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Auerbach-Kotlikoff OLG model in a Huggett heterogeneous agents incomplete markets economy

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Abstract

This paper will investigate A-K OLG model in Huggett heterogeneous agents in incomplete markets setting. Dynamic inefficiency i.e., $r < g$ shows that for this model when government debt is decreasing welfare also decreases. Government transfer payments decrease, and payroll tax revenue is increasing, interest rate has is decreasing dramatically, wages rise and capital and labor also rise. On contrary in A-K model in Hugget economy which is not dynamically inefficient, assets-capital; consumption are age-dependent. Furthermore, RET hypothesis at is shown does not hold since agents cannot replicate risk free payoffs in incomplete markets (proof due to Divino,Orrilo,(2017), in our case Ricardian equivalence fails to hold, we observe that the difference in consumption and government debt does not converge to zero over time, as indicated by the plotted lines.

Keywords: Auerbach-Kotlikoff model,Huggett economy, dynamic inefficiency, heterogeneous agents, incomplete markets

JEL codes: E13; E62;D30

1.Introduction

Auerbach-Kotlikoff dynamic life cycle simulation model (A-K model) has been used to examine a host of policies such as: tax reform, tax cuts, progressive taxation, social security expansion, government spending, monetary policy endogenous growth, human capital accumulation etc., see (Kotlikoff, (1998)).In the original model Auerbach, Alan J.Laurence J. Kotlikoff. (1987), authors pinpoint questions to be addressed by their model:1. savings, welfare and the choice of tax base,2. efficiency gains from dynamic tax reform, crowding out and deficits, business tax incentives, tax progressivity, announcements of policy change and their effects, and demographic shifts. One can see that this model was very useful and fulfilled the authors' prediction for his use. Pioneering OLG models by Samuelson (1958),Diamond (1965),the number of coexisting cohorts amounted to two, the young workers and older retired generation, previous two models were focused on theoretical problems i.e. if there is a role for money and what are the effects of national debt respectively see, Heer,B. Maußner,A.(2005). Before knowing these models one also needs to know Ramsey model see Ramsey (1928). The origins of this model and motivation for this groundwork does come from 1970s literature and Lucas, Sargent which pointed that the basis of theory critique and dismantling of Keynesian theory was based on stressing the importance of connecting macroeconomic outcomes to microeconomic fundamentals, see Lucas (1976), and Lucas,Sargent (1979).Lucas critiques, Firstly, it provided an ultimate criticism of the econometric models like Klein and Goldberger (1955). As Robert Hall puts it, this econometric approach—which was dominant in the 1960s—has been "devastated by the theoretical and empirical force of the Lucas critique", see Hall, (1996). Lucas (1976) addresses following methodological question: How to build econometric models that provide reliable quantitative evaluation of the effects of alternative rules for economic policy? Lucas's answer is: in order to provide a sound expertise, the model parameters must be "structural", i.e their values must be "invariant" with respect to policy changes. In short answer, parameters must be "stable", see Sergi (2018).In short parameters of unrestricted econometric models are not invariant to changes in policy regime. The logic of Lucas critique lies in Samuelsons' correspondence principle, see Samuelson (1941),Samuelson (1941), Samuelson (1947).This

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principle says that the endogenous variables of econometric models can be described as non-trivial functions of the exogenous variable, see Farmer (1989). CGE models were also suffering critique for their reliance of "representative agent" and aggregation procedures. If the representative agents' model is estimated with data from heterogeneous agents' economy under different policy regimes important parameters vary considerably. For instance, the aggregate labor supply elasticity, which was/is often recognized as a crucial parameter for fiscal policy analysis, depends on cross-sectional distribution of reservation wages ,which distribution is in turn a function of fiscal policy regime, see Auerbach, Kotlikof (1987); and Judd (1987); Prescot (2004),and Chang,Kim,Schorfheide(2013). Now, the OLG model is a natural framework to analyze life-cycle problems such as the provision of public pensions, endogenous fertility, or the accumulation of human capital and wealth. Seminal work in this area is the study of dynamic fiscal policy by Auerbach and Kotlikoff (1987). The possibility of inefficient equilibria in these types of models was first studied by Diamond (1965). The previous model was the first to study the phenomenon within a framework of an OLG model with productive capital. This model demonstrated that the market process can lead to an overaccumulation of capital[1] where everyone is made better of if the capital stock is reduced, see Larch (1993). So this problem in OLG models is the one proposed by Diamond (1965) and it's about over saving[2] which occurs when capital accumulation is added to the model. In the terminology of Phelps (1961), the capital stock exceeds the Golden rule level[3].So here corresponding equilibrium will be characterized by interest rate lying below rate of growth of the economy $r < g$ [4] i.e. the capital stock is more than what is called the Golden Rule level. In recent years, there has been a wide range of economic problems studied with the help of OLG models. In addition to the early work by Auerbach and Kotlikoff, subsequent authors have introduced new elements in the study of overlapping generations, for example, stochastic survival probabilities, bequests, or individual income mobility, to name but a few. Huggett; Ventura (2000) look at the determinants of savings and use a calibrated life-cycle model to investigate why high income households save as a group a much higher fraction of income than do low income households as documented by US cross-section data. Huggett (1996) shows that the life-cycle model is able to reproduce the US wealth Gini coefficient and a significant fraction of the wealth inequality within age groups. The model by Mark Huggett (1996), investigated wealth distribution at a quantitative level by pursuing the research program put forward by Atkinson (1971). Two modifications in Huggett (1996) model to previous OLG models are: The two modifications considered are the presence of earnings and lifetime uncertainty and the absence of markets for insuring this uncertainty. Basic life cycle model could not explain savings rate because aggregate savings tend to be low see White (1978), Kotlikoff and Summers (1981) calculate that the vast majority of the US capital stock can be attributed to intergenerational transfers rather than to accumulation out of earnings that are the emphasis of the basic life-cycle model of capital accumulation, and third wealth holding is much more concentrated in the upper tail of the wealth distribution than the basic model predicts. Fourth, wealth is as unequally distributed within an age group in the US as it is in the overall wealth distribution. However, in the basic model wealth only differs across agents in different age groups. Modifications were made in these models by introducing precautionary savings[5] .That is where Krusell, Mukoyama, and Sahin (2010) depart from standard Diamond-Mortensen -Pissarides model, in that workers can insure themselves against job loss by accumulating assets. In short, the employed will save and the unemployed dissave to smooth consumption. Heterogeneity in wealth creates heterogeneity in the value of unemployment. Hence, bilateral bargaining between individual workers and firms leads to a wage schedule that is increasing in wealth. Hugget (1993) economy is of such type where agents experience uninsurable idiosyncratic endowment shocks and smooth consumption by holding a risk-free asset. There has been a considerable amount of work on heterogeneous-agent incomplete-insurance models of asset pricing, see Bewley (1980), Lucas (1980),or in other areas: Imrohoroglu (1989) model that .measures the potential welfare gains from eliminating

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[1] Simulation results show that equilibria with an overaccumulation of capital relative to the Golden Rule are characterized by high population growth, high intertemporal elasticity of substitution and negative rates of time preference.

