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NEW RESULTS ON FIXED POINT THEOREMS IN 2-BANACH SPACES

JAWAD ETTAYB

Abstract. In this article, we prove a common fixed point result for interpolative Kannan type contraction mappings in 2-Banach spaces. On the other hand, we introduce interpolative Dass and Gupta rational type contraction mappings on 2-Banach spaces. In particular, we discuss the existence of a fixed point of such a mapping in 2-Banach spaces.

1. **Introduction**

In a 2-Banach space framework, S. Gähler [5] initiated the study of 2-normed spaces. Recently, A. White [9] initiated and studied the concept of 2-Banach spaces. For more details on 2-normed spaces and 2-Banach spaces, we refer to see [4] and [5].

P. K. Harikrishnan and K. T. Ravindran [6] demonstrated that a contraction mapping has a one fixed point in bounded and closed subsets of a 2-Banach space. However, M. Kir and H. Kiziltunc [7] demonstrated several results on fixed points in 2-Banach spaces.

In [2], J. Ettayb introduced the notions of Meir-Keeler contraction mappings and Cirić contraction mappings on a 2-Banach space. In particular, we discuss the existence and uniqueness of a fixed point of such mappings in a 2-Banach space. On the other hand, he defined the concept of Hardy-Rogers contraction mappings on a 2-Banach space. In particular, he proved the existence and uniqueness of a fixed point of such a mapping in a 2-Banach space. However, several results are demonstrated on fixed point theorems of some mappings in 2-Banach spaces. Recently, J. Ettayb [3] demonstrated a fixed point theorem for generalized weakly contractive mappings in 2-Banach spaces which is a generalization of a Banach contraction mapping principle. On the other hand, he introduced interpolative Boyd-Wong and Matkowski type contractions on 2-Banach spaces. As a result, he proved the existence and uniqueness of a fixed point of such mappings in 2-Banach spaces. In the framework of metric spaces, B. K. Dass and S. Gupta [1] obtained

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an extension of Banach contraction principle through rational expression. On the other hand, M. Noorwali [8] proved a common fixed point result for interpolative Kannan contraction mappings.

The fixed point theory played a crucial role in functional analysis. Moreover, it used in many branches of science such as biology, chemistry, economics, engineering and computer science.

In this article, we prove a common fixed point result for interpolative Kannan type contraction mappings in 2-Banach spaces. On the other hand, we introduce interpolative Dass and Gupta rational type contraction mappings on 2-Banach spaces. In particular, we discuss the existence of a fixed point of such a mapping in 2-Banach spaces.

2. Preliminaries

We begin with preliminaries:

Definition 2.1 ([5]). *Let $\mathcal{X}$ be a real vector space with $\dim \mathcal{X} \geq 2$. A 2-norm on $\mathcal{X}$ is a function $\|\cdot,\cdot\| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ such that*

(i) $\|x, y\| = 0$ *if and only if x and y are linearly dependent.*
(ii) For all $x, y \in \mathcal{X}$, $\|x, y\| = \|y, x\|$.
(iii) For any $x, y \in \mathcal{X}$ *and* $\lambda \in \mathbb{R}$, $\|\lambda x, y\| = |\lambda|\|x, y\|$.
(iv) For each $x, y, z \in \mathcal{X}$, $\|x + y, z\| \leq \|x, z\| + \|y, z\|$.

The pair $(\mathcal{X}, \|\cdot,\cdot\|)$ is called a 2-normed space.

Definition 2.2 ([4]). *Let $\mathcal{X}$ be a 2-normed space. A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be a Cauchy sequence if $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ for any $y \in \mathcal{X}$.*

Definition 2.3 ([9]). *Let $\mathcal{X}$ be a 2-normed space. A sequence $\{x_n\}$ in $\mathcal{X}$ converges in $\mathcal{X}$ if there exists an element $x \in \mathcal{X}$ such that for all $y \in \mathcal{X}$, $\lim_{n\to\infty} \|x_n - x, y\| = 0$. If $\{x_n\}$ converges to x, we write $x_n \to x$ as $n \to \infty$.*

Definition 2.4 ([9]). *A 2-normed space in which every Cauchy sequence converges will be called a 2-Banach space.*

Lemma 2.1 ([4]). *Let $\mathcal{X}$ be a 2-normed space and $x \in \mathcal{X}$. If $\|x, y\| = 0$ for all $y \in \mathcal{X}$, then $x = 0$.*

Definition 2.5 ([6]). *Let $\mathcal{X}$ be a 2-normed space then the mapping $S : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be a contraction if there exists $k \in (0, 1)$ such that*

$$\|Sx - Sy, z\| \leq k\|x - y, z\| \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$.

