

**GOCE DELCEV UNIVERSITY - STIP**  
**FACULTY OF COMPUTER SCIENCE**

ISSN 2545-4803 on line

DOI: 10.46763/BJAMI

**BALKAN JOURNAL  
OF APPLIED MATHEMATICS  
AND INFORMATICS  
(BJAMI)**



YEAR 2025

VOLUME 8, Number 2

**AIMS AND SCOPE:**

BJAMI publishes original research articles in the areas of applied mathematics and informatics.

**Topics:**

1. Computer science;
2. Computer and software engineering;
3. Information technology;
4. Computer security;
5. Electrical engineering;
6. Telecommunication;
7. Mathematics and its applications;
8. Articles of interdisciplinary of computer and information sciences with education, economics, environmental, health, and engineering.

**Managing editor**

**Mirjana Kocaleva Vitanova** Ph.D.

**Zoran Zlatev** Ph.D.

**Editor in chief**

**Biljana Zlatanovska** Ph.D.

**Lectoure**

**Snezana Kirova**

**Technical editor**

**Biljana Zlatanovska** Ph.D.

**Mirjana Kocaleva Vitanova** Ph.D.

**BALKAN JOURNAL  
OF APPLIED MATHEMATICS AND INFORMATICS  
(BJAMI), Vol 8**

**ISSN 2545-4803 on line  
Vol. 8, No. 2, Year 2025**

## EDITORIAL BOARD

- Adelina Plamenova Aleksieva-Petrova**, Technical University – Sofia,  
Faculty of Computer Systems and Control, Sofia, Bulgaria
- Lyudmila Stoyanova**, Technical University - Sofia , Faculty of computer systems and control,  
Department – Programming and computer technologies, Bulgaria
- Zlatko Georgiev Varbanov**, Department of Mathematics and Informatics,  
Veliko Tarnovo University, Bulgaria
- Snezana Scepanovic**, Faculty for Information Technology,  
University “Mediterranean”, Podgorica, Montenegro
- Daniela Veleva Minkovska**, Faculty of Computer Systems and Technologies,  
Technical University, Sofia, Bulgaria
- Stefka Hristova Bouyuklieva**, Department of Algebra and Geometry,  
Faculty of Mathematics and Informatics, Veliko Tarnovo University, Bulgaria
- Vesselin Velichkov**, University of Luxembourg, Faculty of Sciences,  
Technology and Communication (FSTC), Luxembourg
- Isabel Maria Baltazar Simões de Carvalho**, Instituto Superior Técnico,  
Technical University of Lisbon, Portugal
- Predrag S. Stanimirović**, University of Niš, Faculty of Sciences and Mathematics,  
Department of Mathematics and Informatics, Niš, Serbia
- Shcherbacov Victor**, Institute of Mathematics and Computer Science,  
Academy of Sciences of Moldova, Moldova
- Pedro Ricardo Morais Inácio**, Department of Computer Science,  
Universidade da Beira Interior, Portugal
- Georgi Tuparov**, Technical University of Sofia Bulgaria
- Martin Lukarevski**, Faculty of Computer Science, UGD, Republic of North Macedonia
- Ivanka Georgieva**, South-West University, Blagoevgrad, Bulgaria
- Georgi Stojanov**, Computer Science, Mathematics, and Environmental Science Department  
The American University of Paris, France
- Iliya Guerguiev Bouyukliev**, Institute of Mathematics and Informatics,  
Bulgarian Academy of Sciences, Bulgaria
- Riste Škrekovski**, FAMNIT, University of Primorska, Koper, Slovenia
- Stela Zhelezova**, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Bulgaria
- Katerina Taskova**, Computational Biology and Data Mining Group,  
Faculty of Biology, Johannes Gutenberg-Universität Mainz (JGU), Mainz, Germany.
- Dragana Glušac**, Tehnical Faculty “Mihajlo Pupin”, Zrenjanin, Serbia
- Cveta Martinovska-Bande**, Faculty of Computer Science, UGD, Republic of North Macedonia
- Blagoj Delipetrov**, European Commission Joint Research Centre, Italy
- Zoran Zdravev**, Faculty of Computer Science, UGD, Republic of North Macedonia
- Aleksandra Mileva**, Faculty of Computer Science, UGD, Republic of North Macedonia
- Igor Stojanovik**, Faculty of Computer Science, UGD, Republic of North Macedonia
- Saso Koceski**, Faculty of Computer Science, UGD, Republic of North Macedonia
- Natasa Koceska**, Faculty of Computer Science, UGD, Republic of North Macedonia
- Aleksandar Krstev**, Faculty of Computer Science, UGD, Republic of North Macedonia
- Biljana Zlatanovska**, Faculty of Computer Science, UGD, Republic of North Macedonia
- Natasa Stojkovik**, Faculty of Computer Science, UGD, Republic of North Macedonia
- Done Stojanov**, Faculty of Computer Science, UGD, Republic of North Macedonia
- Limonka Koceva Lazarova**, Faculty of Computer Science, UGD, Republic of North Macedonia
- Tatjana Atanasova Pacemska**, Faculty of Computer Science, UGD, Republic of North Macedonia



---

## TABLE OF CONTENTS

<b>Aleksandra Risteska-Kamcheski</b> SOLUTION OF DIDO’S PROBLEM USING VARIATIONS .....	7
<b>Mirjana Kocaleva Vitanova, Elena Karamazova Gelova, Zoran Zlatev, Aleksandar Krstev</b> ENHANCING GEOGRAPHIC INFORMATION SYSTEMS WITH SPATIAL DATA MINING .....	19
<b>Violeta Krcheva, Misa Tomic</b> ADVANCED TOOLPATH VERIFICATION IN CNC DRILLING: APPLYING NEWTON’S INTERPOLATION THROUGH MATLAB .....	31
<b>Martin Tanchev, Saso Koceski</b> WEB-BASED EDUCATIONAL GAME FOR EARLY SCREENING AND SUPPORT OF DYSCALCULIA .....	43
<b>Maja Kukuseva Paneva, Elena Zafirova, Sara Stefanova, Goce Stefanov</b> MONITORING AND TRANSMISSION OF THE PROGRESS PARAMETERS ON AGRO INDUSTRIAL FACILITY IN A GSM NETWORK .....	55
<b>Qazim Tahiri, Natasa Koceska</b> METHODS OF EXTRACTION AND ANALYSIS OF PEOPLE’S SENTIMENTS FROM SOCIAL MEDIA .....	69
<b>Ana Eftimova, Saso Gelev</b> DESIGN AND SIMULATION OF A SCADA – CONTROLLED GREENHOUSE FOR OPTIMIZED ROSE CULTIVATION .....	81
<b>Milka Anceva, Saso Koceski</b> A FHIR – CENTRIC APPROACH FOR INTEROPERABLE REMOTE PATIENT MONITORING .....	93
<b>Jordan Pop-Kartov, Aleksandra Mileva, Cveta Martinovska Bande</b> COMPARATIVE EVALUTION AND ANALYSIS OF DIFFERENT DEEPPFAKE DETECTORS .....	103
<b>Vesna Hristovska, Aleksandar Velinov, Natasa Koceska</b> SECURITY CHALLENGES AND SOLUTIONS IN ROBOTIC AND INTERNET OF ROBOTIC THINGS (IoRT) SYSTEMS: A SCOPING REVIEW .....	115
<b>Violeta Krcheva, Misa Tomic</b> CNC LATHE PROGRAMMING: DESIGN AND DEVELOPMENT OF A PROGRAM CODE FOR SIMULATING LINEAR INTERPOLATION MOTION .....	127
<b>Jawad Ettayb</b> NEW RESULTS ON FIXED POINT THEOREMS IN 2-BANACH SPACES .....	139



