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FIXED POINT THEOREM FOR A CLASS OF CHATTERJEA-TYPE MAPPINGS

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Abstract. In this paper, a new class of mappings in a complete metric space is introduced, which is in a sense analogous to Chatterjea-type mappings, just as the class of Koparde–Waghmode mappings ([7]) is analogous to Kannan-type mappings. Furthermore, a fixed-point theorem and a common fixed-point theorem are proved regarding this class of mappings.

1. **Introduction**

The first result in the fixed-point theory was given by S. Banach in his dissertation in 1920, and it was published in [10] in 1922. Namely, the following theorem holds true.

Theorem 1. Let (X,d) be a complete metric space and $T:X\to X$ be a mapping such that for every $x,y\in X$ the inequality

$$d(Tx,Ty)\le\lambda d(x,y), \tag{1}$$

holds, where $0\le\lambda<1$. Then, T has a unique fixed point. ■

Furthermore, R. Kannan in 1968 in [9] proved the following theorem.

Theorem 2. Let (X,d) be a complete metric space and $T:X\to X$ be a mapping such that for every $x,y\in X$ the inequality

$$d(Tx,Ty)\le\lambda(d(x,Tx)+d(y,Ty)), \tag{2}$$

holds, where $0\le\lambda<\frac{1}{2}$. Then, T has a unique fixed point. ■

Another notable result in the fixed-point theory is that of S. K. Chatterjea, who in 1972 in [11] proved the following theorem.

Theorem 3. Let (X,d) be a complete metric space and $T:X\to X$ be a mapping such that for every $x,y\in X$ the inequality

$$d(Tx,Ty)\le\lambda(d(x,Ty)+d(y,Tx)), \tag{3}$$

holds, where $0\le\lambda<\frac{1}{2}$. Then, T has a unique fixed point. ■

Note that Theorems 1, 2 and 3 are independent, and that the last two characterize the completeness of the metric space (see [8]). The results of Banach, Kannan, and Chatterjea have aroused great interest and have led to many new directions of research, which have led to many generalizations. Such generalizations of Chatterjea's theorem can be seen, for example, in papers [1]-[6] and [12]-[14].

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2. **Main result**

Theorem 4. Let (X,d) be a complete metric space and $T:X\to X$ be a mapping such that for every $x,y\in X$ the inequality

$$d^2(Tx,Ty)\le \lambda(d^2(x,Ty)+d^2(y,Tx)), \tag{4}$$

holds, where $0\le\lambda<\frac{1}{4}$. Then, T has a unique fixed point.

Proof. Let $x_0\in X$ and define a sequence $\{x_n\}$ with $x_{n+1}=Tx_n$, for $n=0,1,2,...$. Then from (4) it holds that

$$\begin{aligned} d^2(x_{n+1},x_n) &= d^2(Tx_n,Tx_{n-1}) \\ &\le \lambda(d^2(x_n,Tx_{n-1})+d^2(x_{n-1},Tx_n)) \\ &= \lambda(d^2(x_n,x_n)+d^2(x_{n-1},x_{n+1})) \\ &= \lambda d^2(x_{n-1},x_{n+1}), \end{aligned}$$

so that is why

$$d(x_{n+1},x_n)\le\sqrt{\lambda}d(x_{n-1},x_{n+1})\le\sqrt{\lambda}(d(x_{n-1},x_n)+d(x_n,x_{n+1})),$$

that is, $d(x_{n+1},x_n)\le\frac{\sqrt{\lambda}}{1-\sqrt{\lambda}}d(x_{n-1},x_n)$. Accordingly, $d(x_{n+1},x_n)\le\alpha d(x_{n-1},x_n)$, for $n=0,1,2,...$, where $\alpha=\frac{\sqrt{\lambda}}{1-\sqrt{\lambda}}<1$. Now, from the last inequality it holds that

$d(x_{n+1},x_n)\le\alpha^n d(x_1,x_0)$, for $n=0,1,2,...$.