[2] Over saving occurs when $s^* > \frac{\left(\frac{df(k)}{dk}\right)}{\frac{f(k)}{k}}$, where s^* represents the golden rule saving

[3] $\frac{dk}{dt} > sf(k) - nk$ or $\frac{dk}{dt} > f(k) - c - nk$,or $f(k) > n + p$ see Appendix 1 for derivation of the results for the Golden Utility growth compared to Golden rule growth and Ramsey exercise.

[4] This has implications for bequest taxation also which is more progressive as $r - g$ is higher, see Farhi, Werning (2013).

[5] Several studies in macroeconomics have extended the basic model of precautionary savings, (see Carroll (1997) , Huggett (1993), Aiyagari (1994)) to incorporate entrepreneurs

aggregate fluctuations. This paper will model A-K economy in Huggett demographic structure i.e. OLG heterogenous agents economy.

2. A-K OLG model (Auerbach,Kotlikoff (1987))

Auerbach, Kotlikoff (1987) (A-K model)extend the two-period OLG model of Allais (1947), Samuelson (1958) and Diamond (1965) to a model with 55 periods. In this model periods correspond to years. Samuelson (1958) introduced a consumption loan model to analyze the interest rate ,with or without social contrivance of money has developed into one of the most significant paradigm of the neoclassical general equilibrium theory ,bypassed Arrow-Debreu(1954) economy, Geanakopolos, (1987) . In the A-K model households live: $T = T^W + T^R = 20 + 40\ years$,the measure of each generation is $1/60$. When $T^W = 40$ agents are workers for 40 years, and their labor supply is l_t^s at age s in period t and their leisure is $1 - l_t^s$,after T^W retirement is mandatory. Agents' lifetime utility is given as:

equation 1

$$\sum_{s=1}^{t} \beta^{s-1} u(c_{t+s}^s, 1 - l_{t+s-1}^s$$

$c_t^s; l_t^s$ denote consumption and labor supply of the year s of the t old generation,β represents the discount factor. Instantaneous utility function of consumption c_s^t and leisure $1 - l_t^s$ is given as:

equation 2

$$u(c, 1 - l) = \frac{\left((c + \psi)(1 - \gamma)\right)^{1-\sigma} - 1}{1 - \sigma}$$

Where parameters $\psi = 0.001$,meaning that utility is finite even for small consumption. Zero initial and terminal wealth:$k_t^1 = k_t^{61} = 0$The real budget constraint is given as:

equation 3

$$a_{t+1}^{s+1} = (1 + r_t)a_t^s + (1 - \tau_t)w_t l_t^s - c_t^s; s = 1, \dots, T^W$$

Where a_t^s-are assets of the s-year old household in period t, r_t is the real interest rate in period t,w_t is the real wage rate at period t, $\tau_t w_t l_t^s$ is the workers social security contribution. Budget constraint on the retiree is given as:

equation 4

$$a_{t+1}^{s+1} = (1 + r_t)a_t^s + pen - c_t^s, s = T^W + 1, \dots, T^W + T^R$$

FOCs for the household are given as:

equation 5

$$\frac{u_l(c_t^s, l_t^s)}{u_x(c_t^s, l_t^s)} = \gamma \frac{c_t^s + \psi}{l_t^s} = (1 - \tau_t)w_t$$

$$\frac{1}{\beta} = \frac{u_c(c_{t+1}^{s+1}, l_{t+1}^{s+1})}{u_c(c_t^s, l_t^s)}[1 + r_{t+1}] = \frac{(c_{t+1}^{s+1} + \psi)^{-\eta}(l_{t+1}^{s+1})^{\gamma(1-\eta)}}{(c_t^s + \psi)^{-\eta}(l_t^s)^{\gamma(1-\eta)}}[1 + r_{t+1}]$$

Production assumes Cobb-Douglas production technology[6]: $Y_t = N_t^{1-\alpha} K_t^{1-\alpha}$. The government uses the revenues from taxing labor to finance its expenditures on social security :

equation 6

$$\tau_t w_t N_t = \frac{T^R}{T + T^R} b$$

[6] marginal products of factors are given as:$w_t = (1 - \alpha)K_t^\alpha N_t^{1-\alpha}$ and $r_t = \alpha K_t^{\alpha-1} N_t^{1-\alpha} - \delta$

capital market equilibrium is given as:

$$A_t = \sum_s a_t^s = K_t$$

Aggregate end-of-periods savings (wealth) are equal to aggregate capital stock, see Heer, Maußner (2005). General equilibrium in his model uses recursive representation following Lucas, Stokey, Prescott (1989).

equation 7

$$V^s(k_t^s, K_t, N_t) = \begin{cases} \max_{k_{t+1}^{s+1}, c_t^s l_t^s} [u(c_t^s, l_t^s) + \beta V^{s+1}(k_{t+1}^{s+1}, K_{t+1}, N_{t+1}] \\ s = 1, .. T \\ \max_{k_{t+1}^{s+1}, c_t^s} [u(c_t^s, 1) + \beta V^{s+1}(k_{t+1}^{s+1}, K_{t+1}, N_{t+1}] \\ s = T+1, \dots, T + T^{R-1} \end{cases}$$

s.t. $V^{T+T^R}\left(k_t^{T+T^R}, K_t^{T+T^R}; N_t^{T+T^R}\right) = u\left(c_t^{T+T^R}, 1\right)$.The value function $V^s(\cdot)$ depends on the state variables $N_t(\cdot), K_t(\cdot)$ to determine the wage rate. Individual and aggregate behavior is consistent:

equation 8

$$N_t = \sum_{s=1}^{T} \frac{n_t^s}{T + T^R}$$

$$K_t = \sum_{s=1}^{T+T^R} \frac{k_t^s}{T + T^R}$$

The goods market clears as:

equation 9

$$Y_t = K_{t_t}^{\alpha L^{1-\alpha}} = \sum_{s=1}^{T} \frac{c_T^s}{T} + K_{t+1} - (1 - \delta) K_t$$