## NEW RESULTS ON FIXED POINT THEOREMS IN 2-BANACH SPACES

JAWAD ETTAYB

**Abstract.** In this article, we prove a common fixed point result for interpolative Kannan type contraction mappings in 2-Banach spaces. On the other hand, we introduce interpolative Dass and Gupta rational type contraction mappings on 2-Banach spaces. In particular, we discuss the existence of a fixed point of such a mapping in 2-Banach spaces.

### 1. Introduction

In a 2-Banach space framework, S. Gähler [5] initiated the study of 2-normed spaces. Recently, A. White [9] initiated and studied the concept of 2-Banach spaces. For more details on 2-normed spaces and 2-Banach spaces, we refer to see [4] and [5].

P. K. Harikrishnan and K. T. Ravindran [6] demonstrated that a contraction mapping has a one fixed point in bounded and closed subsets of a 2-Banach space. However, M. Kir and H. Kiziltunc [7] demonstrated several results on fixed points in 2-Banach spaces.

In [2], J. Ettayb introduced the notions of Meir-Keeler contraction mappings and Ćirić contraction mappings on a 2-Banach space. In particular, we discuss the existence and uniqueness of a fixed point of such mappings in a 2-Banach space. On the other hand, he defined the concept of Hardy-Rogers contraction mappings on a 2-Banach space. In particular, he proved the existence and uniqueness of a fixed point of such a mapping in a 2-Banach space. However, several results are demonstrated on fixed point theorems of some mappings in 2-Banach spaces. Recently, J. Ettayb [3] demonstrated a fixed point theorem for generalized weakly contractive mappings in 2-Banach spaces which is a generalization of a Banach contraction mapping principle. On the other hand, he introduced interpolative Boyd-Wong and Matkowski type contractions on 2-Banach spaces. As a result, he proved the existence and uniqueness of a fixed point of such mappings in 2-Banach spaces. In the framework of metric spaces, B. K. Dass and S. Gupta [1] obtained

---

*Date:* December 19, 2025.

**Keywords.** Fixed point theorems, mappings, 2-Banach spaces, interpolative rational type contractions.

an extension of Banach contraction principle through rational expression. On the other hand, M. Noorwali [8] proved a common fixed point result for interpolative Kannan contraction mappings.

The fixed point theory played a crucial role in functional analysis. Moreover, it used in many branches of science such as biology, chemistry, economics, engineering and computer science.

In this article, we prove a common fixed point result for interpolative Kannan type contraction mappings in 2-Banach spaces. On the other hand, we introduce interpolative Dass and Gupta rational type contraction mappings on 2-Banach spaces. In particular, we discuss the existence of a fixed point of such a mapping in 2-Banach spaces.

## 2. Preliminaries

We begin with preliminaries:

**Definition 2.1** ([5]). Let  $\mathcal{X}$  be a real vector space with  $\dim \mathcal{X} \geq 2$ . A 2-norm on  $\mathcal{X}$  is a function  $\|\cdot, \cdot\| : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$  such that

- (i)  $\|x, y\| = 0$  if and only if  $x$  and  $y$  are linearly dependent.
- (ii) For all  $x, y \in \mathcal{X}$ ,  $\|x, y\| = \|y, x\|$ .
- (iii) For any  $x, y \in \mathcal{X}$  and  $\lambda \in \mathbb{R}$ ,  $\|\lambda x, y\| = |\lambda| \|x, y\|$ .
- (iv) For each  $x, y, z \in \mathcal{X}$ ,  $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ .

The pair  $(\mathcal{X}, \|\cdot, \cdot\|)$  is called a 2-normed space.

**Definition 2.2** ([4]). Let  $\mathcal{X}$  be a 2-normed space. A sequence  $\{x_n\}$  in  $\mathcal{X}$  is said to be a Cauchy sequence if  $\lim_{m, n \rightarrow \infty} \|x_n - x_m, y\| = 0$  for any  $y \in \mathcal{X}$ .

**Definition 2.3** ([9]). Let  $\mathcal{X}$  be a 2-normed space. A sequence  $\{x_n\}$  in  $\mathcal{X}$  converges in  $\mathcal{X}$  if there exists an element  $x \in \mathcal{X}$  such that for all  $y \in \mathcal{X}$ ,  $\lim_{n \rightarrow \infty} \|x_n - x, y\| = 0$ . If  $\{x_n\}$  converges to  $x$ , we write  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

**Definition 2.4** ([9]). A 2-normed space in which every Cauchy sequence converges will be called a 2-Banach space.

**Lemma 2.1** ([4]). Let  $\mathcal{X}$  be a 2-normed space and  $x \in \mathcal{X}$ . If  $\|x, y\| = 0$  for all  $y \in \mathcal{X}$ , then  $x = 0$ .

**Definition 2.5** ([6]). Let  $\mathcal{X}$  be a 2-normed space then the mapping  $S : \mathcal{X} \rightarrow \mathcal{X}$  is said to be a contraction if there exists  $k \in (0, 1)$  such that

$$\|Sx - Sy, z\| \leq k \|x - y, z\| \quad (2.1)$$

for all  $x, y, z \in \mathcal{X}$ .



**Theorem 2.1** ([6]). Let  $\mathcal{X}$  be a 2-normed space and  $\mathcal{F}$  be a nonempty closed and bounded subset of  $\mathcal{X}$ . Let  $S : \mathcal{F} \rightarrow \mathcal{F}$  be a contraction, then  $S$  has a unique fixed point.

**Definition 2.6** ([2]). A mapping  $S$  on a 2-normed space  $\mathcal{X}$  is called a Meir-Keeler contraction if given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for each  $x, y, z \in \mathcal{X}$ ,

$$\varepsilon \leq \|x - y, z\| < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| < \varepsilon. \quad (2.2)$$

**Definition 2.7** ([2]). A mapping  $S$  on a 2-normed space  $\mathcal{X}$  is called a Ćirić contraction if given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for each  $x, y, z \in \mathcal{X}$ ,

$$\varepsilon < \|x - y, z\| < \varepsilon + \delta \Rightarrow \|Sx - Sy, z\| \leq \varepsilon. \quad (2.3)$$

**Theorem 2.2** ([2]). Let  $\mathcal{X}$  be a 2-Banach space and let  $S$  be a Meir-Keeler contraction on  $\mathcal{X}$ , then  $S$  has a unique fixed point in  $\mathcal{X}$ .