Now, from the last inequality and from the triangle inequality, it follows that for $n>m$ holds

$$\begin{aligned} d(x_m,x_n) &\le d(x_m,x_{m+1})+d(x_{m+1},x_{m+2})+...+d(x_{n-1},x_n) \\ &\le(\alpha^m+\alpha^{m+1}+...+\alpha^{n-1})d(x_1,x_0) \\ &=\alpha^m\tfrac{1-\alpha^{n-m}}{1-\alpha}d(x_1,x_0)\le\tfrac{\alpha^m}{1-\alpha}d(x_1,x_0). \end{aligned}$$

But $\alpha<1$, so from the last inequality it follows that the sequence $\{x_n\}$ is Cauchy, and as X is a complete generalized metric space, it is convergent, i.e. it exists $u\in X$ such that $\lim\limits_{n\to\infty}x_n=u$. We will prove that u is a fixed point on T. We have,

$$\begin{aligned} d(u,Tu) &\le d(u,x_{n+1})+d(x_{n+1},Tu)=d(u,x_{n+1})+d(Tx_n,Tu) \\ &\le d(u,x_{n+1})+\sqrt{\lambda(d^2(x_n,Tu)+d^2(u,Tx_n))} \\ &= d(u,x_{n+1})+\sqrt{\lambda(d^2(x_n,Tu)+d^2(u,x_{n+1}))}, \end{aligned}$$

for $n=0,1,2,...$. In the last inequality we will take $n\to\infty$ and we obtain the inequality $d(u,Tu)\le\sqrt{\lambda}d(u,Tu)$. But $\sqrt{\lambda}<1$, so $d(u,Tu)=0$, i.e. $Tu=u$.

Let $u,v\in X$ be two fixed points on T. Then

$$d^2(u,v)=d^2(Tu,Tv)\le\lambda(d^2(u,Tv)+d^2(v,Tu))=2\lambda d^2(u,v)$$

and as $2\lambda<1$, we obtain $d(u,v)=0$, i.e. $u=v$, which means that T has a single fixed point. ■

In the next theorem we will focus on the problem of common fixed points of two mappings from the considered class of mappings.

Theorem 5. Let (X,d) be a complete metric space and $S,T:X\to X$ be a mappings such that for every $x,y\in X$ the inequality

$$d^2(Sx,Ty)\le \lambda(d^2(x,Ty)+d^2(y,Sx))\,, \tag{5}$$

holds, where $0\le\lambda<\frac{1}{4}$. Then, S and T have a unique common fixed point.

Proof. Let $x_0\in X$ and define a sequence $\{x_n\}$ with

$$x_{2n+1}=Sx_{2n}\,,\ x_{2n+2}=Tx_{2n+1}\,,\text{ for } n=0,1,2,\dots\,.$$

If exists $n\ge 0$ such that the equations $x_n=x_{n+1}=x_{n+2}$ holds true, then it is easy to see that $u=x_n$ is a common fixed point for S and T. So, let us assume that the sequence $\{x_n\}$ does not contain three consecutive equal points. Then, using inequality (5), we get that for each $n\ge 1$ holds

$$\begin{aligned} d^2(x_{2n+1},x_{2n}) &= d^2(Sx_{2n},Tx_{2n-1})\\ &\le \lambda(d^2(x_{2n},Tx_{2n-1})+d^2(x_{2n-1},Sx_{2n}))\\ &= \lambda(d^2(x_{2n},x_{2n})+d^2(x_{2n-1},x_{2n+1}))\\ &= \lambda d^2(x_{2n-1},x_{2n+1}), \end{aligned}\,,$$

$$\begin{aligned} d^2(x_{2n-1},x_{2n}) &= d^2(Sx_{2n-2},Tx_{2n-1})\\ &\le \lambda(d^2(x_{2n-2},Tx_{2n-1})+d^2(x_{2n-1},Sx_{2n-2}))\\ &= \lambda(d^2(x_{2n-2},x_{2n})+d^2(x_{2n-1},x_{2n-1}))\\ &= \lambda d^2(x_{2n-2},x_{2n}), \end{aligned}$$

