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SOLUTION OF DIDO'S PROBLEM USING VARIATIONS

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Abstract. Isoperimetric problems constitute a fundamental category within variational analysis where optimization is conducted under specific constraints. The classical problem involves determining the geometric shape with the greatest enclosed area with a fixed perimeter - the solution is a circle. The present study illustrates how the methods from the calculus of variations can be used to resolve such problems, beginning with an adaptation of this classical formulation.

1. Introduction

Numerous mathematical problems are inherently expressed as the task of determining a function that minimizes a particular quantity of interest. A classic example from geometry is the question: what is the shortest path between two points in R^n? Although it is well known that this path is the straight line connecting the two points, providing a rigorous proof requires more subtle reasoning. A more sophisticated problem of the same type arises when, given an open set Ω with prescribed boundary conditions, one seeks a surface defined on this set that satisfies the boundary constraints while minimizing the surface area. The calculus of variations operates on functionals defined over normed vector spaces, particularly spaces of functions. The techniques and results developed within this framework are both elegant and powerful, closely resembling the analytical tools used in finite-dimensional real analysis. One of the most significant outcomes of variational analysis is the Euler–Lagrange equation. This result is fundamental, as a wide range of problems in mathematics, physics, and applied sciences can be reformulated as the minimization or maximization of an integral over a given domain.

2. Derivation and proving of theorems

We will explore for extreme of the functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \quad , \tag{2.1}$$

at the limit points of the allowable set of curves: $y(x_0) = y_0$ and $y(x_1) = y_1$. We assume that the function $F(x, y, y')$ is three times differentiable.

Keywords. : extreme, functional, variation, condition, problem.

We know that the necessary condition for an extreme is the variation in the functional to be equal to zero ([3]). We will now show how the main theorem is applied to the given functional (2.1).

Let's assume that the extreme reached on two times differentiable curve $y = y(x)$ (it is necessary to have a first-order derivative of the other curves, otherwise it may be the curve on which the extreme is reached). We are taking some limit curves $y = \overline{y}(x)$ close to $y = y(x)$ and we include the curves $y = y(x)$ and $y = \overline{y}(x)$ to the family curves with one parameter

$$y(x,\alpha) = y(x) + \alpha(\overline{y}(x) - y(x)) \ .$$

When $\alpha = 0$, we receive the curve $y = y(x)$, while for $\alpha = 1$ we receive $y = \overline{y}(x)$.

As we already know, the difference $\overline{y}(x) - y(x)$ is called the variation of the function $y(x)$ and is noted with the δy .([4])

The variation δy in variational problems plays a role analogous to the role of the increase Δx of an independent variable x in problems for the study of the extreme of function $f(x)$. The variation of function $\delta y = \overline{y}(x) - y(x)$ is a function of the x.

This function can be differentiated one or several times, as $(\delta y)' = \overline{y}'(x) - y'(x) = \delta y'$ is generated if the variance is equal to the variance of the generated, and similarly

$$(\delta y)'' = \overline{y}''(x) - y''(x) = \delta y'',$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(\delta y)^{(k)} = \overline{y}^{(k)}(x) - y^{(k)}(x) = \delta y^{(k)}.$$

Hence, we analyze the family $y = y(x,\alpha)$, where $y(x,\alpha) = y(x) + \alpha\delta y$, containing the $\alpha = 0$ curves, whose extreme is reached, and in some $\alpha = 1$ approximately close or curves called comparison curves.

If we look at the values of functional (2.1), only at the family curves $y = y(x,\alpha)$, the functional turned into function of α :

$$v[y(x,\alpha)] = \varphi(\alpha),$$

As in the case that we consider, $v[y(x,\alpha)]$ is a functional depending on parameter, the value of the parameter α determines the curve of the family $y = y(x,\alpha)$ and by that determines the value of functional $v[y(x,\alpha)]$ as well.