**Theorem 2.3** ([2]). Let  $\mathcal{X}$  be a 2-Banach space and let  $S$  be a Ćirić contraction on  $\mathcal{X}$ , then  $S$  has a unique fixed point in  $\mathcal{X}$ .

**Definition 2.8** ([2]). A mapping  $S$  on a 2-normed space  $\mathcal{X}$  is called a Hardy-Rogers contraction if  $S$  is a self-mapping on  $\mathcal{X}$  satisfying for each  $x, y, z \in \mathcal{X}$ ,

$$\begin{aligned} \|Sx - Sy, z\| &\leq a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| \\ &\quad + e\|y - Sx, z\| + f\|x - y, z\| \end{aligned} \quad (2.4)$$

where  $a, b, c, e, f$  are nonnegative and we put  $\beta = a + b + c + e + f$ .

Consider the following condition

$$\begin{aligned} x \neq y \Rightarrow \|Sx - Sy, z\| &< a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| \\ &\quad + e\|y - Sx, z\| + f\|x - y, z\|. \end{aligned} \quad (2.5)$$

**Theorem 2.4** ([2]). Let  $\mathcal{X}$  be a 2-normed space and  $S$  a self-mapping on  $\mathcal{X}$  satisfying for each  $x, y, z \in \mathcal{X}$ ,

$$\begin{aligned} \|Sx - Sy, z\| &\leq a\|x - Sx, z\| + b\|y - Sy, z\| + c\|x - Sy, z\| \\ &\quad + e\|y - Sx, z\| + f\|x - y, z\| \end{aligned} \quad (2.6)$$

where  $a, b, c, e, f$  are nonnegative and we put  $\beta = a + b + c + e + f$ .

(i) If  $\mathcal{X}$  is complete and  $\beta < 1$ , then  $S$  has a unique fixed point.

(ii) If (2.4) is modified to the condition (2.5) and in this case  $\mathcal{X}$  is compact,  $S$  is continuous and  $\beta = 1$ , then  $S$  has a unique fixed point.

**Theorem 2.5** ([2]). Let  $\mathcal{X}$  be a 2-Banach space,  $a, b, c, e, f$  be monotonically decreasing functions from  $[0, \infty)$  to  $[0, 1)$  and let the sum of these five functions be less than 1. Assume that  $S : \mathcal{X} \rightarrow \mathcal{X}$  satisfies condition (2.4) with  $a = a(\|x - y, z\|), \dots, f = f(\|x - y, z\|)$  for each  $x, y, z \in \mathcal{X}$ . Then  $S$  has a unique fixed point.

**Theorem 2.6** ([2]). *Let  $\mathcal{X}$  be a 2-Banach space and  $S_n : \mathcal{X} \rightarrow \mathcal{X}$ ,  $n = 1, 2, \dots$  satisfy the conditions of Theorem 2.5 with the coefficients  $a, b, c, e, f$ . Let  $S_n x_n = x_n$  and assume that  $S_n \rightarrow S$  pointwise on  $\mathcal{X}$ . Then  $x = \lim_{n \rightarrow \infty} x_n$  is the unique fixed point of  $S$ .*

**Theorem 2.7** ([2]). *Let  $\mathcal{X}$  be a 2-Banach space and  $S_n : \mathcal{X} \rightarrow \mathcal{X}$ ,  $n = 1, 2, \dots$  be functions with at least one fixed point  $x_n$ ,  $n = 1, 2, \dots$ . Let  $S$  satisfy the hypothesis of Theorem 2.5 and  $S_n \rightarrow S$  uniformly on  $\mathcal{X}$ . Then  $x = \lim_{n \rightarrow \infty} x_n$  is the unique fixed point of  $S$ .*

Ettayb [2] established a generalization of Banach's principle of contraction mappings in 2-Banach spaces as follows.

**Theorem 2.8** ([2]). *Let  $\mathcal{X}$  be a 2-Banach space and  $\mathcal{F}$  be a nonempty closed and bounded subset of  $\mathcal{X}$ . If  $S : \mathcal{F} \rightarrow \mathcal{F}$  is a continuous mapping such that  $S^k$  is a contraction for some  $k \geq 1$ , then  $S$  has a unique fixed point.*

### 3. Main results

We present our main results.

**Theorem 3.1.** *Let  $\mathcal{X}$  be a 2-Banach space and let  $S, T$  be self mappings on  $\mathcal{X}$  satisfying for each  $x \in \mathcal{X} \setminus \text{Fix}(S)$ ,  $y \in \mathcal{X} \setminus \text{Fix}(T)$  and  $z \in \mathcal{X}$ ,*

$$\|Sx - Ty, z\| \leq \lambda \|x - Sx, z\|^\alpha \cdot \|y - Ty, z\|^{1-\alpha} \quad (3.1)$$

where  $\lambda \in [0, 1)$  and  $\alpha \in (0, 1)$ , then  $S$  and  $T$  have a common fixed point.

*Proof.* Let  $x_0 \in \mathcal{X}$  and let  $\{x_n\}$  be a sequence in  $\mathcal{X}$  defined by  $x_{2n+1} = Sx_{2n}$  and  $x_{2n+2} = Tx_{2n+1}$  for each  $n \in \mathbb{N}_0$ . If there exists  $n \in \mathbb{N}_0$  such that  $x_n = x_{n+1} = x_{n+2}$ , then  $x_n$  is a common fixed point of  $S$  and  $T$ , so assume that there does not exist three consecutive identical terms in the sequence  $\{x_n\}$  and that  $x_0 \neq x_1$ . Using (3.1), we obtain for all  $z \in \mathcal{X}$ ,

$$\begin{aligned} \|x_{2n+1} - x_{2n+2}, z\| &= \|Sx_{2n} - Tx_{2n+1}, z\| \\ &\leq \lambda \|x_{2n} - x_{2n+1}, z\|^\alpha \cdot \|x_{2n+1} - x_{2n+2}, z\|^{1-\alpha}. \end{aligned}$$

Then

$$\|x_{2n+1} - x_{2n+2}, z\|^\alpha \leq \lambda \|x_{2n} - x_{2n+1}, z\|^\alpha.$$

Hence

$$\begin{aligned} \|x_{2n+1} - x_{2n+2}, z\| &\leq \lambda^{\frac{1}{\alpha}} \|x_{2n} - x_{2n+1}, z\| \\ &\leq \lambda \|x_{2n} - x_{2n+1}, z\|. \end{aligned} \quad (3.2)$$

From (3.2), we obtain for all  $z \in \mathcal{X}$ ,

$$\|x_{2n+1} - x_{2n+2}, z\| \leq \lambda \|x_{2n} - x_{2n+1}, z\| \leq \lambda^2 \|x_{2n-1} - x_{2n}, z\| \leq \dots \leq \lambda^{2n+1} \|x_0 - x_1, z\|.$$