Theorem 2.1 ([6]). *Let $\mathcal{X}$ be a 2-normed space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. Let $S : \mathcal{F} \longrightarrow \mathcal{F}$ be a contraction, then S has a unique fixed point.*

Definition 2.6 ([2]). *A mapping S on a 2-normed space $\mathcal{X}$ is called a Meir-Keeler contraction if given $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x, y, z \in \mathcal{X}$,*

$$\varepsilon \leq \|x - y, z\| < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| < \varepsilon. \tag{2.2}$$

Definition 2.7 ([2]). *A mapping S on a 2-normed space $\mathcal{X}$ is called a Ćirić contraction if given $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x, y, z \in \mathcal{X}$,*

$$\varepsilon < \|x - y, z\| < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| \leq \varepsilon. \tag{2.3}$$

Theorem 2.2 ([2]). *Let $\mathcal{X}$ be a 2-Banach space and let S be a Meir-Keeler contraction on $\mathcal{X}$, then S has a unique fixed point in $\mathcal{X}$.*

Theorem 2.3 ([2]). *Let $\mathcal{X}$ be a 2-Banach space and let S be a Ćirić contraction on $\mathcal{X}$, then S has a unique fixed point in $\mathcal{X}$.*

Definition 2.8 ([2]). *A mapping S on a 2-normed space $\mathcal{X}$ is called a Hardy-Rogers contraction if S is a self-mapping on $\mathcal{X}$ satisfying for each $x, y, z \in \mathcal{X}$,*

$$\begin{aligned} \|Sx - Sy, z\| \leq a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| \\ + e\|y - Sx, z\| + f\|x - y, z\| \end{aligned} \tag{2.4}$$

where a, b, c, e, f are nonnegative and we put $\beta = a + b + c + e + f$.

Consider the following condition

$$\begin{aligned} x \neq y \Longrightarrow \|Sx - Sy, z\| < a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| \\ + e\|y - Sx, z\| + f\|x - y, z\|. \end{aligned} \tag{2.5}$$

Theorem 2.4 ([2]). *Let $\mathcal{X}$ be a 2-normed space and S a self-mapping on $\mathcal{X}$ satisfying for each $x, y, z \in \mathcal{X}$,*

$$\begin{aligned} \|Sx - Sy, z\| \leq a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| \\ + e\|y - Sx, z\| + f\|x - y, z\| \end{aligned} \tag{2.6}$$

where a, b, c, e, f are nonnegative and we put $\beta = a + b + c + e + f$.

- *(i) If $\mathcal{X}$ is complete and $\beta < 1$, then S has a unique fixed point.*
- *(ii) If (2.4) is modified to the condition (2.5) and in this case $\mathcal{X}$ is compact, S is continuous and $\beta = 1$, then S has a unique fixed point.*

Theorem 2.5 ([2]). *Let $\mathcal{X}$ be a 2-Banach space, a, b, c, e, f be monotonically decreasing functions from $[0, \infty)$ to $[0, 1)$ and let the sum of these five functions be less than 1. Assume that $S : \mathcal{X} \longrightarrow \mathcal{X}$ satisfies condition (2.4) with $a = a(\|x - y, z\|), \cdots, f = f(\|x - y, z\|)$ for each $x, y, z \in \mathcal{X}$. Then S has a unique fixed point.*

Theorem 2.6 ([2]). *Let $\mathcal{X}$ be a 2-Banach space and $S_n : \mathcal{X} \longrightarrow \mathcal{X}$, $n = 1, 2, \cdots$ satisfy the conditions of Theorem 2.5 with the coefficients a, b, c, e, f. Let $S_n x_n = x_n$ and assume that $S_n \longrightarrow S$ pointwise on $\mathcal{X}$. Then $x = \lim_{n\longrightarrow\infty} x_n$ is the unique fixed point of S.*

Theorem 2.7 ([2]). *Let $\mathcal{X}$ be a 2-Banach space and $S_n : \mathcal{X} \longrightarrow \mathcal{X}$, $n = 1, 2, \cdots$ be functions with at least one fixed point $x_n, n = 1, 2, \cdots$. Let S satisfy the hypothesis of Theorem 2.5 and $S_n \longrightarrow S$ uniformly on $\mathcal{X}$. Then $x = \lim_{n\longrightarrow\infty} x_n$ is the unique fixed point of S.*

Ettayb [2] established a generalization of Banach's principle of contraction mappings in 2-Banach spaces as follows.

Theorem 2.8 ([2]). *Let $\mathcal{X}$ be a 2-Banach space and $\mathcal{F}$ be a nonempty closed and bounded subset of $\mathcal{X}$. If $S : \mathcal{F} \longrightarrow \mathcal{F}$ is a continuous mapping such that S^k is a contraction for some $k \geq 1$, then S has a unique fixed point.*

3. Main results

We present our main results.

Theorem 3.1. *Let $\mathcal{X}$ be a 2-Banach space and let S, T be self mappings on $\mathcal{X}$ satisfying for each $x \in \mathcal{X}\backslash Fix(S), y \in \mathcal{X}\backslash Fix(T)$ and $z \in \mathcal{X}$,*

$$\|Sx - Ty, z\| \leq \lambda\|x - Sx, z\|^{\alpha} \cdot \|y - Ty, z\|^{1-\alpha} \tag{3.1}$$

where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, then S and T have a common fixed point.