Constant distribution of capital stock over generations is given as:

equation 10

$$\{a_t^s\}_{s=1}^{60} = \{a_{t+1}^s\}_{s=1}^{60} = \{a^s\}_{s=1}^{60}$$
$$\{l_t^s\}_{s=1}^{40} = \{l_{t+1}^s\}_{s=1}^{40} = \{l^s\}_{s=1}^{40}$$

Replacement ratio of pensions with respect to average net wage income is given as:

equation 11

$$\theta = \frac{pen}{(1 - \tau) w l} = 0.3$$

And assum9ing constant replacement rate of pensions relative to net wages:

equation 12

$$pen = \theta (1 - \tau) w \bar{l}$$

The budget of social security implies[7] :

$$\tau w L = \frac{20}{60} \theta (1 - \tau) w \bar{l}$$

Next, we will present the steady-state computation in Auerbach-Kotlikoff model. Now first we will reduce the number of equilibrium conditions (we will do so by elimination of consumption c^s form the individual

[7] $L = \frac{40}{60} \bar{l}$ which can be solved for τ

budget constraints) as a result we can the workers' steady-state equations for $s = 1, \dots T^W - 1$ by the following expression:

equation 13

$$(1-\tau)w = \gamma \frac{(1+r) + (1-\tau)wl^s - a^{s+1} - a^{s+1} + \psi}{1-l^s}$$
$$\frac{1}{\beta} = \frac{\left((1+r)a^{s+1} + (1-\tau)wl^{s+1} - a^{s+2} + \psi\right)^{-\sigma}}{\left((1+r)a^s + (1-\tau)wl^s - a^{s+1} + \psi\right)^{-\sigma}} \times \frac{(1-l^{s+1})^{\gamma(1-\sigma)}}{(1-l^s)^{\gamma(1-\sigma)}}[1+r]$$
$$\frac{1}{\beta} = \frac{\left((1+r)a^{s+1} + pen - a^{s+2} + \psi\right)^{-\sigma}}{\left((1+r)a^s + pen - a^{s+1} + \psi\right)^{-\sigma}}[1+r]$$

Previous represents 59 equations with 59 unknowns $\{a^s\}_{s=2}^{60}$ and $\{l^s\}_{s=1}^{40}$ Now, with the help from this $\frac{1}{\beta} = \frac{\left((1+r)a^{s+1}+pen-a^{s+2}+\psi\right)^{-\sigma}}{\left((1+r)a^s+pen-a^{s+1}+\psi\right)^{-\sigma}}[1+r]$ we can compute a^{59} given a^{60} and $a^{61} = 0$

equation 14

$$\frac{1}{\beta} = \frac{\left((1+r)a^{60} + pen - a^{61} + \psi\right)^{-\sigma}}{\left((1+r)a^{59} + pen - a^{60} + \psi\right)^{-\sigma}}[1+r]$$

Which is solved by Newton-Rhapson algorithm.In the Newton's method the algorithm can be applied iteratively to obtain:$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_{n-1})}$,if $\lim_{x_{n+1}\to x^*} \frac{f(x_n)}{f'(x_n)} = x_n$,and $x_n = x^* + \epsilon_n$,where $\epsilon_{n+1} = \frac{f''(x^*)}{2\cdot f'(x^*)}\epsilon_n^2$ .Fixed point theorem states that if $\exists f(x) \in [a,b]$,then $\exists x \in [a,b]$, and $f(x) - x = 0 \Rightarrow f(x) = x$, see (Rosenlicht 1968).Computation can be done also with secant method:

equation 15

$$x_{s+2} = x_{s+1} - \frac{x_{s+1} - x_s}{f(x_{s+1}) - f(x_s)} f(x_s)$$

About the value function iteration here we present discrete-time programming in finite time. The newborn wants to maximize lifetime utility at the beginning of age 1 in period t:

equation 16

$$\sum_{j=1}^{T} \beta^{j-1} \frac{\left((c^j + \psi)(1-l^j)\right)^{1-\sigma} - 1}{1-\sigma}$$

s.t. budget constraint:

equation 17

$$a^{j+1} = (1+r)a^j + (1-\tau)wl^j - c^j, j = 1, ., , T^W$$
$$a^{j+1} == (1+r)a^j + pen - c^j, j = 1, \dots, T^W + T^R$$

a^s denotes the state variable at age s and u^s denotes the control variables l^s ; c^s value function $V^s(a^s)$ is:

equation 18

$$V^s(a^s) \equiv \sup_{u^s(\hbar^s)} \sum_{j=s}^{T} \beta^{s-j} \frac{\left((c^j + \psi)(1-l^j)\right)^{1-\sigma} - 1}{1-\sigma}$$

Where $u^s(\hbar^s)$ are the set of all rules that are feasible for choosing the controls u^j at age $j, j > s$.Now lets consider 60 years old retiree in his last period of life he simply maximizes utility in the last period by consuming all his income from pension and savings:

equation 19

$$V^{60} = \frac{\left((c^{60} + \psi)(1 - l^{60}\ )\gamma\right)^{1-\sigma} - 1}{1 - \sigma}$$
$$c^{60} = pen + (1 + r)a^{60}$$

Bellman equation HJB (Hamilton-Jacobi-Bellman) equation was a result of the theory of dynamic programing pioneered by Richard Bellman (namely Bellman(1954),Bellman(1957),Bellman, Dreyfus,(1959)), this equation for the worker $s = 1, \dots, 40$ is given as:

equation 20

$$V^s(a^s) = \max_{c^s, l^s} \frac{\left((c^s + \psi)(1 - l^s\ )\gamma\right)^{1-\sigma} - 1}{1 - \sigma} + \beta V^{s+1}(a^{s+1})$$

s.t.