Then

$$\|x_{2n+1} - x_{2n+2}, z\| \leq \lambda^{2n+1} \|x_0 - x_1, z\|. \quad (3.3)$$

Similarly,

$$\begin{aligned} \|x_{2n+1} - x_{2n}, z\| &= \|Sx_{2n} - Tx_{2n-1}, z\| \\ &\leq \lambda \|x_{2n} - x_{2n+1}, z\|^\alpha \cdot \|x_{2n-1} - x_{2n}, z\|^{1-\alpha}. \end{aligned}$$

Then

$$\|x_{2n+1} - x_{2n}, z\|^{1-\alpha} \leq \lambda \|x_{2n-1} - x_{2n}, z\|^{1-\alpha}.$$

Hence

$$\begin{aligned} \|x_{2n+1} - x_{2n}, z\| &\leq \lambda^{\frac{1}{1-\alpha}} \|x_{2n-1} - x_{2n}, z\| \\ &\leq \lambda \|x_{2n-1} - x_{2n}, z\|. \end{aligned}$$

Thus

$$\|x_{2n+1} - x_{2n}, z\| \leq \lambda \|x_{2n-1} - x_{2n}, z\| \leq \lambda^2 \|x_{2n-2} - x_{2n-1}, z\| \leq \cdots \leq \lambda^{2n} \|x_0 - x_1, z\|.$$

Then

$$\|x_{2n+1} - x_{2n}, z\| \leq \lambda^{2n} \|x_0 - x_1, z\|. \quad (3.4)$$

Using (3.3) and (3.4), we obtain

$$\|x_n - x_{n+1}, z\| \leq \lambda^n \|x_0 - x_1, z\|. \quad (3.5)$$

Now, by (3.5) we demonstrate that  $\{x_n\}$  is Cauchy. Let  $m, n > 0$  with  $m > n$ . Put  $m = n + l$ , hence for any  $z \in \mathcal{X}$ ,

$$\begin{aligned} \|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\ &= \| (x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z \| \\ &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{n+l-1} - x_{n+l}, z\| \\ &\leq \lambda^n \|x_0 - x_1, z\| + \lambda^{n+1} \|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1} \|x_0 - x_1, z\| \\ &= \lambda^n (1 + \lambda + \cdots + \lambda^{l-1}) \|x_0 - x_1, z\| \\ &\leq \frac{\lambda^n}{1 - \lambda} \|x_0 - x_1, z\|. \end{aligned}$$

Letting  $n, m \rightarrow \infty$ , we get

$$\lim_{m, n \rightarrow \infty} \|x_n - x_m, z\| = 0.$$

Hence  $\{x_n\}$  is a Cauchy sequence in  $\mathcal{X}$ . Then  $\{x_n\}$  converges to some  $x \in \mathcal{X}$ . Now, we demonstrate that  $x \in \mathcal{X}$  is a fixed point of  $S$ . Hence for each  $z \in \mathcal{X}$ ,

$$\begin{aligned} \|Sx - x_{2n+2}, z\| &= \|Sx - Tx_{2n+1}, z\| \\ &\leq \lambda \|x - Sx, z\|^\alpha \cdot \|x_{2n+1} - x_{2n+2}, z\|^{1-\alpha}. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we get

$$\|Sx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

Using Lemma 2.1, we obtain  $Sx = x$ . Similarly

$$\begin{aligned} \|x_{2n+1} - Tx, z\| &= \|Sx_{2n} - Tx, z\| \\ &\leq \lambda \|x_{2n} - x_{2n+1}, z\|^\alpha \cdot \|x - Tx, z\|^{1-\alpha}. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we get

$$\|Tx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

From Lemma 2.1, we have  $x = Tx$ . Then  $x = Sx = Tx$ .  $\square$

**Theorem 3.2.** *Let  $\mathcal{X}$  be a 2-Banach space and let  $S$  be a self mapping on  $\mathcal{X}$  satisfying for each  $x, y, z \in \mathcal{X}$ ,*

$$\|Sx - Sy, z\| \leq k_1 \|x - y, z\| + k_2 \frac{\|x - Sx, z\| \|y - Sy, z\|}{1 + \|x - y, z\|} \quad (3.6)$$

where  $k_1, k_2 \geq 0$  such that  $k_1 + k_2 < 1$ , then  $S$  has a unique fixed point in  $\mathcal{X}$ .

*Proof.* Let  $x_0 \in \mathcal{X}$ . Define the sequence  $\{x_n\}$  in  $\mathcal{X}$  by  $x_n = Sx_{n-1} = S^n x_0$  for each  $n \in \mathbb{N}$ . Putting  $x = x_{n-1}$  and  $y = x_n$  in (3.6), we obtain for all  $z \in \mathcal{X}$ ,

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Sx_{n-1} - Sx_n, z\| \\ &\leq k_1 \|x_{n-1} - x_n, z\| + k_2 \frac{\|x_{n-1} - Sx_{n-1}, z\| \|x_n - Sx_n, z\|}{1 + \|x_{n-1} - x_n, z\|} \\ &= k_1 \|x_{n-1} - x_n, z\| + k_2 \frac{\|x_{n-1} - x_n, z\| \|x_n - x_{n+1}, z\|}{1 + \|x_{n-1} - x_n, z\|} \\ &\leq k_1 \|x_{n-1} - x_n, z\| + k_2 \|x_n - x_{n+1}, z\|. \end{aligned}$$

Then

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\| \quad (3.7)$$

where  $\lambda = \frac{k_1}{1-k_2}$ . Since  $\lambda = \frac{k_1}{1-k_2} < 1$ , we deduce that the sequence  $\{\|x_n - x_{n+1}, z\|\}$  is decreasing. From (3.7), we obtain for all  $z \in \mathcal{X}$ ,

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\| \leq \lambda^n \|x_0 - x_1, z\|. \quad (3.8)$$

Letting  $n \rightarrow \infty$  in (3.8), we obtain  $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}, z\| = 0$  for all  $z \in \mathcal{X}$ . Now, we demonstrate that  $\{x_n\}$  is Cauchy. Let  $m, n > 0$  with  $m > n$ . Put  $m = n + l$ , hence for any  $z \in \mathcal{X}$ ,

$$\begin{aligned} \|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\ &= \| (x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z \| \\ &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{n+l-1} - x_{n+l}, z\| \\ &\leq \lambda^n \|x_0 - x_1, z\| + \lambda^{n+1} \|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1} \|x_0 - x_1, z\| \end{aligned}$$

$$\begin{aligned}
&= \lambda^n(1 + \lambda + \cdots + \lambda^{l-1})\|x_0 - x_1, z\| \\
&\leq \frac{\lambda^n}{1 - \lambda}\|x_0 - x_1, z\|.
\end{aligned}$$