Proof. Let $x_0 \in \mathcal{X}$ and let $\{x_n\}$ be a sequence in $\mathcal{X}$ defined by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for each $n \in \mathbb{N}_0$. If there exists $n \in \mathbb{N}_0$ such that $x_n = x_{n+1} = x_{n+2}$, then x_n is a common fixed point of S and T, so assume that there does not exist three consecutive identical terms in the sequence $\{x_n\}$ and that $x_0 \neq x_1$. Using (3.1), we obtain for all $z \in \mathcal{X}$,

$$\begin{aligned} \|x_{2n+1} - x_{2n+2}, z\| &= \|Sx_{2n} - Tx_{2n+1}, z\| \\ &\leq \lambda\|x_{2n} - x_{2n+1}, z\|^{\alpha} \cdot \|x_{2n+1} - x_{2n+2}, z\|^{1-\alpha}. \end{aligned}$$

Then

$$\|x_{2n+1} - x_{2n+2}, z\|^{\alpha} \leq \lambda\|x_{2n} - x_{2n+1}, z\|^{\alpha}.$$

Hence

$$\begin{aligned} \|x_{2n+1} - x_{2n+2}, z\| &\leq \lambda^{\frac{1}{\alpha}}\|x_{2n} - x_{2n+1}, z\| \\ &\leq \lambda\|x_{2n} - x_{2n+1}, z\|. \end{aligned} \tag{3.2}$$

From (3.2), we obtain for all $z \in \mathcal{X}$,

$$\|x_{2n+1}-x_{2n+2}, z\| \leq \lambda\|x_{2n}-x_{2n+1}, z\| \leq \lambda^2\|x_{2n-1}-x_{2n}, z\| \leq \cdots \leq \lambda^{2n+1}\|x_0-x_1, z\|.$$

Then

$$\|x_{2n+1} - x_{2n+2}, z\| \leq \lambda^{2n+1}\|x_0 - x_1, z\|. \tag{3.3}$$

Similarly,

$$\begin{aligned} \|x_{2n+1} - x_{2n}, z\| &= \|Sx_{2n} - Tx_{2n-1}, z\| \\ &\leq \lambda\|x_{2n} - x_{2n+1}, z\|^{\alpha} \cdot \|x_{2n-1} - x_{2n}, z\|^{1-\alpha}. \end{aligned}$$

Then

$$\|x_{2n+1} - x_{2n}, z\|^{1-\alpha} \leq \lambda\|x_{2n-1} - x_{2n}, z\|^{1-\alpha}.$$

Hence

$$\begin{aligned} \|x_{2n+1} - x_{2n}, z\| &\leq \lambda^{\frac{1}{1-\alpha}}\|x_{2n-1} - x_{2n}, z\| \\ &\leq \lambda\|x_{2n-1} - x_{2n}, z\|. \end{aligned}$$

Thus

$$\|x_{2n+1}-x_{2n}, z\| \leq \lambda\|x_{2n-1}-x_{2n}, z\| \leq \lambda^2\|x_{2n-2}-x_{2n-1}, z\| \leq \cdots \leq \lambda^{2n}\|x_0-x_1, z\|.$$

Then

$$\|x_{2n+1} - x_{2n}, z\| \leq \lambda^{2n}\|x_0 - x_1, z\|. \tag{3.4}$$

Using (3.3) and (3.4), we obtain

$$\|x_n - x_{n+1}, z\| \leq \lambda^n\|x_0 - x_1, z\|. \tag{3.5}$$

Now, by (3.5) we demonstrate that $\{x_n\}$ is Cauchy. Let $m, n > 0$ with $m > n$. Put $m = n + l$, hence for any $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\ &= \||(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z\| \\ &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{n+l-1} - x_{n+l}, z\| \\ &\leq \lambda^n\|x_0 - x_1, z\| + \lambda^{n+1}\|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1}\|x_0 - x_1, z\| \\ &= \lambda^n(1 + \lambda + \cdots + \lambda^{l-1})\|x_0 - x_1, z\| \\ &\leq \frac{\lambda^n}{1-\lambda}\|x_0 - x_1, z\|. \end{aligned}$$

Letting $n, m \longrightarrow \infty$, we get

$$\lim_{m,n\to\infty} \|x_n - x_m, z\| = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$. Then $\{x_n\}$ converges to some $x \in \mathcal{X}$. Now, we demonstrate that $x \in \mathcal{X}$ is a fixed point of S. Hence for each $z \in \mathcal{X}$,

$$\begin{aligned} \|Sx - x_{2n+2}, z\| &= \|Sx - Tx_{2n+1}, z\| \\ &\leq \lambda\|x - Sx, z\|^{\alpha} \cdot \|x_{2n+1} - x_{2n+2}, z\|^{1-\alpha}. \end{aligned}$$

Letting $n \longrightarrow \infty$, we get

$$\|Sx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

Using Lemma 2.1, we obtain $Sx = x$. Similarly

$$\begin{aligned} \|x_{2n+1} - Tx, z\| &= \|Sx_{2n} - Tx, z\| \\ &\leq \lambda \|x_{2n} - x_{2n+1}, z\|^{\alpha} \cdot \|x - Tx, z\|^{1-\alpha}. \end{aligned}$$

Letting $n \longrightarrow \infty$, we get

$$\|Tx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

From Lemma 2.1, we have $x = Tx$. Then $x = Sx = Tx$. □

Theorem 3.2. *Let $\mathcal{X}$ be a 2-Banach space and let S be a self mapping on $\mathcal{X}$ satisfying for each $x, y, z \in \mathcal{X}$,*

$$\|Sx - Sy, z\| \leq k_1 \|x - y, z\| + k_2 \frac{\|x - Sx, z\| \|y - Sy, z\|}{1 + \|x - y, z\|} \tag{3.6}$$

where $k_1, k_2 \geq 0$ such that $k_1 + k_2 < 1$, then S has a unique fixed point in $\mathcal{X}$.

