, M. Boro* (2001) On existence and construction of a general polynomial solution of a class of second order differential equations' systems, Bulletin mathematique de la societe des mathematiciens de la Republique de Macedoine, tome 25, (43-48)

[4] *Piperevski, M. Boro* (1990) One transformation of a class of linear differential equations of the second order, Department of Electrical Engineering, Skopje, tome 6-7, (27-34)

[5] *Piperevski, M. Boro* (1997) ON THE EXISTENCE AND CONSTRUCTION OF A RACIONAL SOLUTIONS OF A CLASS OF LINEAR DIFFERENTIAL EQUATION OF THE SECOND ORDER WITH POLYNOMIAL COEFFICIENTS, Bulletin mathematique de la societe des mathematiciens de la Republique de Macedoine, tome 21, (21-26)

[6] *Piperevski, M. Boro, Zlatanovska B.* (2020) For a correlation between a class of second order linear differential equations and a class of systems of first order differential equations, In: E-Proceedings Paper. I Congress of Differential Equations, Mathematical Analysis, IX Seminar of Differential equations and Analysis, Codema 2020, (5-12).

[7] *Piperevski, M. Boro, Zlatanovska B.* (2020) ABOUT ONE B.S. POPOV'S RESULT, BALKAN JOURNAL OF APPLIED MATHEMATICS AND INFORMATICS (BJAMI), Vol 3, ISSN 2545-4803 online, Vol. 3, No. 2, (15-24)

Biljana Zlatanovska
Goce Delcev University,
Faculty of Computer Sciences,
"Krste Misirkov" No.10-A, Stip,
R. North Macedonia
E-mail: biljana.zlatanovska@ugd.edu.mk

Boro M. Piperevski
Ss Cyril and Methodius
University
Faculty of Electrical
Engineering and Information
Technologies,
Karpos bb, P.O. Box 574, 1000
Skopje,
R. North Macedonia
Email:borom@feit.ukim.edu.mk