Letting  $n, m \rightarrow \infty$ , we get

$$\lim_{m, n \rightarrow \infty} \|x_n - x_m, z\| = 0.$$

Hence  $\{x_n\}$  is Cauchy. Then  $\{x_n\}$  converges to some  $x \in \mathcal{X}$ . Now, we demonstrate that  $x \in \mathcal{X}$  is a fixed point of  $S$ . Hence for each  $z \in \mathcal{X}$ ,

$$\begin{aligned}
\|x_{n+1} - Sx, z\| &= \|Sx_n - Sx, z\| \\
&\leq k_1\|x_n - x, z\| + k_2 \frac{\|x_n - Sx_n, z\|\|x - Sx, z\|}{1 + \|x_n - x, z\|}.
\end{aligned}$$

Letting  $n \rightarrow \infty$ , we get

$$\|Sx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

By Lemma 2.1, we have  $Sx = x$ . Then  $x$  is a fixed point of  $S$ . Suppose that  $x_1$  and  $x_2$  are two distinct fixed points of  $S$ . From for all  $z \in \mathcal{X}$ ,

$$\begin{aligned}
\|x_1 - x_2, z\| &= \|Sx_1 - Sx_2, z\| \\
&\leq k_1\|x_1 - x_2, z\| + k_2 \frac{\|x_1 - Sx_1, z\|\|x_2 - Sx_2, z\|}{1 + \|x_1 - x_2, z\|} \\
&= k_1\|x_1 - x_2, z\| + k_2 \frac{\|x_1 - x_1, z\|\|x_2 - x_2, z\|}{1 + \|x_1 - x_2, z\|} \\
&= k_1\|x_1 - x_2, z\|
\end{aligned}$$

which contradicts that  $k_1 < 1$ . Therefore  $\|x_1 - x_2, z\| = 0$  for all  $z \in \mathcal{X}$ . Then by Lemma 2.1, it is established that  $x_1 = x_2$ .  $\square$

Similarly to the proof of Theorem 3.2, we obtain:

**Theorem 3.3.** *Let  $\mathcal{X}$  be a 2-Banach space and let  $S$  be a self mapping on  $\mathcal{X}$  satisfying for each  $x, y, z \in \mathcal{X}$ ,*

$$\|Sx - Sy, z\| \leq k_1\|x - y, z\| + k_2 \frac{\|x - Sy, z\|\|y - Sx, z\|}{1 + \|x - y, z\|} \quad (3.9)$$

where  $k_1, k_2 \geq 0$  such that  $k_1 + k_2 < 1$ , then  $S$  has a unique fixed point in  $\mathcal{X}$ .

Now, we introduce an interpolative Dass and Gupta rational type contraction in a 2-normed space as follows.

**Definition 3.1.** A continuous self-mapping  $S$  on a 2-normed space  $\mathcal{X}$  is called an interpolative Dass and Gupta rational type contraction if there exist  $\lambda \in [0, 1)$  and  $\alpha \in (0, 1)$  such that

$$\|Sx - Sy, z\| \leq \lambda \left[ \frac{\|x - Sx, z\| \|y - Sy, z\|}{\|x - y, z\|} \right]^\alpha \cdot [\|x - y, z\|]^{1-\alpha} \quad (3.10)$$

for each  $x, y \in \mathcal{X} \setminus \text{Fix}(S)$  with  $x \neq y$  and  $z \in \mathcal{X}$  where  $\text{Fix}(S) = \{u \in \mathcal{X} : Su = u\}$ .

**Theorem 3.4.** Let  $\mathcal{X}$  be a 2-Banach space and let  $S$  be an interpolative Dass and Gupta rational type contraction on  $\mathcal{X}$ , then  $S$  has a fixed point in  $\mathcal{X}$ .

*Proof.* Let  $x_0 \in \mathcal{X}$ . We will set a constructive sequence  $\{x_n\}$  by  $x_n = Sx_{n-1} = S^n x_0$  for each  $n \in \mathbb{N}$ . Putting  $x = x_n$  and  $y = x_{n-1}$  in (3.10), we obtain for all  $z \in \mathcal{X}$ ,

$$\begin{aligned} \|x_{n+1} - x_n, z\| &= \|Sx_n - Sx_{n-1}, z\| \\ &\leq \lambda \left[ \frac{\|x_n - Sx_n, z\| \|x_{n-1} - Sx_{n-1}, z\|}{\|x_n - x_{n-1}, z\|} \right]^\alpha \cdot [\|x_n - x_{n-1}, z\|]^{1-\alpha} \\ &\leq \lambda [\|x_n - x_{n+1}, z\|]^\alpha \cdot [\|x_n - x_{n-1}, z\|]^{1-\alpha}. \end{aligned}$$

Then

$$\|x_n - x_{n+1}, z\|^{1-\alpha} \leq \lambda \|x_{n-1} - x_n, z\|^{1-\alpha}. \quad (3.11)$$

Hence

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\|. \quad (3.12)$$

Thus, we deduce that the sequence  $\{\|x_n - x_{n+1}, z\|\}$  is decreasing. As a result, there exists  $M \in \mathbb{R}_+$  such that  $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}, z\| = M$  for all  $z \in \mathcal{X}$ . We will indicate that  $M = 0$ . Indeed, by (3.12), we derive that

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\| \leq \lambda^n \|x_0 - x_1, z\|. \quad (3.13)$$

Letting  $n \rightarrow \infty$  in (3.13), we obtain  $M = 0$ . Now, we demonstrate that  $\{x_n\}$  is Cauchy. Let  $m, n > 0$  with  $m > n$ . Put  $m = n + l$ , hence for any  $z \in \mathcal{X}$ ,

$$\begin{aligned} \|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\ &= \| (x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z \| \\ &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{n+l-1} - x_{n+l}, z\| \\ &\leq \lambda^n \|x_0 - x_1, z\| + \lambda^{n+1} \|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1} \|x_0 - x_1, z\| \\ &= \lambda^n (1 + \lambda + \cdots + \lambda^{l-1}) \|x_0 - x_1, z\| \\ &\leq \frac{\lambda^n}{1 - \lambda} \|x_0 - x_1, z\|. \end{aligned}$$

Letting  $n, m \rightarrow \infty$ , we get

$$\lim_{m, n \rightarrow \infty} \|x_n - x_m, z\| = 0.$$

Hence  $\{x_n\}$  is Cauchy. Then  $\{x_n\}$  converges to some  $x \in \mathcal{X}$ . Now, we demonstrate that  $x \in \mathcal{X}$  is a fixed point of  $S$ . Since  $S$  is continuous on  $\mathcal{X}$ , we obtain

$$x = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} Sx_n = S(\lim_{n \rightarrow \infty} x_n) = Sx.$$

Consequently  $x$  is a fixed point of  $S$ .  $\square$

We obtain the following theorem.