Theorem 2.1. *If functional* $v(y) = \int_{x_0}^{x_1} F(x, y, y')dx$ *has a local extreme in* y *, the necessary condition for extreme of functional is*

$$\int_{x_0}^{x_1} [F_y - \frac{d}{dx} F_{y'}] \delta y \, dx = 0, \text{ ([3])} \tag{2.2}$$

Proof. We analyze the function $\varphi(\alpha)$. It reaches its extreme at $\alpha = 0$. When $\alpha = 0$, we receive $y = y(x)$, and the functional, in assumption, reaches the extreme compared with any permissible curve, and in particular, in terms of the nearly families curves $y = y(x,\alpha)$.

The necessary condition for extreme of the function $\varphi(\alpha)$ at $\alpha = 0$ is its derivative to be equal to zero at $\alpha = 0$ ([3]), i.e.

$$\varphi'(0) = 0 \ .$$

Since

$$\varphi(\alpha) = \int_{x_0}^{x_1} F(x, y(x,\alpha), y_x'(x,\alpha)) \, dx,$$

Its derivative is

$$\varphi'(\alpha) = \int_{x_0}^{x_1} \left[F_y' \frac{\partial}{\partial \alpha} y(x,\alpha) + F_{y'}' \frac{\partial}{\partial \alpha} y'(x,\alpha)) \right] dx,$$

Where

$$F_y' = \frac{\partial}{\partial y} F(x, y(x,\alpha), y'(x,\alpha)),$$

$$F_{y'}' = \frac{\partial}{\partial y'} F(x, y(x,\alpha), y'(x,\alpha)),$$

$$\frac{\partial}{\partial \alpha} y(x,\alpha) = \frac{\partial}{\partial \alpha} [y(x) + \alpha \delta y] = \delta y$$

$$\frac{\partial}{\partial \alpha} y'(x,\alpha) = \frac{\partial}{\partial \alpha} [y'(x) + \alpha \delta y'] = \delta y',$$

and we obtain

$$\varphi'(\alpha) = \int_{x_0}^{x_1} \left[F_y(x, y(x,\alpha), y'(x,\alpha)) \delta y + F_{y'}(x, y(x,\alpha), y'(x,\alpha)) \delta y' \right] dx,$$

$$\varphi'(0) = \int_{x_0}^{x_1} \left[F_y(x, y(x), y'(x)) \delta y + F_{y'}(x, y(x), y'(x)) \delta y' \right] dx \quad (\alpha = 0).$$

As we know, $\varphi'(0)$ is called the variation of functional and noted δv .([3])

The necessary condition for extreme of functional is its variation to be equal to zero

$$\delta v = 0 \text{ ([8]).}$$

For the functional (2.1) this condition has a type of

$$\int_{x_0}^{x_1} [F_y{}'\delta y + F'_{y'}\delta y'] \, dx = 0 \tag{2.3}$$

Integrating the equation (2.3) in parts, whereas $\delta y' = (\delta y)'$, we obtain

$$\delta v = [F'_{y'}\delta y']\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} [F_y{}' - \frac{d}{dx} F'_{y'}]\delta y \, dx =$$

$$= \int_{x_0}^{x_1} F'_y \delta y \, dx + F'_{y'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))\delta y(x_1) - F'_{y'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))\delta y(x_0) =$$

$$= \int_{x_0}^{x_1} F'_y \delta y \, dx + F'_{y'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))(\overline{y}(x_1) - y(x_1))$$

$$-F'_{y'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))(\overline{y}(x_0) - y(x_0)) - \int_{x_0}^{x_1} (\delta y) dF'_{y'} =$$

$$= \int_{x_0}^{x_1} F'_y \delta y \, dx + F'_{y'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))(0)$$

$$-F'_{y'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))(0) - \int_{x_0}^{x_1} (\delta y) \frac{d}{dx} F'_{y'}$$

Since all of the possible (permissible) curves in the given problem pass through fixed limit points, we obtain

$$\delta v = \int_{x_0}^{x_1} [F_y{}' - \frac{d}{dx} F'_{y'}]\delta y \, dx \ .$$

■

Note. The first multiplier $F_y{}' - \frac{d}{dx} F'_{y'}$ of the curve $y = y(x)$ reaches the extreme of the continuous function, and the second multiplier δy, with arbitrary comparison $y = \overline{y}(x)$ is arbitrary function having passed only certain general conditions, namely: the function δy in the boundary points $x = x_0$, and $x = x_1$ is equal to zero, continuous and differentiable one or several times, δy or both δy and $\delta y'$ are small in absolute value.