Proof. Let $x_0 \in \mathcal{X}$. Define the sequence $\{x_n\}$ in $\mathcal{X}$ by $x_n = Sx_{n-1} = S^n x_0$ for each $n \in \mathbb{N}$. Putting $x = x_{n-1}$ and $y = x_n$ in (3.6), we obtain for all $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Sx_{n-1} - Sx_n, z\| \\ &\leq k_1 \|x_{n-1} - x_n, z\| + k_2 \frac{\|x_{n-1} - Sx_{n-1}, z\| \|x_n - Sx_n, z\|}{1 + \|x_{n-1} - x_n, z\|} \\ &= k_1 \|x_{n-1} - x_n, z\| + k_2 \frac{\|x_{n-1} - x_n, z\| \|x_n - x_{n+1}, z\|}{1 + \|x_{n-1} - x_n, z\|} \\ &\leq k_1 \|x_{n-1} - x_n, z\| + k_2 \|x_n - x_{n+1}, z\|. \end{aligned}$$

Then

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\| \tag{3.7}$$

where $\lambda = \frac{k_1}{1-k_2}$. Since $\lambda = \frac{k_1}{1-k_2} < 1$, we deduce that the sequence $\{\|x_n - x_{n+1}, z\|\}$ is decreasing. From (3.7), we obtain for all $z \in \mathcal{X}$,

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\| \leq \lambda^n \|x_0 - x_1, z\|. \tag{3.8}$$

Letting $n \longrightarrow \infty$ in (3.8), we obtain $\lim_{n \to \infty} \|x_n - x_{n+1}, z\| = 0$ for all $z \in \mathcal{X}$. Now, we demonstrate that $\{x_n\}$ is Cauchy. Let $m, n > 0$ with $m > n$. Put $m = n + l$, hence for any $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\ &= \||(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z\| \\ &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{n+l-1} - x_{n+l}, z\| \\ &\leq \lambda^n \|x_0 - x_1, z\| + \lambda^{n+1} \|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1} \|x_0 - x_1, z\| \end{aligned}$$

$$\begin{aligned} &= \lambda^n(1+\lambda+\cdots+\lambda^{l-1})\|x_0-x_1,z\| \\ &\leq \frac{\lambda^n}{1-\lambda}\|x_0-x_1,z\|. \end{aligned}$$

Letting $n, m \longrightarrow \infty$, we get

$$\lim_{m,n\to\infty}\|x_n-x_m,z\|=0.$$

Hence $\{x_n\}$ is Cauchy. Then $\{x_n\}$ converges to some $x \in \mathcal{X}$. Now, we demonstrate that $x \in \mathcal{X}$ is a fixed point of S. Hence for each $z \in \mathcal{X}$,

$$\begin{aligned} \|x_{n+1}-Sx,z\| &= \|Sx_n-Sx,z\| \\ &\leq k_1\|x_n-x,z\|+k_2\frac{\|x_n-Sx_n,z\|\|x-Sx,z\|}{1+\|x_n-x,z\|}. \end{aligned}$$

Letting $n \longrightarrow \infty$, we get

$$\|Sx-x,z\|=0 \quad \text{for each } z\in\mathcal{X}.$$

By Lemma 2.1, we have $Sx = x$. Then x is a fixed point of S. Suppose that x_1 and x_2 are two distinct fixed points of S. From for all $z \in \mathcal{X}$,

$$\begin{aligned} \|x_1-x_2,z\| &= \|Sx_1-Sx_2,z\| \\ &\leq k_1\|x_1-x_2,z\|+k_2\frac{\|x_1-Sx_1,z\|\|x_2-Sx_2,z\|}{1+\|x_1-x_2,z\|} \\ &= k_1\|x_1-x_2,z\|+k_2\frac{\|x_1-x_1,z\|\|x_2-x_2,z\|}{1+\|x_1-x_2,z\|} \\ &= k_1\|x_1-x_2,z\| \end{aligned}$$

which contradicts that $k_1 < 1$. Therefore $\|x_1 - x_2, z\| = 0$ for all $z \in \mathcal{X}$. Then by Lemma 2.1, it is established that $x_1 = x_2$. □

Similarly to the proof of Theorem 3.2, we obtain:

Theorem 3.3. *Let $\mathcal{X}$ be a 2-Banach space and let S be a self mapping on $\mathcal{X}$ satisfying for each $x, y, z \in \mathcal{X}$,*

$$\|Sx-Sy,z\|\leq k_1\|x-y,z\|+k_2\frac{\|x-Sy,z\|\|y-Sx,z\|}{1+\|x-y,z\|} \tag{3.9}$$

where $k_1, k_2 \geq 0$ such that $k_1 + k_2 < 1$, then S has a unique fixed point in $\mathcal{X}$.

Now, we introduce an interpolative Dass and Gupta rational type contraction in a 2-normed space as follows.

Definition 3.1. *A continuous self-mapping S on a 2-normed space $\mathcal{X}$ is called an interpolative Dass and Gupta rational type contraction if there exist $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that*

$$\|Sx - Sy, z\| \leq \lambda[\frac{\|x - Sx, z\|\|y - Sy, z\|}{\|x - y, z\|}]^{\alpha} \cdot [\|x - y, z\|]^{1-\alpha} \tag{3.10}$$

for each $x, y \in \mathcal{X} \backslash Fix(S)$ with $x \neq y$ and $z \in \mathcal{X}$ where $Fix(S) = \{u \in \mathcal{X} : Su = u\}$.