**Theorem 3.5.** *Let  $\mathcal{X}$  be a 2-Banach space and let  $S, T$  be self mappings on  $\mathcal{X}$  satisfying for each  $x, y \in \mathcal{X} \setminus \text{Fix}(S)$  and  $z \in \mathcal{X}$ ,*

$$\|Sx - Sy, z\| \leq \lambda \left[ \frac{\|x - Sx, z\| \|y - Sy, z\|}{1 + \|x - y, z\|} \right]^\alpha \cdot [\|x - y, z\|]^{1-\alpha} \quad (3.14)$$

where  $\lambda, \alpha \in (0, 1)$ , then  $S$  has a fixed point in  $\mathcal{X}$ .

*Proof.* Let  $x_0 \in \mathcal{X}$ . We will set a constructive sequence  $\{x_n\}$  by  $x_n = S^n x_0$  for each  $n \in \mathbb{N}$ . Putting  $x = x_n$  and  $y = x_{n-1}$  in (3.14), we obtain for all  $z \in \mathcal{X}$ ,

$$\begin{aligned} \|x_{n+1} - x_n, z\| &= \|Sx_n - Sx_{n-1}, z\| \\ &\leq \lambda \left[ \frac{\|x_n - Sx_n, z\| \|x_{n-1} - Sx_{n-1}, z\|}{1 + \|x_n - x_{n-1}, z\|} \right]^\alpha \cdot [\|x_n - x_{n-1}, z\|]^{1-\alpha} \\ &\leq \lambda [\|x_n - x_{n+1}, z\|]^\alpha \cdot [\|x_n - x_{n-1}, z\|]^{1-\alpha}. \end{aligned}$$

Then

$$\|x_n - x_{n+1}, z\|^{1-\alpha} \leq \lambda \|x_{n-1} - x_n, z\|^{1-\alpha}. \quad (3.15)$$

Hence

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\|. \quad (3.16)$$

Thus, we deduce that the sequence  $\{\|x_n - x_{n+1}, z\|\}$  is decreasing. As a result, there exists  $M \in \mathbb{R}_+$  such that  $\lim_{n \rightarrow \infty} \|x_n - x_{n+1}, z\| = M$  for all  $z \in \mathcal{X}$ . We will indicate that  $M = 0$ . Indeed, by (3.16), we derive that

$$\|x_n - x_{n+1}, z\| \leq \lambda \|x_{n-1} - x_n, z\| \leq \lambda^n \|x_0 - x_1, z\|. \quad (3.17)$$

Letting  $n \rightarrow \infty$  in (3.17), we obtain  $M = 0$ . Now, we demonstrate that  $\{x_n\}$  is Cauchy. Let  $m, n > 0$  with  $m > n$ . Put  $m = n + l$ , hence for any  $z \in \mathcal{X}$ ,

$$\begin{aligned} \|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\ &= \| (x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z \| \\ &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{n+l-1} - x_{n+l}, z\| \\ &\leq \lambda^n \|x_0 - x_1, z\| + \lambda^{n+1} \|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1} \|x_0 - x_1, z\| \\ &= \lambda^n (1 + \lambda + \cdots + \lambda^{l-1}) \|x_0 - x_1, z\| \\ &\leq \frac{\lambda^n}{1 - \lambda} \|x_0 - x_1, z\|. \end{aligned}$$

Letting  $n, m \rightarrow \infty$ , we get

$$\lim_{m, n \rightarrow \infty} \|x_n - x_m, z\| = 0.$$

Hence  $\{x_n\}$  is Cauchy. Then  $\{x_n\}$  converges to some  $x \in \mathcal{X}$ . Now, we demonstrate that  $x \in \mathcal{X}$  is a fixed point of  $S$ . Hence for each  $z \in \mathcal{X}$ ,

$$\|x_{n+1} - Sx, z\| = \|Sx_n - Sx, z\| \leq \lambda \left[ \frac{\|x_n - Sx_n, z\| \|x - Sx, z\|}{1 + \|x_n - x, z\|} \right]^\alpha \cdot [\|x_n - x, z\|]^{1-\alpha}.$$

Letting  $n \rightarrow \infty$ , we get

$$\|Sx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

From Lemma 2.1, it follows that  $Sx = x$ . Consequently  $x$  is a fixed point of  $S$ .  $\square$

Now, we prove the next theorem.

**Theorem 3.6.** *Let  $\mathcal{X}$  be a 2-Banach space and let  $S, T$  be self mappings on  $\mathcal{X}$  satisfying for each  $x \in \mathcal{X} \setminus \text{Fix}(S), y \in \mathcal{X} \setminus \text{Fix}(T)$  and  $z \in \mathcal{X}$ ,*

$$\|Sx - Ty, z\| \leq \lambda \left[ \frac{\|x - Sx, z\| \|y - Ty, z\|}{1 + \|x - y, z\|} \right]^\alpha \cdot [\|x - y, z\|]^{1-\alpha} \quad (3.18)$$

where  $\lambda, \alpha \in (0, 1)$ , then  $S$  and  $T$  have a common fixed point.

*Proof.* Let  $x_0 \in \mathcal{X}$ . Define a sequence  $\{x_n\}$  in  $\mathcal{X}$  by  $x_{2n+1} = Sx_{2n}$  and  $x_{2n+2} = Tx_{2n+1}$  for each  $n \in \mathbb{N}_0$ . If there exists  $n \in \mathbb{N}_0$  such that  $x_{2n} = x_{2n+1} = x_{2n+2}$ , then  $x_{2n}$  is a common fixed point of  $S$  and  $T$ , so assume that there does not exist three consecutive identical terms in the sequence  $\{x_n\}$  and that  $x_0 \neq x_1$ . Using (3.18), we obtain for all  $z \in \mathcal{X}$ ,

$$\begin{aligned} \|x_{2n+1} - x_{2n+2}, z\| &= \|Sx_{2n} - Tx_{2n+1}, z\| \\ &\leq \lambda \left[ \frac{\|x_{2n} - Sx_{2n}, z\| \|x_{2n+1} - Tx_{2n+1}, z\|}{1 + \|x_{2n} - x_{2n+1}, z\|} \right]^\alpha \cdot \|x_{2n} - x_{2n+1}, z\|^{1-\alpha} \\ &\leq \lambda \|x_{2n+1} - x_{2n+2}, z\|^\alpha \cdot \|x_{2n} - x_{2n+1}, z\|^{1-\alpha}. \end{aligned}$$

Then

$$\|x_{2n+1} - x_{2n+2}, z\|^{1-\alpha} \leq \lambda \|x_{2n} - x_{2n+1}, z\|^{1-\alpha}. \quad (3.19)$$

Hence

$$\begin{aligned} \|x_{2n+1} - x_{2n+2}, z\| &\leq \lambda^{\frac{1}{1-\alpha}} \|x_{2n} - x_{2n+1}, z\| \\ &\leq \lambda \|x_{2n} - x_{2n+1}, z\|. \end{aligned} \quad (3.20)$$

From (3.20), we obtain for all  $z \in \mathcal{X}$ ,

$$\|x_{2n+1} - x_{2n+2}, z\| \leq \lambda \|x_{2n} - x_{2n+1}, z\| \leq \lambda^2 \|x_{2n-1} - x_{2n}, z\| \leq \dots \leq \lambda^{2n+1} \|x_0 - x_1, z\|.$$

