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CONTENT

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GENERALIZATION OF APPLICATION OF FUNDAMENTAL LEMMA OF VARIATIONAL CALCULUS

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Abstract. Variational calculus studied methods for finding maximum and minimum values of a functional. It has its inception in 1696 year by Johan Bernoulli with its glorious problem for the brachistochrone: to find a curve connecting two points, A and B, which does not lie vertically, such that heavy point descending along this curve from position A to position B takes the least time. In functional analysis, variational calculus occupies the same space as the theory of maxima and minima in classical analysis. In this paper we will prove a theorem for a functional where it proves that the necessary condition for the extreme of a functional is that the variation of a functional is equal to zero.

1. Introduction

Many problems in mathematics are naturally formulated in terms of identifying a function that minimizes some quantity of interest. A natural example from geometry is the seemingly simple question: which is the shortest length path between two points in R^n? While everyone knows that such a path is the straight-line segment connecting the two points, proving that is more subtle. A more complex example of a question in the same vein is to ask: given some open set Ω, and some boundary conditions, can we identify a surface defined on this set, satisfying the boundary conditions, that have the minimum possible area?

The setting of the calculus of variations is over functionals on general normed vector spaces, specifically vector spaces of functions, the methods of results of the calculus of variations are remarkably simple and powerful and bear a great deal of resemblance to the machinery of finite-dimensional real analysis.

The Euler-Lagrange equations are a very useful result in variational analysis, since many naturally occurring problems in mathematics, physics and other domains of application can be formulated in terms of minimizing or maximizing an integral on a given domain. [1]

2. Derivation and proving of theorems

We will explore for the extreme of the functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))\ dx, \ (1)$$

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With the limit points of the allowable set of curves: $y(x_0) = y_0$ and $y(x_1) = y_1$ we will assume that the function $F(x, y, y')$ is three times differentiable. We know that the necessary condition for the extreme is the variation in the functional to be equal to zero. We will now show how the main theorem is applied to the given functional (1).

Let us assume that the extreme is reached on two times differentiable curve $y = y(x)$ (it is necessary to have a first-order derivative of the other curves, otherwise it may be the curve on which the extremum is reached). We are taking some limit curves $y = \overline{y}(x)$ close to $y = y(x)$ and include curves $y = y(x)$ and $y = \overline{y}(x)$ to the family curves with one parameter

$$y(x,\alpha) = y(x) + \alpha(\overline{y}(x) - y(x)) \quad ([1]).$$

When $\alpha = 0$, we receive the curve $y = y(x)$, when $\alpha = 1$, we receive $y = \overline{y}(x)$.

As we already know, the difference $\overline{y}(x) - y(x)$ is called the variation of the function $y(x)$ and it is noted with the δy.

The variation δy in variational problems plays a role analogous to the role of the increase Δx of an independent variable x in problems for the study of extreme of function $f(x)$. The variation of the function $\delta y = \overline{y}(x) - y(x)$ is a function of the x.

This function can be differentiated one or several times, as $(\delta y)' = \overline{y}'(x) - y'(x) = \delta y'$ generated of the variance is equal to the variance of the generated, and similarly

$$(\delta y)'' = \overline{y}''(x) - y''(x) = \delta y'',$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(\delta y)^{(k)} = \overline{y}^{(k)}(x) - y^{(k)}(x) = \delta y^{(k)}.$$

Hence, we analyze the family $y = y(x,\alpha)$, where $y(x,\alpha) = y(x) + \alpha\delta y$, containing the $\alpha = 0$ curves, whose extreme is reached, and in some $\alpha = 1$ approximately close or curves called comparison curves.

If we look at the values of the functional (1), only at the family curves $y = y(x,\alpha)$ the functional turned into the function of α :

$$v[y(x,\alpha)] = \varphi(\alpha),$$

As in the case that we consider $v[y(x,\alpha)]$ is functional depending on the parameter, the value of the parameter α determines the curve of the family $y = y(x,\alpha)$ and so determined also the value of the functional $v[y(x,\alpha)]$.

Theorem 1.

If the functional $v(y) = \int_{x_0}^{x_1} F(x, y, y')dx$ has a local extreme in y , the necessary condition for the extreme of the functional is ([3])

$$\int_{x_0}^{x_1} [F_y - \frac{d}{dx} F_{y'}]\delta y \, dx = 0, \qquad (2)$$

Proof of theorem 1 .