Theorem 3.4. *Let $\mathcal{X}$ be a 2-Banach space and let S be an interpolative Dass and Gupta rational type contraction on $\mathcal{X}$, then S has a fixed point in $\mathcal{X}$.*

Proof. Let $x_0 \in \mathcal{X}$. We will set a constructive sequence $\{x_n\}$ by $x_n = Sx_{n-1} = S^n x_0$ for each $n \in \mathbb{N}$. Putting $x = x_n$ and $y = x_{n-1}$ in (3.10), we obtain for all $z \in \mathcal{X}$,

$$\begin{aligned} \|x_{n+1} - x_n, z\| &= \|Sx_n - Sx_{n-1}, z\| \\ &\leq \lambda[\frac{\|x_n - Sx_n, z\|\|x_{n-1} - Sx_{n-1}, z\|}{\|x_n - x_{n-1}, z\|}]^{\alpha} \cdot [\|x_n - x_{n-1}, z\|]^{1-\alpha} \\ &\leq \lambda[\|x_n - x_{n+1}, z\|]^{\alpha} \cdot [\|x_n - x_{n-1}, z\|]^{1-\alpha}. \end{aligned}$$

Then

$$\|x_n - x_{n+1}, z\|^{1-\alpha} \leq \lambda \|x_{n-1} - x_n, z\|^{1-\alpha}. \tag{3.11}$$

Hence

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\|. \tag{3.12}$$

Thus, we deduce that the sequence $\{\|x_n - x_{n+1}, z\|\}$ is decreasing. As a result, there exists $M \in \mathbb{R}_+$ such that $\lim_{n\to\infty} \|x_n - x_{n-1}, z\| = M$ for all $z \in \mathcal{X}$. We will indicate that $M = 0$. Indeed, by (3.12), we derive that

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\| \leq \lambda^n \|x_0 - x_1, z\|. \tag{3.13}$$

Letting $n \longrightarrow \infty$ in (3.13), we obtain $M = 0$. Now, we demonstrate that $\{x_n\}$ is Cauchy. Let $m, n > 0$ with $m > n$. Put $m = n + l$, hence for any $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\ &= \||(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z\| \\ &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{n+l-1} - x_{n+l}, z\| \\ &\leq \lambda^n \|x_0 - x_1, z\| + \lambda^{n+1} \|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1} \|x_0 - x_1, z\| \\ &= \lambda^n (1 + \lambda + \cdots + \lambda^{l-1}) \|x_0 - x_1, z\| \\ &\leq \frac{\lambda^n}{1-\lambda} \|x_0 - x_1, z\|. \end{aligned}$$

Letting $n, m \longrightarrow \infty$, we get

$$\lim_{m,n\to\infty} \|x_n - x_m, z\| = 0.$$

Hence $\{x_n\}$ is Cauchy. Then $\{x_n\}$ converges to some $x \in \mathcal{X}$. Now, we demonstrate that $x \in \mathcal{X}$ is a fixed point of S. Since S is continuous on $\mathcal{X}$, we obtain

$$x = \lim_{n \longrightarrow \infty} x_{n+1} = \lim_{n \longrightarrow \infty} Sx_n = S(\lim_{n \longrightarrow \infty} x_n) = Sx.$$

Consequently x is a fixed point of S. □

We obtain the following theorem.

Theorem 3.5. *Let $\mathcal{X}$ be a 2-Banach space and let S, T be self mappings on $\mathcal{X}$ satisfying for each $x, y \in \mathcal{X} \backslash Fix(S)$ and $z \in \mathcal{X}$,*

$$\|Sx - Sy, z\| \leq \lambda[\frac{\|x - Sx, z\| \|y - Sy, z\|}{1 + \|x - y, z\|}]^{\alpha} \cdot [\|x - y, z\|]^{1-\alpha} \tag{3.14}$$

where $\lambda, \alpha \in (0, 1)$, then S has a fixed point in $\mathcal{X}$.

Proof. Let $x_0 \in \mathcal{X}$. We will set a constructive sequence $\{x_n\}$ by $x_n = S^n x_0$ for each $n \in \mathbb{N}$. Putting $x = x_n$ and $y = x_{n-1}$ in (3.14), we obtain for all $z \in \mathcal{X}$,

$$\begin{aligned} \|x_{n+1} - x_n, z\| &= \|Sx_n - Sx_{n-1}, z\| \\ &\leq \lambda[\frac{\|x_n - Sx_n, z\| \|x_{n-1} - Sx_{n-1}, z\|}{1 + \|x_n - x_{n-1}, z\|}]^{\alpha} \cdot [\|x_n - x_{n-1}, z\|]^{1-\alpha} \\ &\leq \lambda[\|x_n - x_{n+1}, z\|]^{\alpha} \cdot [\|x_n - x_{n-1}, z\|]^{1-\alpha}. \end{aligned}$$

Then

$$\|x_n - x_{n+1}, z\|^{1-\alpha} \leq \lambda \|x_{n-1} - x_n, z\|^{1-\alpha}. \tag{3.15}$$

Hence

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\|. \tag{3.16}$$

Thus, we deduce that the sequence $\{\|x_n - x_{n+1}, z\|\}$ is decreasing. As a result, there exists $M \in \mathbb{R}_+$ such that $\lim_{n \to \infty} \|x_n - x_{n-1}, z\| = M$ for all $z \in \mathcal{X}$. We will indicate that $M = 0$. Indeed, by (3.16), we derive that

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\| \leq \lambda^n \|x_0 - x_1, z\|. \tag{3.17}$$