equation 21

$$a^{s+1} = (1 + r)a^s + (1 - \tau)wl^s - c^s$$

As for the value function iterations we can write the Bellman equation for workers as :

equation 22

$$V^s(a^s) = \max_{a^s, l^s} \left\{ \frac{\left(((1 + r)a^s + (1 - \tau)wl^s - a^{s+1} + \psi)(1 - l^s)^\gamma\right)^{1-\sigma}}{1 - \sigma} + \beta V^{s+1}(a^{s+1}) \right\}$$

FOCs for the envelope condition of the previous problem are given as:

equation 23

$$u_c(c^s, 1 - l^s)(1 - \tau)w = u_{1-l}(c^s, l^s)$$
$$u_s(c^s, 1 - l^s) = \beta V^{s+1'}(a^{s+1})$$
$$V^{s'}(a^s) = (1 + r)u_c(c^s, 1 - l^s)$$

Definite version for the working agent Bellman equation can be computed as:

equation 24

$$\frac{\left((1 - \tau)wl^s + (1 + r)a_i - a^{s+1} + \psi)\ (1 - l^s)^\gamma\right)^{1-\sigma} - 1}{1 - \sigma} \beta V^{s+1}(a^{s+1})$$
$$l^s = \max\left\{0, \frac{1}{1 + \gamma}\left(1 - \frac{\gamma}{(1 - \tau)w}(\psi + (1 + r)a^s - a^{s+1})\right)\right\}$$

3. Huggett (1993) and Huggett (1996) model of wealth distribution in life cycle economies

Continuous version of Huggett (1993) economy should be simplest of all heterogeneous agents models, that captures features of more complicated models as per Achdou et al.(2022).This model can be represented as follows:

equation 25

$$\rho v_1(a) = \max_c\ u(c) + v'(a)(z_1 + ra - c)\_\lambda_1\left(v_2(a) - v_1(a)\right)$$
$$\rho(v_2) = \max_c + v_2'(a)(z_2 + ra - c) + \lambda_2\left(c_1(a) - v_2(a)\right)$$

Next equations from the model are presented in the next page:

equation 26

$$0 = -\frac{d}{da}[s_1(a)g_1(a)] - \lambda_1 g_1(a) + \lambda_2 g_2(a)$$
$$0 = -\frac{d}{da}[s_2(a)g_2(a)] - \lambda_1 g_2(a) + \lambda_2 g_1(a)$$
$$1 = \int_{\underline{a}}^{\infty} g_1(a)da + \int_{\underline{a}}^{\infty} g_2(a)da$$
$$1 = \int_{\underline{a}}^{\infty} a g_1(a)da + \int_{\underline{a}}^{\infty} a g_2(a)da \equiv S(r)$$

Synthetized two basic equations for Huggett economy are :

Two basic equations to explain Huggett economy are :

equation 27

$$\begin{pmatrix} \rho v_1(a) = \max_c u(c) + v_1'(a)(z_1 + ra - c) + \lambda_1\big(v_2(a) - v_1(a)\big) \\ \rho v_2(a) = \max_c u(c) + v_2'(a)(z_2 + ra - c) + \lambda_2\big(v_1(a) - v_2(a)\big) \end{pmatrix}$$

Where $\rho \geq 0$ represents the discount factor for the future consumption c_t (Individuals have standard preferences over utility flows), a represents wealth in form of bonds that evolve according to :

equation 28

$$\dot{a} = y_t + r_t a_t - c_t$$

y_t is the income of individual, which is endowment of economy's final good, and r_t represents the interest rate. Equilibrium in this Huggett (1993) economy is given as:

equation 29

$$\int_{\underline{a}}^{\infty} a g_1(a,t)da + \int_{\underline{a}}^{\infty} a g_2(a,t)da = B$$

Where in previous expression $0 \leq B \leq \infty$ and when $B = 0$ that means that bonds are zero net supply. So the finite difference method approx. to $\begin{pmatrix} \rho v_1(a) = \max_c u(c) + v_1'(a)(z_1 + ra - c) + \lambda_1\big(v_2(a) - v_1(a)\big) \\ \rho v_2(a) = \max_c u(c) + v_2'(a)(z_2 + ra - c) + \lambda_2\big(v_1(a) - v_2(a)\big) \end{pmatrix}$ is given as:

equation 30

$$\rho v_{i,j} = u\big(c_{i,j}\big) + v_{i,j}'\big(z_j + ra_i + c_{i,j}\big) + \lambda_j\big(v_{i,-j} - v_{i,j}\big), j = 1,2$$
$$c_{i,j} = (u')^{-1}\big(v_{i,j}'\big)$$

This algorithm uses Broyden method see Broyden (1965). In the secant method, we replace the first derivative f' at x_n with the finite-difference approximation. Newton method for system of non-linear equations is:

Definition 1 : Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and $x_0 \in \mathbb{R}^n$ at each iteration k solve:

equation 31

$$J(x_k)s_k = -F(x_k)$$
$$x_{k+1} = x_k + s_k$$

Theorem 1: Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in an open convex set $D \subset \mathbb{R}^n$.also $\exists\, x_* \in \mathbb{R}^n$ and $r, \beta > 0$ so that $N(x_*, r) \subset D, F(x_*) = 0, \exists J(x_*)^{-1}$; $\| J(x_*)^{-1} \| \leq \beta, \wedge J \in Lip_\gamma(N(x_*, r))$. So $\exists \varepsilon > 0; \forall x_0 \in N(x_*, \varepsilon)$ the sequence $x_1, x_2, \dots$ generated by :

equation 32

$$x_{k+1} = x_k - J(x_k)^{-1} F(x_k), k = 0,1, \dots$$

And converges to x_* :

equation 33

$$\| x_{k+1} - x_* \| \leq \beta\gamma \| x_k - x_* \|^2, k = 0,1, \dots.$$

Proof: we choose ε so that Jacobian[8] $J(x)$ is non-singular $\forall x \in N(x_*, \varepsilon)$,and because local error in Newton method is $\mathcal{O}(\| x_k - x_* \|^2)$ the convergence is quadratic q

equation 34

$$\varepsilon = \min\left\{r, \frac{1}{2\beta\gamma}\right\}$$

By induction of k can be shown also that:

$$\| x_k - x_* \| \leq \frac{1}{2} \| x_k - x_* \| \Rightarrow x_{k+1} \in N(x_*, \varepsilon)$$