To simplify the obtained necessary condition (2.2), we will use the following lemma:

Lemma 2.1. (Fundamental lemma of the variational calculus) *If for any continuous function* $\eta(x)$ *it is true that*

$$\int_{x_0}^{x_1} \Phi(x)\eta(x)\,dx = 0,$$

where the function $\Phi(x)$ *is continuous on the interval* $[x_0, x_1]$ *, it*

$$\Phi(x) \equiv 0 \ ,$$

in this interval. ([4], [9]).

Proof. We assume that, at the point $x = \bar{x}$, lying in the interval (x_0, x_1) , $\Phi(x) \neq 0$, is a contradiction.

Indeed, from the continuity of the function $\Phi(x)$, it follows that if $\Phi(\bar{x}) \neq 0$, $\Phi(x)$ keeps the sign in the vicinity of $\bar{x}$ ($x_0 \leq x \leq x_1$). We choose function $\eta(x)$ which also retains the sign in that vicinity and is equal to zero outside of this vicinity. We receive

$$\int_{x_0}^{x_1} \Phi(x)\eta(x)\,dx = \int_{\bar{x}_0}^{\bar{x}_1} \Phi(x)\eta(x)\,dx \neq 0,$$

since the product $\Phi(x)\eta(x)$ retains its sign in the interval $x_0 \leq x \leq x_1$ and is equal to zero in the same interval.

And so, we come to a contradiction, therefore $\Phi(x) \equiv 0$. ■

Note.The adoption of lemma and its proof remains unchanged if the function $\eta(x)$ requires the following restrictions:

$$\eta(x_0) = \eta(x_1) = 0,$$

$\eta(x)$ There is a continuous derived to line n ,

$$\left|\eta^{(s)}(x)\right| < \varepsilon, \quad (s = 0,1,\ldots,q; q \leq n) \ .$$

The function $\eta(x)$ can be selected, e.g. :

$$\eta(x) = \begin{cases} k(x - \bar{x}_0)^{2n}(x - \bar{x}_1)^{2n}, & x \in [\bar{x}_0, \bar{x}_1] \\ 0 & x \in [x_0, x_1] \setminus [\bar{x}_0, \bar{x}_1] \end{cases} ,$$

where n is a positive number, and k is a constant.

Apparently, the function $\eta(x)$ satisfies the above conditions: it is a continuous, there is a continuous derivative to line $2n-1$, at the points x_0 and x_1 it is equal to zero and by reducing the factor by k we can do $\left|\eta^{(s)}(x)\right| < \varepsilon$ for the $\forall x \in [x_0, x_1]$.

Now we will apply the fundamental lemma of variational calculus to simplify the above necessary condition for the extreme (2.2) of functional (2.1).

Consequence 2.1. *If functional* $v(y) = \int_{x_0}^{x_1} F(x, y, y')dx$ *reaches the extreme of the curve* $y = y(x)$ *, and* $F_y^{'}$ *and* $\frac{d}{dx}F_{y'}^{'}$ *are continuous, then* $y = y(x)$ *is a solution to the differential equation (equation of Euler)*

$$F_y - \frac{d}{dx}F_{y'} = 0,$$

Or in an expanded form

$$F_y - F_{xy'} - F_{yy'}y' - F_{y'y'}y'' = 0 \ .$$

Proof. The proof of consequence 1.1 follows immediately from the fundamental lemma of variational calculus.([8], [9]). ■

This equation is called the equation of Euler (1744 year). Integral curve $y = y(x, C_1, C_2)$ equation of Euler is called extreme.([1])

To find a curve, which reaches the extreme of functional (2.1) we integrate the equation of Euler and spell out random constants, satisfying the general solution of this equation, of the boundary conditions $y(x_0) = y_0, y(x_1) = y_1$.

Only if they are satisfied with these conditions, the extreme of functional can be reached. However, in order to determine whether they are really extremes (maximum or minimum), a sufficient conditions for extreme must be studied as well.