Letting $n \longrightarrow \infty$ in (3.17), we obtain $M = 0$. Now, we demonstrate that $\{x_n\}$ is Cauchy. Let $m, n > 0$ with $m > n$. Put $m = n + l$, hence for any $z \in \mathcal{X}$,

$$\begin{aligned} \|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\ &= \||(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z\| \\ &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{n+l-1} - x_{n+l}, z\| \\ &\leq \lambda^n \|x_0 - x_1, z\| + \lambda^{n+1} \|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1} \|x_0 - x_1, z\| \\ &= \lambda^n (1 + \lambda + \cdots + \lambda^{l-1}) \|x_0 - x_1, z\| \\ &\leq \frac{\lambda^n}{1 - \lambda} \|x_0 - x_1, z\|. \end{aligned}$$

Letting $n, m \longrightarrow \infty$, we get

$$\lim_{m,n\to\infty} \|x_n - x_m, z\| = 0.$$

Hence $\{x_n\}$ is Cauchy. Then $\{x_n\}$ converges to some $x \in \mathcal{X}$. Now, we demonstrate that $x \in \mathcal{X}$ is a fixed point of S. Hence for each $z \in \mathcal{X}$,

$$\|x_{n+1} - Sx, z\| = \|Sx_n - Sx, z\| \leq \lambda[\frac{\|x_n - Sx_n, z\|\|x - Sx, z\|}{1 + \|x_n - x, z\|}]^\alpha \cdot [\|x_n - x, z\|]^{1-\alpha}.$$

Letting $n \longrightarrow \infty$, we get

$$\|Sx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

From Lemma 2.1, it follows that $Sx = x$. Consequently x is a fixed point of S. □

Now, we prove the next theorem.

Theorem 3.6. *Let $\mathcal{X}$ be a 2-Banach space and let S, T be self mappings on $\mathcal{X}$ satisfying for each $x \in \mathcal{X}\backslash Fix(S), y \in \mathcal{X}\backslash Fix(T)$ and $z \in \mathcal{X}$,*

$$\|Sx - Ty, z\| \leq \lambda[\frac{\|x - Sx, z\|\|y - Ty, z\|}{1 + \|x - y, z\|}]^\alpha \cdot [\|x - y, z\|]^{1-\alpha} \tag{3.18}$$

where $\lambda, \alpha \in (0, 1)$, then S and T have a common fixed point.

Proof. Let $x_0 \in \mathcal{X}$. Define a sequence $\{x_n\}$ in $\mathcal{X}$ by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for each $n \in \mathbb{N}_0$. If there exists $n \in \mathbb{N}_0$ such that $x_{2n} = x_{2n+1} = x_{2n+2}$, then x_{2n} is a common fixed point of S and T, so assume that there does not exist three consecutive identical terms in the sequence $\{x_n\}$ and that $x_0 \neq x_1$. Using (3.18), we obtain for all $z \in \mathcal{X}$,

$$\begin{aligned}\|x_{2n+1} - x_{2n+2}, z\| &= \|Sx_{2n} - Tx_{2n+1}, z\| \\ &\leq \lambda[\frac{\|x_{2n} - Sx_{2n}, z\|\|x_{2n+1} - Tx_{2n+1}, z\|}{1 + \|x_{2n} - x_{2n+1}, z\|}]^\alpha \cdot \|x_{2n} - x_{2n+1}, z\|^{1-\alpha} \\ &\leq \lambda\|x_{2n+1} - x_{2n+2}, z\|^\alpha \cdot \|x_{2n} - x_{2n+1}, z\|^{1-\alpha}.\end{aligned}$$

Then

$$\|x_{2n+1} - x_{2n+2}, z\|^{1-\alpha} \leq \lambda\|x_{2n} - x_{2n+1}, z\|^{1-\alpha}. \tag{3.19}$$

Hence

$$\begin{aligned}\|x_{2n+1} - x_{2n+2}, z\| &\leq \lambda^{\frac{1}{1-\alpha}}\|x_{2n} - x_{2n+1}, z\| \\ &\leq \lambda\|x_{2n} - x_{2n+1}, z\|.\end{aligned} \tag{3.20}$$

From (3.20), we obtain for all $z \in \mathcal{X}$,

$$\|x_{2n+1} - x_{2n+2}, z\| \leq \lambda\|x_{2n} - x_{2n+1}, z\| \leq \lambda^2\|x_{2n-1} - x_{2n}, z\| \leq \cdots \leq \lambda^{2n+1}\|x_0 - x_1, z\|.$$

Then

$$\|x_{2n+1} - x_{2n+2}, z\| \leq \lambda^{2n+1}\|x_0 - x_1, z\|. \tag{3.21}$$

Similarly,

$$\begin{aligned}\|x_{2n+1}-x_{2n},z\| &= \|Sx_{2n}-Tx_{2n-1},z\|\\ &\leq \lambda[\frac{\|x_{2n}-Sx_{2n},z\|\|x_{2n-1}-Tx_{2n-1},z\|}{1+\|x_{2n}-x_{2n-1},z\|}]^{\alpha}\cdot[\|x_{2n-1}-x_{2n},z\|]^{1-\alpha}\\ &= \lambda[\frac{\|x_{2n}-x_{2n+1},z\|\|x_{2n-1}-x_{2n},z\|}{1+\|x_{2n}-x_{2n-1},z\|}]^{\alpha}\cdot[\|x_{2n-1}-x_{2n},z\|]^{1-\alpha}.\end{aligned}$$