Then

$$\|x_{2n+1} - x_{2n+2}, z\| \leq \lambda^{2n+1} \|x_0 - x_1, z\|. \quad (3.21)$$



Similarly,

$$\begin{aligned}
 \|x_{2n+1} - x_{2n}, z\| &= \|Sx_{2n} - Tx_{2n-1}, z\| \\
 &\leq \lambda \left[ \frac{\|x_{2n} - Sx_{2n}, z\| \|x_{2n-1} - Tx_{2n-1}, z\|}{1 + \|x_{2n} - x_{2n-1}, z\|} \right]^\alpha \cdot [\|x_{2n-1} - x_{2n}, z\|]^{1-\alpha} \\
 &= \lambda \left[ \frac{\|x_{2n} - x_{2n+1}, z\| \|x_{2n-1} - x_{2n}, z\|}{1 + \|x_{2n} - x_{2n-1}, z\|} \right]^\alpha \cdot [\|x_{2n-1} - x_{2n}, z\|]^{1-\alpha}.
 \end{aligned}$$

Then

$$\|x_{2n+1} - x_{2n}, z\|^{1-\alpha} \leq \lambda \|x_{2n-1} - x_{2n}, z\|^{1-\alpha}.$$

Hence

$$\begin{aligned}
 \|x_{2n+1} - x_{2n}, z\| &\leq \lambda^{\frac{1}{1-\alpha}} \|x_{2n-1} - x_{2n}, z\| \\
 &\leq \lambda \|x_{2n-1} - x_{2n}, z\|.
 \end{aligned}$$

Thus

$$\|x_{2n+1} - x_{2n}, z\| \leq \lambda \|x_{2n-1} - x_{2n}, z\| \leq \lambda^2 \|x_{2n-2} - x_{2n-1}, z\| \leq \dots \leq \lambda^{2n} \|x_0 - x_1, z\|.$$

Then

$$\|x_{2n+1} - x_{2n}, z\| \leq \lambda^{2n} \|x_0 - x_1, z\|. \quad (3.22)$$

Using (3.21) and (3.22), we obtain

$$\|x_n - x_{n+1}, z\| \leq \lambda^n \|x_0 - x_1, z\|. \quad (3.23)$$

Now, by (3.23) we demonstrate that  $\{x_n\}$  is Cauchy. Let  $m, n > 0$  with  $m > n$ . Put  $m = n + l$ , hence for any  $z \in \mathcal{X}$ ,

$$\begin{aligned}
 \|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\
 &= \| (x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \dots + (x_{n+l-1} - x_{n+l}), z \| \\
 &\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \dots + \|x_{n+l-1} - x_{n+l}, z\| \\
 &\leq \lambda^n \|x_0 - x_1, z\| + \lambda^{n+1} \|x_0 - x_1, z\| + \dots + \lambda^{n+l-1} \|x_0 - x_1, z\| \\
 &= \lambda^n (1 + \lambda + \dots + \lambda^{l-1}) \|x_0 - x_1, z\| \\
 &\leq \frac{\lambda^n}{1 - \lambda} \|x_0 - x_1, z\|.
 \end{aligned}$$

Letting  $n, m \rightarrow \infty$ , we get

$$\lim_{m, n \rightarrow \infty} \|x_n - x_m, z\| = 0.$$

Hence  $\{x_n\}$  is Cauchy. Then  $\{x_n\}$  converges to some  $x \in \mathcal{X}$ . Now, we demonstrate that  $x \in \mathcal{X}$  is a fixed point of  $S$ . Hence for each  $z \in \mathcal{X}$ ,

$$\begin{aligned}
 \|Sx - x_{2n+2}, z\| &= \|Sx - Tx_{2n+1}, z\| \\
 &\leq \lambda \left[ \frac{\|x - Sx, z\| \|x_{2n+1} - Tx_{2n+1}, z\|}{1 + \|x - x_{2n+1}, z\|} \right]^\alpha \cdot [\|x - x_{2n+1}, z\|]^{1-\alpha}.
 \end{aligned}$$

Letting  $n \rightarrow \infty$ , we get

$$\|Sx - x, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

By Lemma 2.1, we infer that  $Sx = x$ . Similarly

$$\begin{aligned} \|x_{2n+1} - Tx, z\| &= \|Sx_{2n} - Tx, z\| \\ &\leq \lambda \left[ \frac{\|x_{2n} - Sx_{2n}, z\| \|x - Tx, z\|}{1 + \|x - x_{2n}, z\|} \right]^\alpha \cdot [\|x - x_{2n}, z\|]^{1-\alpha}. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we get

$$\|x - Tx, z\| = 0 \quad \text{for each } z \in \mathcal{X}.$$

By applying Lemma 2.1, we conclude that  $x = Tx$ . Then  $x = Sx = Tx$ .  $\square$

We finish with the following theorem.

**Theorem 3.7.** *Let  $\mathcal{X}$  be a 2-Banach space and let  $S, T$  be continuous self mappings on  $\mathcal{X}$  satisfying for each  $x \in \mathcal{X} \setminus \text{Fix}(S)$ ,  $y \in \mathcal{X} \setminus \text{Fix}(T)$  and  $z \in \mathcal{X}$ ,*

$$\|Sx - Ty, z\| \leq \lambda \left[ \frac{\|x - Sx, z\| \|y - Ty, z\|}{\|x - y, z\|} \right]^\alpha \cdot [\|x - y, z\|]^{1-\alpha} \quad (3.24)$$

where  $\lambda \in [0, 1)$  and  $\alpha \in (0, 1)$ , then  $S$  and  $T$  have a common fixed point.

*Proof.* Let  $x_0 \in \mathcal{X}$ . Define a sequence  $\{x_n\}$  in  $\mathcal{X}$  by  $x_{2n+1} = Sx_{2n}$  and  $x_{2n+2} = Tx_{2n+1}$  for each  $n \in \mathbb{N}_0$ . If there exists  $n \in \mathbb{N}_0$  such that  $x_{2n} = x_{2n+1} = x_{2n+2}$ , then  $x_{2n}$  is a common fixed point of  $S$  and  $T$ , so assume that there does not exist three consecutive identical terms in the sequence  $\{x_n\}$  and that  $x_0 \neq x_1$ . Using (3.24), we obtain for all  $z \in \mathcal{X}$ ,

$$\begin{aligned} \|x_{2n+1} - x_{2n+2}, z\| &= \|Sx_{2n} - Tx_{2n+1}, z\| \\ &\leq \lambda \left[ \frac{\|x_{2n} - Sx_{2n}, z\| \|x_{2n+1} - Tx_{2n+1}, z\|}{\|x_{2n} - x_{2n+1}, z\|} \right]^\alpha \cdot \|x_{2n} - x_{2n+1}, z\|^{1-\alpha} \\ &\leq \lambda \|x_{2n+1} - x_{2n+2}, z\|^\alpha \cdot \|x_{2n} - x_{2n+1}, z\|^{1-\alpha}. \end{aligned}$$