$J(x_0)$ is non-singular and from $\| x_0 - x_* \| \leq \varepsilon$ the Lipschitz continuity[9] of J at x_* and $\varepsilon = \min\left\{r, \frac{1}{2\beta\gamma}\right\}$

follows that :

equation 35

$$\| J(x_*)^{-1} [J(x_0) - J(x_*)] \| \leq \| J(x_*)^{-1} \| \| J(x_0) - J(x_*) \| \leq \beta \cdot \gamma \| x_0 - x_* \| \leq \beta \cdot \gamma \cdot \varepsilon \leq \frac{1}{2}$$

$J(x_0)$ is non-singular and :

equation 36

$$\| J(x_0)^{-1} \| \leq \frac{\| J(x_*)^{-1} \|}{1 - \| J(x_0)^{-1} [J(x_0) - J(x_*)] \|} \leq 2 \| J(x_*)^{-1} \| \leq 2 \cdot \beta$$

Furthermore:

equation 37

$$\| x_k - x_* \| \leq \| J(x_0)^{-1} \| \| F(x_*) - F(x_0) - J(x_0)(x_* - x_0) \| \leq 2\beta \cdot \frac{\gamma}{2} \| x_0 - x_* \|^2$$

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[8] Given set of equations $\mathbf{y} = \mathbf{f}(\mathbf{x})$ with n variables ,Jacobian matrix is : $J(x_1, \dots x_n) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$ see, Simon and Blume (1994)

[9] For $\forall \epsilon > 0\ \exists \delta > 0 \forall x_0 \in \mathbb{R}^l \forall x \in \mathbb{R}^l [|x - x_0| < \delta$ |and $f(x) - f(x_0) \geq \epsilon|]$.Now if $f: [a, b] \to R$ and for some constant K which is called Lipschitz constant for $\forall x_1, x_2 \in [a, b]$, then : $d_R(f(x_1) - f(x_2)) \leq K d_{\mathbb{R}^l}(x_1 - x_2)$.Distance d from x_1 to x_2 is $|x_1 - x_2|$. Then the functions is called Lipschitz function and one can write $\in \mathrm{Lip}(a, b)$. Inverse mapping of Lipschitz is $f^{-1}: f(x) = x$.

Previous used following lemma :

Lemma 1: Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in an open convex set $D \subset \mathbb{R}^n$, for $x \in D$ and any non-zero perturbation $p \in \mathbb{R}^n$, the directional derivative of f at x in the direction of p defined by :

equation 38

$$\frac{\partial f}{\partial p}(x) \equiv \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon\, p) - f(x)}{\varepsilon}$$

$$\exists \frac{\partial f}{\partial p}(x) = \nabla f(x)^T, \forall x, x + p \in D$$

equation 39

$$f(x+p) = f(x) + \int_0^1 \nabla f(x + tp)^T p dt$$
$$\equiv f(x) + \int_x^{x+p} \nabla f(z) dz \,; \exists z \in (x, x+p) \to f(x+p) = f(x) + \nabla (f(z)^T p$$

Proof: if we parametrize f along the line $(x, x+p)$; $g: \mathbb{R} \to \mathbb{R}, g(t) = f(x + tp)$,and if we define $x(t) = x + tp$ by the chain rule for $0 \le \alpha \le 1$

equation 40

$$\frac{dg}{dt}(\alpha) = \sum_{i=1}^{n} \frac{\partial\big(x(t)\big)}{\partial\big(x(t)\big)_i} \left(x(\alpha) \frac{dx(t)_i}{dt}(\alpha) \right) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big(x(\alpha)\big) \cdot p = \nabla f(x + \alpha p)^T \cdot p_i$$

If $\alpha = 0$ then $\frac{\partial f}{\partial x_i} = \nabla f(x)^T p$, by the Newton's theorem (fund.theorem of calculus) :

equation 41

$$g(1) = g(0) + \int_0^1 g'(t) dt$$

Which is equivalent to :

equation 42

$f(x+p) = f(x) + \int_0^1 \nabla f(x+tp)^T p dt \,; g(1) = g(0) + g'(\varepsilon); \varepsilon \in (0,1)$■ see Dennis,J.E.,Schnabel,R.B.(1987).

Huggett (1996) economy is described as:

equation 43

$$E\left[\sum_{i=1}^{n} \beta^t \big(\Pi_{j=1}^t s_j\big) u(c_t)\right]$$

Utility function is $u(c_t)$ is CRRA [10],constant returns of scale is the production function in this economy:

equation 44

$$Y = F(K, L) = AK^{\alpha} L^{1-\alpha}$$

Optimization in this economy is given as:

---

[10] $u(c) = \begin{cases} \frac{c^{1-\eta}}{1-\eta}; \eta \ge 0, \eta \ne 1 \\ \ln(c), \eta = 1 \end{cases}$ c is consumption

equation 45

$$V(x, t0 = \max_{c,a'} u(c) + \beta s_{t+1} E[V(a', z', t+1)|x]$$

s.t.

equation 46

$$c + a' \leq a\big(1 + r(1-\tau)\big) + (1 \pm \theta - \tau)e(z,t)w + T + b$$
$$c \geq 0. a' \geq a, \wedge\, a' \geq 0\, ; t = N$$

In previous a are asset holdings ; $e(z,t)$ is labor endowment ,lump-sum transfer is T and age dependent social net-benefits b , asset holdings pay a risk-free interest rate r , capital and labor income are taxed by tax rate τ . In this economy probability space is :$(X, B(X), \psi_t]$, where $X = [a, \infty] \times Z$ and $B(X)$ is σ algebra on X. The distribution of tastes across economy is:

equation 47

$$\psi_t(B) = \int_x P(x, t-1, B) d\psi_{t-1}, \forall B \in B(X)$$

In this economy markets clear at :

equation 48

$$\sum_t \mu_t \int_x (c(x,t) + a(x,t) d\psi_t + G = F(K,L) + (1-\delta)K$$
$$\sum_t \mu_t \int_x a(x,t) d\psi_t = (1+n)K$$
$$\sum_t \mu_t \int_x e(z,t) d\psi_t = L$$