We analyze the function $\varphi(\alpha)$. It reaches its extreme at $\alpha = 0$, and when $\alpha = 0$, we receive $y = y(x)$, and the functional, in assumption, reaches the extreme compared with any permissible curve, and in particular, in terms of the nearly families curves $y = y(x,\alpha)$.

The necessary condition for the extreme of the function $\varphi(\alpha)$ at $\alpha = 0$, as is known, is its derivative equal to zero at $\alpha = 0$ ([3]), i.e.

$\varphi'(0) = 0$.

Since

$$\varphi(\alpha) = \int_{x_0}^{x_1} F(x, y(x,\alpha), y_x'(x,\alpha)) \, dx,$$

it

$$\varphi'(\alpha) = \int_{x_0}^{x_1} \left[F_y' \frac{\partial}{\partial\alpha} y(x,\alpha) + F_{y'}' \frac{\partial}{\partial\alpha} y'(x,\alpha)) \right] dx,$$

where

$$F_y{}' = \frac{\partial}{\partial y} F(x, y(x,\alpha), y'(x,\alpha)),$$

$$F_{y'}' = \frac{\partial}{\partial y'} F(x, y(x,\alpha), y'(x,\alpha)),$$

$$\frac{\partial}{\partial \alpha} y(x,\alpha) = \frac{\partial}{\partial \alpha}[y(x) + \alpha \delta y] = \delta y$$

$$\frac{\partial}{\partial \alpha} y'(x,\alpha) = \frac{\partial}{\partial \alpha}[y'(x) + \alpha \delta y'] = \delta y',$$

and we get

$$\varphi'(\alpha) = \int_{x_0}^{x_1} \left[F_y(x, y(x,\alpha), y'(x,\alpha))\delta y + F_{y'}(x, y(x,\alpha), y'(x,\alpha))\delta y' \right] dx,$$

$$\varphi'(0) = \int_{x_0}^{x_1} \left[F_y(x, y(x), y'(x))\delta y + F_{y'}(x, y(x), y'(x))\delta y' \right] dx \quad (npu\ \alpha = 0).$$

when $\alpha = 0$.

As we know, $\varphi'(0)$ is called a variation of a functional and means δv.([3])
The necessary condition for the extreme of the functional is that its variation is equal to zero

$$\delta v = 0 \ .$$

For the functional (1), this condition has a type of

$$\int_{x_0}^{x_1} [F_y{}' \delta y + F_{y'}' \delta y']\, dx = 0 \qquad (3)$$

Integrating the equation (3) in parts, whereas $\delta y' = (\delta y)'$, we get

$$\delta v = [F_{y'}^{'}\delta y']\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} [F_y{}' - \frac{d}{dx}F_{y'}^{'}]\delta y \, dx =$$

$$= \int_{x_0}^{x_1} F_y^{'}\delta y \, dx + F_{y'}^{'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))\delta y(x_1) - F_{y'}^{'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))\delta y(x_0) =$$

$$= \int_{x_0}^{x_1} F_y^{'}\delta y \, dx + F_{y'}^{'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))(\overline{y}(x_1) - y(x_1))$$

$$-F_{y'}^{'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))(\overline{y}(x_0) - y(x_0)) - \int_{x_0}^{x_1} (\delta y) dF_{y'}^{'} =$$

$$= \int_{x_0}^{x_1} F_y^{'}\delta y \, dx + F_{y'}^{'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))(0)$$

$$-F_{y'}^{'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))(0) - \int_{x_0}^{x_1} (\delta y)\frac{d}{dx}F_{y'}^{'}$$

Since, all of the possible (permissible) curves in the given problem pass through fixed limit points, we get

$$\delta v = \int_{x_0}^{x_1} [F_y{}' - \frac{d}{dx}F_{y'}^{'}]\delta y \, dx$$

.

Note.

The first multiplier $F_y{}' - \frac{d}{dx}F_{y'}^{'}$ of the curve $y = y(x)$ reaches the extreme of the continuous function, and the second multiplier δy, with an arbitrary comparison curve $y = \overline{y}(x)$, is an arbitrary function having passed only certain general conditions, namely: the function δy in the boundary points $x = x_0$, and $x = x_1$ is equal to zero, continuous and differentiable one or several times, δy or δy and $\delta y'$ are small in absolute value. To simplify the obtained necessary condition (2), we will use the following lemma:

Fundamental lemma of the variational calculus

If for any continuous function $\eta(x)$ is true

$$\int_{x_0}^{x_1} \Phi(x)\,\eta(x)\,dx = 0,$$

where the function $\Phi(x)$ is continuous in the interval $[x_0, x_1]$, it is

$$\Phi(x) \equiv 0\ ,$$

in this interval ([6],[7]).