from which we get that for each $n\ge 1$ holds

$$d(x_{2n+1},x_{2n})\le\sqrt{\lambda}d(x_{2n-1},x_{2n+1})\le\sqrt{\lambda}(d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1}))\,,$$
$$d(x_{2n-1},x_{2n})\le\sqrt{\lambda}d(x_{2n-2},x_{2n})\le\sqrt{\lambda}(d(x_{2n-2},x_{2n-1})+d(x_{2n-1},x_{2n}))\,.$$

From the last two inequalities it follows that each $n\ge 1$ holds

$$d(x_{n+1},x_n)\le\alpha d(x_n,x_{n-1})\,,$$

where $\alpha=\frac{\sqrt{\lambda}}{1-\sqrt{\lambda}}<1$. Now, analogously to the proof of Theorem 4, we conclude that the sequence $\{x_n\}$ is Cauchy and as X is a complete metric space, it is convergent, i.e. exists $u\in X$ such that $\lim\limits_{n\to\infty}x_n=u$. We will prove that u is a fixed point for S. We have,

$$\begin{aligned} d(u,Su) &\le d(u,x_{2n+2})+d(x_{2n+2},Su)\\ &= d(u,x_{2n+2})+d(Tx_{2n+1},Su)\\ &\le d(u,x_{2n+2})+\sqrt{\lambda(d^2(x_{2n+1},Su)+d^2(u,Tx_{2n+1}))}\\ &= d(u,x_{2n+2})+\sqrt{\lambda(d^2(x_{2n+1},Su)+d^2(u,x_{2n+2}))}, \end{aligned}$$

for $n = 0,1,2,...$. In the last inequality we will take $n \to \infty$ and we obtain the inequality $d(u,Su) \le \sqrt{\lambda} d(u,Su)$. But $\sqrt{\lambda} < 1$, so $d(u,Su) = 0$, i.e. $Su = u$. In a completely identical way, it is proven that u is a fixed point for T .

Let $v \in X$ be another fixed point on T , i.e. let $Tv = v$. Then

$$d^2(u,v) = d^2(Su,Tv) \le \lambda(d^2(u,Tv) + d^2(v,Su)) = 2\lambda d^2(u,v)$$

and as $2\lambda < 1$, we obtain $d(u,v) = 0$, i.e. $u = v$. ■

Corollary 1. Let (X,d) be a complete metric space, $p,q \in R$ and $S,T : X \to X$ be a mappings such that for every $x, y \in X$ the inequality

$$d^2(S^p x, T^q y) \le \lambda(d^2(x,T^q y) + d^2(y,S^p x)), \tag{6}$$

holds, where $0 \le \lambda < \frac{1}{4}$. Then, S and T have a unique common fixed point.

Proof. According to Theorem 5, the mappings S^p and T^q have a unique common fixed point $u \in X$. Accordingly, $S^p u = u$, from where it follows

$$Su = S(S^p u) = S^{p+1}u = S^p(Su),$$

i.e., Su is a fixed point for S^p . Analogue, from $T^q u = u$, from where it follows

$$Tu = T(T^q u) = T^{q+1}u = T^q(Tu),$$

i.e. Tu is a fixed point for T^q . But from the proof of Theorem 5 it follows that each of the mappings S^p and T^q has a unique fixed point. Therefore, $Su = u$ and $Tu = u$. So, $u \in X$ is a common fixed point for mappings S and T .

Finally, if $v \in X$ is another common fixed point for S and T , then v is a common fixed point for S^p and T^q . But S^p and T^q have a unique common fixed point, so $u = v$. ■

3. Discussion

In this paper we have proved several theorems about fixed points and common fixed points of a Chatterjea-type mapping. The question arises whether analogous results hold for mappings of this type defined on generalized metric spaces or on 2-Banach spaces.

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