Distribution of tastes is consistent with the individual behavior :

equation 49

$$\psi_{t+1}(B) = \int_x P(x,t,B) d\psi_t\, , t = 1, \ldots, N-1\, , \forall B \in B(X)$$

Government budget constraint is :

equation 50

$$G = \tau(rK + wL)$$

Social security benefits equal taxes:

equation 51

$$\theta w L = b\left(\sum_{t=R}^{N} \mu_t\right)$$

Transfers equal accidental bequests :

equation 52

$$T = \frac{\left[\sum_t \mu_t (1 - s_{t+1}) \int_x a(x,t)\big(1 + r(1-\tau)\big) d\psi_t\right]}{1-n}$$

N is the terminal age, n is some age μ_t is fraction of agents in this economy, x is the state of any agent.This model is solved by Kolmogorov Forward (Fokker-Planck[11]) equation we have folowing: let x be a scalar diffusion

equation 53

$$dx = \mu(x)dt + \sigma(x)dW, x(0) = x_0$$

Let's suppose that we are interested in the evolution of the distribution of $x, f(x,t)$ and $\lim_{t\to\infty} f(x,t)$ .So, given an initial distribution $f(x,0) = f_0(x), f(x,t)$ satisfies PDE :

equation 54

$$\frac{\partial f(x,t)}{\partial t} = -\frac{\partial}{\partial x}[\mu(x)f(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(x)f(x,t)]$$

Previous PDE is called "Kolmogorov Forward Equation" or "Fokker-Planck Equation".

Corollary 1: if a stationary equilibrium exists $\lim_{t\to\infty} f(x,t) \ = f(x)$, it satisfies ODE

equation 55

$$0 - \frac{d}{dx}[\mu(x)f(x)] + \frac{1}{2}\frac{d^2}{dx^2}[\sigma^2(x)f(x)]$$

In the multivariate case Kolmogorov Forward Equation is given as:

equation 56

$$\frac{\partial f(x,t)}{\partial t} = -\sum_{i=1}^{m}\frac{\partial}{\partial x_i}[\mu(x)f(x,t)] + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^2}{\partial x^2}[\sigma_{ij}^2(x)f(x,t)]$$

These models als are solved by usine **Ornstein–Uhlenbeck process-** The Ornstein-Uhlenbeck process is a stochastic process that satisfies the following stochastic differential equation:

equation 57

$$dx_\tau = k(\theta - x_\tau)d\tau + \sigma dW_\tau$$

$k > 0$ is the mean rate of reversion; θ is the long term mean of the process, $\sigma > 0$ is the volatility or average magnitude, per square-root time, of the random fluctuations that are modelled as Brownian motions.

Mean reverting property-where $dx_\tau = k(\theta - x)$:

equation 58

$$\frac{\theta - x_\tau}{\theta - x_0} = e^{-k(\tau-\tau_0)}, x_\tau = \theta + (x_0 - \theta)e^{-k(\tau-\tau_0)}$$

Solution for $\forall \tau > s \geq 0$ is given as:

equation 59

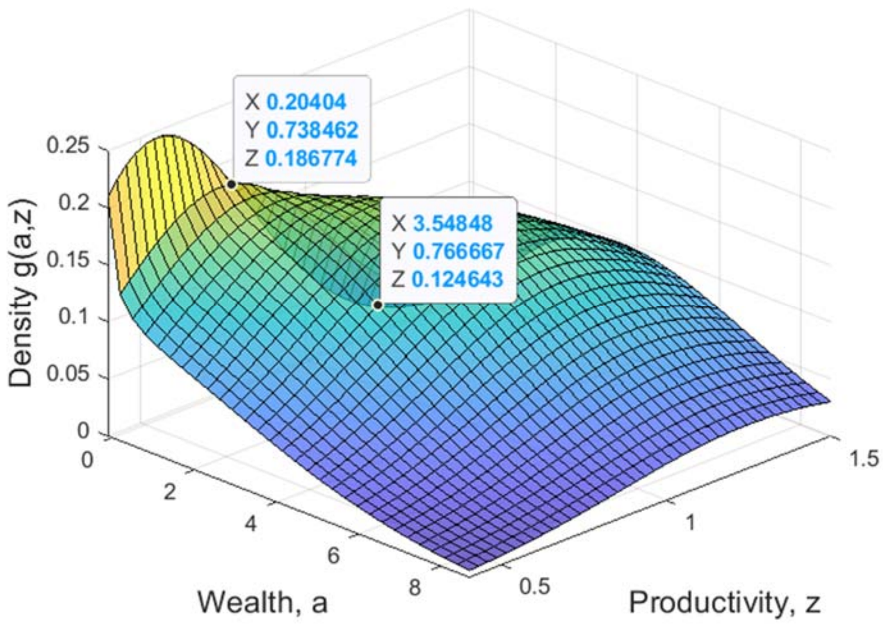
$$x_\tau = \theta + (x_s - \theta)e^{-k(\tau-s)} + \sigma\int_s^\tau e^{-k(\tau-u)}dW_u$$

[11] See Fokker (1914), Planck (1917), Kolmogorov (1931).

See Jacobsen.M(1996) .So now partial differential equation $\frac{\partial c_{j,\tau}}{\partial \tau} = Ac_{j,\tau}(a) - c_{j,\tau}(\underline{a})(a)$ is the solution to $c'_{j,\tau}(a) = \mathbb{E}\left[\int_0^\tau c_j(a_t)dt \,\middle|\, a_0 = a, y_0 = y_j\right)\right]$ ■.