<u>Proof of the fundamental lemma of variational calculus.</u>

We assume that in the point $x = \bar{x}$ lies in the interval (x_0, x_1) , $\Phi(x) \neq 0$, is a contradiction.

From the continuity of the function $\Phi(x)$, it follows that if $\Phi(\bar{x}) \neq 0$ then $\Phi(x)$ keeps its characteristics in vicinity of $\bar{x}$ ($x_0 \leq x \leq x_1$). We choose the function $\eta(x)$ which also retains the mark in that vicinity and is equal to zero outside of this vicinity. We receive

$$\int_{x_0}^{x_1} \Phi(x)\,\eta(x)\,dx = \int_{\bar{x}_0}^{\bar{x}_1} \Phi(x)\,\eta(x)\,dx \neq 0,$$

since the product $\Phi(x)\eta(x)$ retains its mark in the interval $x_0 \leq x \leq x_1$ and is equal to zero in the same interval.

And so, we come to a contradiction, therefore $\Phi(x) \equiv 0$.

Note.

The adoption of the lemma and its proof remain unchanged if the function $\eta(x)$ requires the following restrictions:

$\eta(x_0) = \eta(x_1) = 0,$

the function $\eta(x)$ is continuously differentiable to order n ,

$\left|\eta^{(s)}(x)\right| < \varepsilon, \quad (s = 0, 1, \ldots, q; q \leq n)$ ([5]).

The function $\eta(x)$ can be selected, e.g. :

$$\eta(x) = \begin{cases} k(x-\bar{x}_0)^{2n}(x-\bar{x}_1)^{2n}, & x \in [\bar{x}_0, \bar{x}_1] \\ 0 & x \in [x_0, x_1] \setminus [\bar{x}_0, \bar{x}_1] \end{cases} ,$$

where n is a positive number, and k is a constant.

Apparently, the function $\eta(x)$ satisfies the above conditions: it is a continuous, is continuously differentiable to order $2n-1$, in the points x_0 and x_1 it is equal to zero and by reducing the factor by k, we can do $\left|\eta^{(s)}(x)\right| < \varepsilon$ for the $\forall x \in [x_0, x_1]$.

Now we will apply the fundamental lemma of variational calculus to simplify the above necessary condition for the extreme (2) of the functional (1).

Consequence1.1.

If the functional $v(y) = \int_{x_0}^{x_1} F(x, y, y')dx$ reaches the extreme of the curve $y = y(x)$, and F_y' and $\frac{d}{dx}F_{y'}'$ are continuous, then $y = y(x)$ is a solution to the differential equation (the equation of Euler)

$$F_y - \frac{d}{dx}F_{y'} = 0,$$

or in an expanded form

$$F_y - F_{xy'} - F_{yy'}y' - F_{y'y'}y'' = 0$$ ([2]).

Proof of consequence 1.1 .

The proof of consequence 1.1 follows immediately from the fundamental lemma of the variational calculus.

This equation is called the equation of Euler (1744 year) ([3],[4]). The integral curve of Euler's equation $y = y(x, C_1, C_2)$ is called the extreme.

To find a curve which reached the extreme of the functional (1) we integrate the equation of Euler and explain random constants that are satisfying the general solution of this equation, of the boundary conditions $y(x_0) = y_0, y(x_1) = y_1$.

Only if this is satisfied, can the extreme of the functional be reached.
However, to determine whether they are extreme (maximum or minimum), the sufficient conditions for the extreme must be studied.
Let us recall that the boundary problem

$$F_y{}' - \frac{d}{dx} F'_{y'} = 0, \quad y(x_0) = y_0, \quad y(x_1) = y_1 \quad ,$$

not always has a solution, and if there is a solution, then this may not be sole.
It should be considered that in many variational problems the existence of solutions is evident, from physical or consideration of the problem, and, in the solution of the equations of Euler satisfying the boundary conditions, only a single extreme may be the solution of the given problem.

3. Conclusion

It should be considered that, in many variational problems, the existence of solutions is evident from the physical or geometrical sense of the problem. Furthermore, in the solution of the Euler equations that satisfy the boundary conditions, only a single extremum may be the solution to the given problem.

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