Then

$$\|x_{2n+1} - x_{2n+2}, z\|^{1-\alpha} \leq \lambda \|x_{2n} - x_{2n+1}, z\|^{1-\alpha}. \quad (3.25)$$

Hence

$$\begin{aligned} \|x_{2n+1} - x_{2n+2}, z\| &\leq \lambda^{\frac{1}{1-\alpha}} \|x_{2n} - x_{2n+1}, z\| \\ &\leq \lambda \|x_{2n} - x_{2n+1}, z\|. \end{aligned} \quad (3.26)$$

From (3.26), we obtain for all  $z \in \mathcal{X}$ ,

$$\|x_{2n+1} - x_{2n+2}, z\| \leq \lambda \|x_{2n} - x_{2n+1}, z\| \leq \lambda^2 \|x_{2n-1} - x_{2n}, z\| \leq \dots \leq \lambda^{2n+1} \|x_0 - x_1, z\|.$$

Then

$$\|x_{2n+1} - x_{2n+2}, z\| \leq \lambda^{2n+1} \|x_0 - x_1, z\|. \quad (3.27)$$

Similarly,

$$\begin{aligned}
\|x_{2n+1} - x_{2n}, z\| &= \|Sx_{2n} - Tx_{2n-1}, z\| \\
&\leq \lambda \left[ \frac{\|x_{2n} - Sx_{2n}, z\| \|x_{2n-1} - Tx_{2n-1}, z\|}{\|x_{2n} - x_{2n-1}, z\|} \right]^\alpha \cdot [\|x_{2n-1} - x_{2n}, z\|]^{1-\alpha} \\
&= \lambda \left[ \frac{\|x_{2n} - x_{2n+1}, z\| \|x_{2n-1} - x_{2n}, z\|}{\|x_{2n} - x_{2n-1}, z\|} \right]^\alpha \cdot [\|x_{2n-1} - x_{2n}, z\|]^{1-\alpha}.
\end{aligned}$$

Then

$$\|x_{2n+1} - x_{2n}, z\|^{1-\alpha} \leq \lambda \|x_{2n-1} - x_{2n}, z\|^{1-\alpha}.$$

Hence

$$\begin{aligned}
\|x_{2n+1} - x_{2n}, z\| &\leq \lambda^{\frac{1}{1-\alpha}} \|x_{2n-1} - x_{2n}, z\| \\
&\leq \lambda \|x_{2n-1} - x_{2n}, z\|.
\end{aligned}$$

Thus

$$\|x_{2n+1} - x_{2n}, z\| \leq \lambda \|x_{2n-1} - x_{2n}, z\| \leq \lambda^2 \|x_{2n-2} - x_{2n-1}, z\| \leq \cdots \leq \lambda^{2n} \|x_0 - x_1, z\|.$$

Then

$$\|x_{2n+1} - x_{2n}, z\| \leq \lambda^{2n} \|x_0 - x_1, z\|. \quad (3.28)$$

Using (3.27) and (3.28), we obtain

$$\|x_n - x_{n+1}, z\| \leq \lambda^n \|x_0 - x_1, z\|. \quad (3.29)$$

Now, by (3.29) we demonstrate that  $\{x_n\}$  is Cauchy. Let  $m, n > 0$  with  $m > n$ . Put  $m = n + l$ , hence for any  $z \in \mathcal{X}$ ,

$$\begin{aligned}
\|x_n - x_m, z\| &= \|x_n - x_{n+l}, z\| \\
&= \| (x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \cdots + (x_{n+l-1} - x_{n+l}), z \| \\
&\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{n+l-1} - x_{n+l}, z\| \\
&\leq \lambda^n \|x_0 - x_1, z\| + \lambda^{n+1} \|x_0 - x_1, z\| + \cdots + \lambda^{n+l-1} \|x_0 - x_1, z\| \\
&= \lambda^n (1 + \lambda + \cdots + \lambda^{l-1}) \|x_0 - x_1, z\| \\
&\leq \frac{\lambda^n}{1 - \lambda} \|x_0 - x_1, z\|.
\end{aligned}$$

Letting  $n, m \rightarrow \infty$ , we get

$$\lim_{m, n \rightarrow \infty} \|x_n - x_m, z\| = 0.$$

Hence  $\{x_n\}$  is Cauchy. Then  $\{x_n\}$  converges to some  $x \in \mathcal{X}$ . Now, we demonstrate that  $x \in \mathcal{X}$  is a fixed point of  $S$ . Since  $S$  and  $T$  are continuous on  $\mathcal{X}$ , we obtain

$$x = \lim_{n \rightarrow \infty} x_{2n+1} = \lim_{n \rightarrow \infty} Sx_{2n} = S(\lim_{n \rightarrow \infty} x_{2n}) = Sx.$$

Then  $Sx = x$ . Similarly

$$x = \lim_{n \rightarrow \infty} x_{2n+2} = \lim_{n \rightarrow \infty} Tx_{2n+1} = T\left(\lim_{n \rightarrow \infty} x_{2n+1}\right) = Tx.$$

Hence  $x = Tx$ . Consequently  $x = Sx = Tx$ .  $\square$

#### REFERENCES

- [1] B. K. Dass, S. Gupta, *An extension of Banach contraction principle through rational expression*, Indian Journal of Pure and Applied Mathematics, 6 (1975), 1455-1458.
- [2] J. Ettayb, *Fixed point theorems for some mappings in 2-Banach spaces*, Mathematical Analysis and its Contemporary Applications, Vo. 7, No. 3 (2025), 67-75.
- [3] J. Ettayb, *On interpolative Boyd-Wong and Matkowski type contractions in 2-Banach spaces*, Eur. J. Math. Appl. Vol. 5 (2025), Article ID 15, 10 pp.
- [4] R. W. Freese, Y. J. Cho, *Geometry of linear 2-normed spaces*, Nova Publishers, Inc., New York, 2001.
- [5] S. Gähler, *Lineare 2-Normierte Räume*, Math. Nachr. 28 (1964), 1-43.
- [6] P. K. Harikrishnan, K. T. Ravindran, *Some Properties of Accretive operators in Linear 2-Normed Spaces*, Int. Math. Forum, Vo. 6 No. 59 (2011), 2941-2947.
- [7] M. Kir, H. Kiziltunc, *Some New Fixed Point Theorems in 2-Normed Spaces*, Int. Journal of Math. Analysis, Vo. 58 No. 7 (2013), 2885-2890.
- [8] M. Noorwali, *Common fixed point for Kannan type contractions via interpolation*, J. Math. Anal. Vo. 9 No. 6 (2018), 92-94.
- [9] A. White, *2-Banach spaces*, Math. Nachr. 42 (1969), 43-60.

JAWAD ETTAYB

INDEPENDENT RESEARCHER,

HAD SOUALEM,

MOROCCO.

Email address: jawad.ettayb@gmail.com