Then

$$\|x_{2n+1}-x_{2n},z\|^{1-\alpha}\leq\lambda\|x_{2n-1}-x_{2n},z\|^{1-\alpha}.$$

Hence

$$\begin{aligned}\|x_{2n+1}-x_{2n},z\| &\leq \lambda^{\frac{1}{1-\alpha}}\|x_{2n-1}-x_{2n},z\|\\ &\leq \lambda\|x_{2n-1}-x_{2n},z\|.\end{aligned}$$

Thus

$$\|x_{2n+1}-x_{2n},z\|\leq\lambda\|x_{2n-1}-x_{2n},z\|\leq\lambda^2\|x_{2n-2}-x_{2n-1},z\|\leq\cdots\leq\lambda^{2n}\|x_0-x_1,z\|.$$

Then

$$\|x_{2n+1}-x_{2n},z\|\leq\lambda^{2n}\|x_0-x_1,z\|. \tag{3.22}$$

Using (3.21) and (3.22), we obtain

$$\|x_n-x_{n+1},z\|\leq\lambda^{n}\|x_0-x_1,z\|. \tag{3.23}$$

Now, by (3.23) we demonstrate that $\{x_n\}$ is Cauchy. Let $m,n>0$ with $m>n$. Put $m=n+l$, hence for any $z\in\mathcal{X}$,

$$\begin{aligned}\|x_n-x_m,z\| &= \|x_n-x_{n+l},z\|\\ &= \||(x_n-x_{n+1})+(x_{n+1}-x_{n+2})+\cdots+(x_{n+l-1}-x_{n+l}),z\|\\ &\leq \|x_n-x_{n+1},z\|+\|x_{n+1}-x_{n+2},z\|+\cdots+\|x_{n+l-1}-x_{n+l},z\|\\ &\leq \lambda^n\|x_0-x_1,z\|+\lambda^{n+1}\|x_0-x_1,z\|+\cdots+\lambda^{n+l-1}\|x_0-x_1,z\|\\ &= \lambda^n(1+\lambda+\cdots+\lambda^{l-1})\|x_0-x_1,z\|\\ &\leq \frac{\lambda^n}{1-\lambda}\|x_0-x_1,z\|.\end{aligned}$$

Letting $n,m\longrightarrow\infty$, we get

$$\lim_{m,n\to\infty}\|x_n-x_m,z\|=0.$$

Hence $\{x_n\}$ is Cauchy. Then $\{x_n\}$ converges to some $x\in\mathcal{X}$. Now, we demonstrate that $x\in\mathcal{X}$ is a fixed point of S. Hence for each $z\in\mathcal{X}$,

$$\begin{aligned}\|Sx-x_{2n+2},z\| &= \|Sx-Tx_{2n+1},z\|\\ &\leq \lambda[\frac{\|x-Sx,z\|\|x_{2n+1}-Tx_{2n+1},z\|}{1+\|x-x_{2n+1},z\|}]^{\alpha}\cdot[\|x-x_{2n+1},z\|]^{1-\alpha}.\end{aligned}$$

Letting $n \longrightarrow \infty$, we get

$$\|Sx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

By Lemma 2.1, we infer that $Sx = x$. Similarly

$$\begin{aligned}\|x_{2n+1} - Tx, z\| &= \|Sx_{2n} - Tx, z\|\\ &\leq \lambda[\frac{\|x_{2n} - Sx_{2n}, z\|\|x - Tx, z\|}{1 + \|x - x_{2n}, z\|}]^{\alpha} \cdot [\|x - x_{2n}, z\|]^{1-\alpha}.\end{aligned}$$

Letting $n \longrightarrow \infty$, we get

$$\|x - Tx, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

By applying Lemma 2.1, we conclude that $x = Tx$. Then $x = Sx = Tx$. □

We finish with the following theorem.

Theorem 3.7. *Let $\mathcal{X}$ be a 2-Banach space and let S, T be continuous self mappings on $\mathcal{X}$ satisfying for each $x \in \mathcal{X} \backslash Fix(S), y \in \mathcal{X} \backslash Fix(T)$ and $z \in \mathcal{X}$,*

$$\|Sx - Ty, z\| \leq \lambda[\frac{\|x - Sx, z\|\|y - Ty, z\|}{\|x - y, z\|}]^{\alpha} \cdot [\|x - y, z\|]^{1-\alpha} \tag{3.24}$$

where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, then S and T have a common fixed point.

Proof. Let $x_0 \in \mathcal{X}$. Define a sequence $\{x_n\}$ in $\mathcal{X}$ by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for each $n \in \mathbb{N}_0$. If there exists $n \in \mathbb{N}_0$ such that $x_{2n} = x_{2n+1} = x_{2n+2}$, then x_{2n} is a common fixed point of S and T, so assume that there does not exist three consecutive identical terms in the sequence $\{x_n\}$ and that $x_0 \neq x_1$. Using (3.24), we obtain for all $z \in \mathcal{X}$,

$$\begin{aligned}\|x_{2n+1} - x_{2n+2}, z\| &= \|Sx_{2n} - Tx_{2n+1}, z\|\\ &\leq \lambda[\frac{\|x_{2n} - Sx_{2n}, z\|\|x_{2n+1} - Tx_{2n+1}, z\|}{\|x_{2n} - x_{2n+1}, z\|}]^{\alpha} \cdot \|x_{2n} - x_{2n+1}, z\|^{1-\alpha}\\ &\leq \lambda\|x_{2n+1} - x_{2n+2}, z\|^{\alpha} \cdot \|x_{2n} - x_{2n+1}, z\|^{1-\alpha}.\end{aligned}$$