To recall, that boundary problem

$$F_y' - \frac{d}{dx}F_{y'}^{'} = 0, \quad y(x_0) = y_0, \quad y(x_1) = y_1 \ ,$$

not always has a solution, and if there is a solution, then this may not be sole.

It should be considered that in many variational problems the existence of solutions is evident, from physical or geometrical sense of the problem, and in the solution of the equations of Euler satisfying boundary conditions, only a single extreme may be the solution of the given problem.

3. **Different forms of the Euler Lagrange equation**

Let's suppose that $y(x)$ is an extremizer of $I(y)$
Since $F = F(x, y, y')$

$$\frac{d}{dx}(F) = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{dy'}{dx}$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' \qquad (3.1)$$

Consider

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}y'' \qquad (3.2)$$

Subtracting (3.1) – (3.2) we obtain,

$$\frac{dF}{dx} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y}\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)y'$$

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = y'\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]$$

As $y(x)$ is an extremizer and by using the Euler-Lagrange equation, we obtain

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0\ ,$$

which is another form of the Euler-Lagrange Equation.

Special Cases. Extremize

$$I(y) = \int_{x1}^{x2}(x, y, y')\,dx \quad y(x1) = y1;\ y(x2) = y2.$$

(i) When x does not appear in F explicitly

$$\frac{\partial F}{\partial x} = 0$$

Hence,

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0,$$

becomes

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0,$$

or

$$\frac{\partial F}{\partial y'} = \text{const.}$$

This is known as Beltrami Identity. ([2])

(ii) When y does not appear in F explicitly

$$\frac{\partial F}{\partial y\prime} = 0.$$

Hence,

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

reduces to

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0,$$

or

$$\frac{\partial F}{\partial y'} = \text{const.}$$

(iii) When y' does not appear in F explicitly

$$\frac{\partial F}{\partial y'} = 0,$$

then

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

reduces to

$$\frac{\partial F}{\partial y} = 0.$$

4. Constrained Extremization Problem (Isoparametric Problems)

In certain problems of calculus of variations, while extremizing a given functional $I(y)$, along with the end conditions $y(x1) = y1,\ y(x2) = y2$, we also need the extremizing function which must satisfy an additional integral constraint as we see in the following Dido's Problem.

Dido's Problem. Find the plane curve of fixed perimeter which has a maximum area above x - axis.

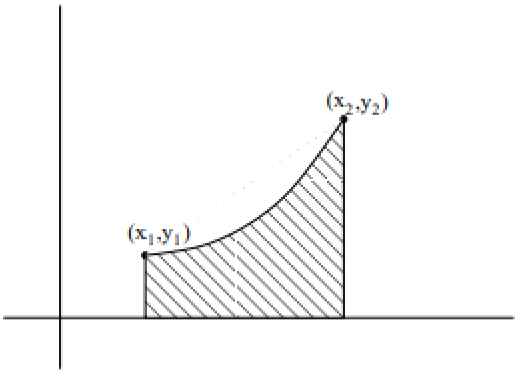

Figure 1.

The perimeter and the area under the curve are given by

$$\text{Perimeter} = \text{Arc length} = \int_{x1}^{x2} \sqrt{1 + y'^2}\, dx$$

$$\text{Area under the curve} = \int_{x1}^{x2} y(x)dx.$$

Variational Problem. Maximize

$$\mathrm{I(y)} = \int_{x2}^{x1} \mathrm{y(x)dx}$$

subject to the constraints

$$\int_{x1}^{x2} \sqrt{1 + y'^2 dx = L}\,,$$

and with boundary conditions

$$y(x1) = y1 \text{ and } y(x2) = y2.$$

General Problem. Extremize

$$\mathrm{I(y)} = \int_{\boldsymbol{x1}}^{x2} \mathrm{F(x, y, y')dx}$$

subject to the integral constraint

$$\int_{x2}^{x1} G(x, y, y')\, dx = L = \text{constant},$$

and with boundary conditions

$$y(x1) = y1 \text{ and } y(x2) = y2$$

Lagrange Multiplier Technique:
Convert the constrained optimization problem into an unconstrained optimization problem by the Lagrange Multiplier Technique.
Define a new functional H by

$$H(x, y, y') = F(x, y, y') + \lambda\, G(x, y, y')$$

and optimize

$$\int_{x1}^{x2} H(x, y, y')dx\ ,$$

without constraints. It means to optimize

$$I(y) = \int_{x1}^{x2} F(x, y, y') + \lambda G(x, y, y')dx$$

with boundary condition $y(x1) = y1$ and $y(x2) = y2$.