One version of this models is given in solution as:

Figure 1 Huggett economy :density,wealth,productivity per Achdou et al.(2022



Source: authors' own calculation based on Benjamin Moll codes: https://benjaminmoll.com/codes/

4. Incomplete markets: Arrow securities and Bond markets (per Mukoyama (2021))

In this economy there are two types of consumers type I and type II. Arrow security[12] does not exist for the irregular state although the consumers recognize the possibility of the irregular state in the future. A Type-I consumer's problem is given as:

equation 60

$$\max_{c_1, c_2, \tilde{c}_2, a} u(c_1) + (1-\pi)u(c_2) + \pi u(\tilde{c}_2)$$

s.t. $c_1 + pa = 0; c_2 = 2 + a \; ; \tilde{c}_2 = 2 - \tau \;;$

where a denotes holding Arrow securities, regular state occurs with probability $1-\pi$, irregular state occurs with probability π where $\pi \in (0,1)$. Type I receives $1-\tau$, Type II consumer receives $(1+\tau)$ where $\tau \in (0,1)$ in irregular state transfer occurs from Type I to type II consumer. Utility $u(\cdot)$ is strictly increasing, strictly concave, and continuously differentiable. Robbin, Joel W. (2010), here states that f is said to be continuous on $\mathbb{R}^l$ if :

equation 61

$$\forall x_0 \in \mathbb{R}^l \forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \in \mathbb{R}^l [|x - x_0| < \delta \; |\Rightarrow f(x) - f(x_0) < \epsilon|]$$

[12] An Arrow security is an instrument with a fixed payout of one unit in a specified state and no payout in other states, see Arrow (1953)

In previous condition ϵ is trimmed price space [13], x_0 is vector parameter, hence why the PDF is of a form $f_{x_0}(x) = (x - x_0)$. Next, for type II consumer we have:

$$\max_{c'_1, c'_2. \tilde{c}'_2, a'} u(c'_1) + (1-\pi)u(c'_2) + \pi u(\tilde{c}'_2)$$

This is the maximization problem for consumer Type II $c'_1 + pa' = 2; c'_2 = \alpha \ ; \tilde{c}'_2 = \tau$.The competitive equilibrium here is :$(c_1, c_1', c_2, c_2', \tilde{c}_2, \tilde{c}_2') = (1,1,1,1,2-\tau,\tau)$. Thus the limit is given as:

equation 62

$$li\, m_{\pi\to 0}(c_1, c_1', c_2, c_2', \tilde{c}_2, \tilde{c}_2') = (1,1,2-\tau,\tau)$$

Where p is the price of Arrow security. In the Bond markets this version of the model is given as with quadratic utility function:

equation 63

$$u(c) = \alpha c - \frac{\gamma}{2}c^2$$

Where $\alpha > 0; \gamma > 0$, the value of $\alpha \gg 0$ so that utility is increasing in c for relevant range.Type I consumer problem in this economy is given as:

$$\max_{c_1, c_2. \tilde{c}_2, b} u(c_1) + (1-\pi)u(c_2) + \pi u(\tilde{c}_2)$$

s.t. $c_1 + qb = 1; c_2 = 1 + b \ ; \tilde{c}_2 = 1 - \tau + b$; where q represents the bond price and b is the bond holding. Now, a type I consumer problem and bond demand after FOC is given as:

equation 64

$$b = \frac{q(\gamma-\alpha) + \alpha - \gamma(1-\pi\tau)}{\gamma(q^2+1)}$$

Type II consumer problem is given as :

$$\max_{c_1, c_2. \tilde{c}_2, b} u(c'_1) + (1-\pi)u(c'_2) + \pi u(\tilde{c}'_2)$$

s.t. $c'_1 + qb' = 1; c'_2 = 1 + b' \ ; \tilde{c}'_2 = 1 - \tau + b'$ .The bond demand for Type II consumer is given as:

equation 65

$$b = \frac{q(\gamma-\alpha) + \alpha - \gamma(1+\pi\tau)}{\gamma(q^2+1)}$$

The bond price q demand is zero here is set so that :$b + b' = 0$. Now, $q = 1, (b, b') = \left(\frac{\pi}{2}\tau, -\frac{\pi}{2}\tau\right)$. The resulting consumption functions are :

equation 66

$$(c_1, c_1', c_2, c_2'. \tilde{c}_2, \tilde{c}_2') = \left(1 - \frac{\pi}{2}\tau, 1 + \frac{\pi}{2}\tau, 1 + \frac{\pi}{2}\tau, 1 - \frac{\pi}{2}\tau, 1 + \left(\frac{\pi}{2} - 1\right)\tau, 1 + \left(1 - \frac{\pi}{2}\right)\tau\right)$$

[13] Trimmed space as a location parameter class of probability functions that is parametrized by scalar or vector valued parameter x_0 which determines distributions or shift of the distribution.

In the limit $\pi \to 0$, the consumption profile when irregular state takes place in period 2 approach:

equation 67

$$\lim_{\pi \to 0}(c_1, c_1', \tilde{c}_2, \tilde{c}_2') = (1,1,1-\tau, 1+\tau)$$

Now in an Arrow security economy if there is MIT shock[14], because the irregular state is not spanned by the Arrow security, the ex-post allocation will be given as: $\tilde{c}_2' = 2-\tau$; $\tilde{c}_2' = \tau$ where tilde $(\tilde{\ })$ denotes irregular state. The entire ex-post allocation with MIT shock is: $(c_1, c_1', \tilde{c}_2, c_2') = (1,1,2-\tau,\tau)$. The unique competitive equilibrium before the shock was: $p = 1, a = 1, a' = 1, c_1 = c_1' = c_2 = c_2' = 1$. In the bond economy post MIT shock allocation would be : $\tilde{c}_2 = 1-\tau; \tilde{c}_2 = 1+\tau$ .The unique competitive equilibrium before the shock was: $q = 1, b = -1, b' = 1$; $c_1 = c_1' = c_2 = c_2' = 1$.

5. OLGHA moldel as in Auerbach ,Kotlikoff (1987) with a precise demographic structure and incomplete markets as in Huggett (1996).