Then

$$\|x_{2n+1} - x_{2n+2}, z\|^{1-\alpha} \leq \lambda\|x_{2n} - x_{2n+1}, z\|^{1-\alpha}. \tag{3.25}$$

Hence

$$\begin{aligned}\|x_{2n+1} - x_{2n+2}, z\| &\leq \lambda^{\frac{1}{1-\alpha}}\|x_{2n} - x_{2n+1}, z\|\\ &\leq \lambda\|x_{2n} - x_{2n+1}, z\|.\end{aligned} \tag{3.26}$$

From (3.26), we obtain for all $z \in \mathcal{X}$,

$$\|x_{2n+1} - x_{2n+2}, z\| \leq \lambda\|x_{2n} - x_{2n+1}, z\| \leq \lambda^2\|x_{2n-1} - x_{2n}, z\| \leq \cdots \leq \lambda^{2n+1}\|x_0 - x_1, z\|.$$

Then

$$\|x_{2n+1} - x_{2n+2}, z\| \leq \lambda^{2n+1}\|x_0 - x_1, z\|. \tag{3.27}$$

Similarly,

$$\begin{aligned}\|x_{2n+1}-x_{2n},z\| &= \|Sx_{2n}-Tx_{2n-1},z\|\\ &\leq \lambda[\frac{\|x_{2n}-Sx_{2n},z\|\|x_{2n-1}-Tx_{2n-1},z\|}{\|x_{2n}-x_{2n-1},z\|}]^{\alpha}\cdot[\|x_{2n-1}-x_{2n},z\|]^{1-\alpha}\\ &= \lambda[\frac{\|x_{2n}-x_{2n+1},z\|\|x_{2n-1}-x_{2n},z\|}{\|x_{2n}-x_{2n-1},z\|}]^{\alpha}\cdot[\|x_{2n-1}-x_{2n},z\|]^{1-\alpha}.\end{aligned}$$

Then

$$\|x_{2n+1}-x_{2n},z\|^{1-\alpha}\leq\lambda\|x_{2n-1}-x_{2n},z\|^{1-\alpha}.$$

Hence

$$\begin{aligned}\|x_{2n+1}-x_{2n},z\| &\leq \lambda^{\frac{1}{1-\alpha}}\|x_{2n-1}-x_{2n},z\|\\ &\leq \lambda\|x_{2n-1}-x_{2n},z\|.\end{aligned}$$

Thus

$$\|x_{2n+1}-x_{2n},z\|\leq\lambda\|x_{2n-1}-x_{2n},z\|\leq\lambda^2\|x_{2n-2}-x_{2n-1},z\|\leq\cdots\leq\lambda^{2n}\|x_0-x_1,z\|.$$

Then

$$\|x_{2n+1}-x_{2n},z\|\leq\lambda^{2n}\|x_0-x_1,z\|. \tag{3.28}$$

Using (3.27) and (3.28), we obtain

$$\|x_n-x_{n+1},z\|\leq\lambda^n\|x_0-x_1,z\|. \tag{3.29}$$

Now, by (3.29) we demonstrate that $\{x_n\}$ is Cauchy. Let $m,n>0$ with $m>n$. Put $m=n+l$, hence for any $z\in\mathcal{X}$,

$$\begin{aligned}\|x_n-x_m,z\| &= \|x_n-x_{n+l},z\|\\ &= \|\|(x_n-x_{n+1})+(x_{n+1}-x_{n+2})+\cdots+(x_{n+l-1}-x_{n+l}),z\|\\ &\leq \|x_n-x_{n+1},z\|+\|x_{n+1}-x_{n+2},z\|+\cdots+\|x_{n+l-1}-x_{n+l},z\|\\ &\leq \lambda^n\|x_0-x_1,z\|+\lambda^{n+1}\|x_0-x_1,z\|+\cdots+\lambda^{n+l-1}\|x_0-x_1,z\|\\ &= \lambda^n(1+\lambda+\cdots+\lambda^{l-1})\|x_0-x_1,z\|\\ &\leq \frac{\lambda^n}{1-\lambda}\|x_0-x_1,z\|.\end{aligned}$$

Letting $n,m\longrightarrow\infty$, we get

$$\lim_{m,n\to\infty}\|x_n-x_m,z\|=0.$$

Hence $\{x_n\}$ is Cauchy. Then $\{x_n\}$ converges to some $x\in\mathcal{X}$. Now, we demonstrate that $x\in\mathcal{X}$ is a fixed point of S. Since S and T are continuous on $\mathcal{X}$, we obtain

$$x=\lim_{n\longrightarrow\infty}x_{2n+1}=\lim_{n\longrightarrow\infty}Sx_{2n}=S(\lim_{n\longrightarrow\infty}x_{2n})=Sx.$$

Then $Sx = x$. Similarly

$$x = \lim_{n\longrightarrow\infty} x_{2n+2} = \lim_{n\longrightarrow\infty} Tx_{2n+1} = T(\lim_{n\longrightarrow\infty} x_{2n+1}) = Tx.$$

Hence $x = Tx$. Consequently $x = Sx = Tx$. □

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