The problem is solved by solving the Euler Lagrange Equation:

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$
$$y(x1) = y1,\ \ y(x2) = y2$$

Solution of Dido's Problem. Maximize

$$I(y) = \int_{x1}^{x2} y(x)dx$$

subject to

$$\int_{x1}^{x2} \sqrt{1 + y'^2 dx}\ dx = L,$$

with boundary condition $y(x1) = y1$ and $y(x2) = y2$.

From

$$F(x, y, y') = y(x) \qquad G(x, y, y') = \sqrt{1 + y'^2}$$
$$H(x, y, y') = y + \lambda\sqrt{1 + y'^2}$$
$$\frac{\partial H}{\partial y} = 1,\ \ \frac{\partial H}{\partial y'} = \frac{\lambda y'}{\sqrt{1 + y'^2}}$$

the Euler's Equation is:

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1 + y'^2}}\right) = 1$$
$$\Longrightarrow \quad \frac{\lambda y'}{\sqrt{1 + y'^2}} = x + a$$

$$\Longrightarrow \quad \frac{\lambda y'}{x+a} = \sqrt{1+y'^2}$$
$$\Longrightarrow \quad \lambda^2 y'^2 = (1+y'^2)(x+a)^2$$
$$\Longrightarrow \quad y'^2\big(\lambda^2-(x+a)\big)^2 = (x+a)^2$$
$$y' = \frac{x+a}{\sqrt{\lambda^2-(x+a)^2}}$$
$$\Longrightarrow \quad y = -\sqrt{\lambda^2-(x+a)^2}+b$$
$$(y-b)^2 = \lambda^2-(x+a)^2$$
$$\Longrightarrow \quad (x+a)^2+(y-b)^2 = \lambda^2$$

which is a circle, where the constants a, b, λ can be obtained from three conditions, namely, two boundary conditions and one constraint condition.

3. Conclusion

In many variational problems, the existence of a solution can often be inferred from the underlying physical or geometric sense of the formulation. However, when solving the Euler equations with prescribed boundary conditions, it is often the case that only one extremal function constitutes the valid solution to the problem.

References

[1] *Assoc. Dr. B. Златанов* - metric spaces;
[2]*L.Э.Эльсгольц* - ДИФФЕРЕНЦИАЛЬНЫЕ equations AND ВАРИАЦИОННОЕ ИСЧИСЛЕНИЕ;
[3] *Russak I.B.* - Calculus of variations (MA4311, 1996);
[4] *Brunt B.* - The calculus of variations (ISBN 0387402470), Springer, 2004);
[5] *Byerly W.* - Introduction to the calculus of variations (Cambridge, 1917);
[6] *Dr. Hosein Naderpour* - CALCULUS OF VARIATIONS
[7] *PROF. ARNOLD ARTHURS* CALCULUS OF VARIATIONS
[8] *Risteska, Aleksandra and Kocaleva, Mirjana (2023)* Application of fundamental lemma of variational calculus to the problem of plateau. Balkan Journal of Applied Mathematics and Informatics, 6 (1). ISSN 2545-4083 (https://eprints.ugd.edu.mk/cgi/users/home?screen=EPrint%3A%3AView&eprintid=32016)
[9] *Risteska, Aleksandra (2023)* Generalization of the application of a fundamental lemma of variational calculus to revolutionize transportation by using the solution of brachistochrone. Balkan Journal of Applied Mathematics and Informatics, VI (1). ISSN 2545-4083 (https://eprints.ugd.edu.mk/cgi/users/home?screen=EPrint%3A%3AView&eprintid=32017)

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