This model is inspired by the Auerbach, Kotlikoff (1987) with a demographic structure as in Huggett (1996). $N_{i,t}$ are the number of agents of age i in period t.Total population is $N_t = \sum_i N_{i,t}$ each period $1+n$ households of age 0 are born. Household stochastic survival rates are $\phi_{i-1,i}$ for $i \geq 1$ .Compound survival rate is :

equation 68

$$\Phi_i = \Pi_{k=1}^{i} \phi_{k-1,k}$$

Assuming there is one zero-year old date $t = 0$, the population of each age at time t is:

equation 69

$$N_t = \sum_{(i=0)}^{T} N_{i,t} = (1+n)^t \sum_{i=0}^{T} \frac{\Phi_i}{(1+n)^i}$$

Population shares are $\pi_i = \frac{N_{i,t}}{N_t}$; where $\pi_i = \left(\sum_{i=0}^{T} \frac{\Phi_i}{(1+n)^i}\right)^{-1}$.Households economic life starts at T^W and they retire at T^R and die at age T. The maximize their life-time utility:

equation 70

$$\sum_{i=T^W}^{T} \beta_i \Phi_i u(c_i)$$

Here Φ_i is probability of surviving up to the age i .Budget constraint of agents is :

equation 71

$$c_i + a_{i+1} = \frac{1+r}{\Phi_{i-1,i}} a_i + y_i$$

[14] *"An "MIT shock" is an unexpected shock that hits an economy at its steady state, leading to a transition path back towards the economy's steady state……"*.Mukoyama (2021) also follows Boppart et al. (2018) definition:"…. the probability of the shock is considered zero, and no prior (contingent) arrangement is possible for the occurrence of the MIT shock"…..The dynamic analysis that was using exogenous shocks or policy changes has been used in the literature with the earlier examples including: Abel,Blanchard (1983), Auerbach, Kotlikoff (1983), and Judd (1985).And more recent examples being: Boppart et al. (2018), Kaplan et al. (2018), Boar ,Midrigan (2020), Guerrieri et al. (2020).

Before retirement households provide supply of labor $\bar{h}_i$ inelastically and are subject to idiosyncratic shocks ε .Once when they are retired households receive social benefits b_i income is given as:

equation 72

$$y_{i,t} = \begin{cases} (1-\tau_i) w_i \bar{h}_i \varepsilon ; i < T^R \\ b_t \, ; i \leq T^R \end{cases}$$

Where $(1-\tau_i)w_i$ is the real wage net of the payroll tax used to finance the social benefits. We assume that ε follows Markov chain with transition probability matrix Π.Household borrowing constraint is $a_{t+1} \leq \underline{a}$. On the production side firms produce output with effective labor $L_t = \sum_i N_{i,t}\bar{h}_i$ with capital K_t and the technology is : $Y_t = F(K_t, L_t) = K_t^a L_t^{1-a}$ in a perfect competition competitive prices are :

equation 73

$$w_t = (1-a)\left(\frac{K_t}{L_t}\right)^a$$
$$r_t = a\left(\frac{K_t}{L_t}\right)^{a-1} - \delta$$

The government finances social security benefits b_t with payroll tax τ_t such that:

equation 74

$$\tau_t \sum_{i=T^W}^{T^R-1} w_t N_{i,t} \bar{h}_i = \sum_{i=T^R}^{T} b_t N_{i,t}$$

Where $\tau_t \ \sum_{i=T^W}^{T^R-1} w_t N_{i,t} \bar{h}_i$ are taxes and $\sum_{i=T^R}^{T} b_t N_{i,t}$ are transfers. Stationary equilibrium is a collection of policy functions $c(a,\varepsilon,i)$ and $a^t(a,\varepsilon,i)$ which represents the initial allocation of asset holdings and demographic structure n and Φ_i and government social policy τ_t, b_t and relative prices r_t, w_t :

equation 75

$$L_t = \sum_i N_{i,t}\bar{h}_{it}; A_t = \sum_{(i=0)}^{T} N_{i,t} a_{it}; C_t = \sum_{(i=0)}^{T} N_{i,t} c_{i,t}$$

When computing policy functions, household problem is :

equation 76

$$V(\varepsilon, a, i) = \max_{c, a_{i+1}} \frac{c^{1-\sigma}}{1-\sigma} + \beta \phi_{i,i+1} \mathbb{E}([V(\varepsilon', a', i+1]|\varepsilon)$$
$$s.t.\, c_i + a_{i+1} = \frac{1+r}{\phi_{(i-1,i)}} a_i + y_i$$
$$a_{i+1} \geq \underline{a}$$

The Euler equation is formulated for this problem as :

equation 77

$$c_i^{-\sigma} \geq \beta(1+r)\mathbb{E}[c_{i+1}^{-\sigma}]$$

Cash on hand for a given asset grid a is given as:

equation 78

$$z_i(a,\varepsilon) = z_T(a,\varepsilon); a_T = 0$$

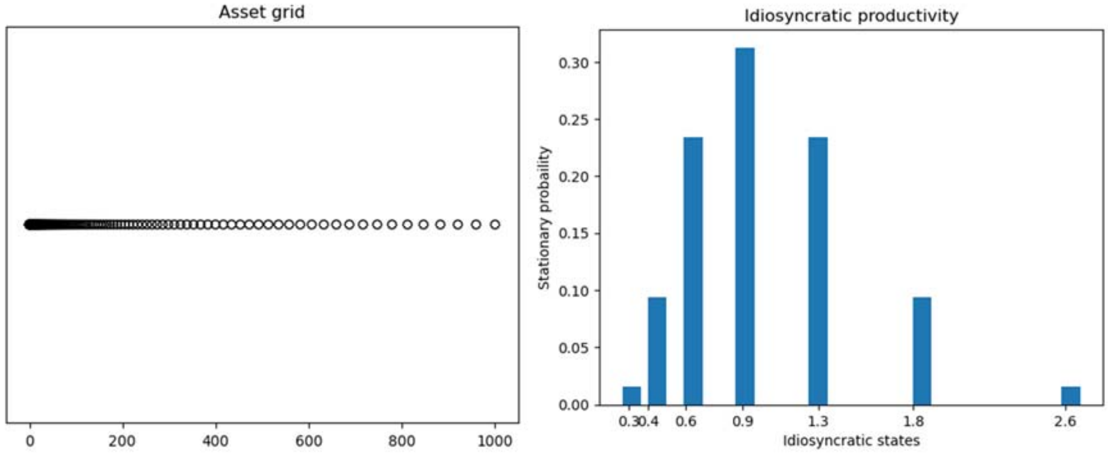
With backward recursion previous is :

equation 79

$$c(i,\varepsilon,a)^{-\sigma} = \beta(1+r)\mathbb{E}[c(i+1,\varepsilon',a',(i,\varepsilon,a)^{-\sigma}|\varepsilon]$$

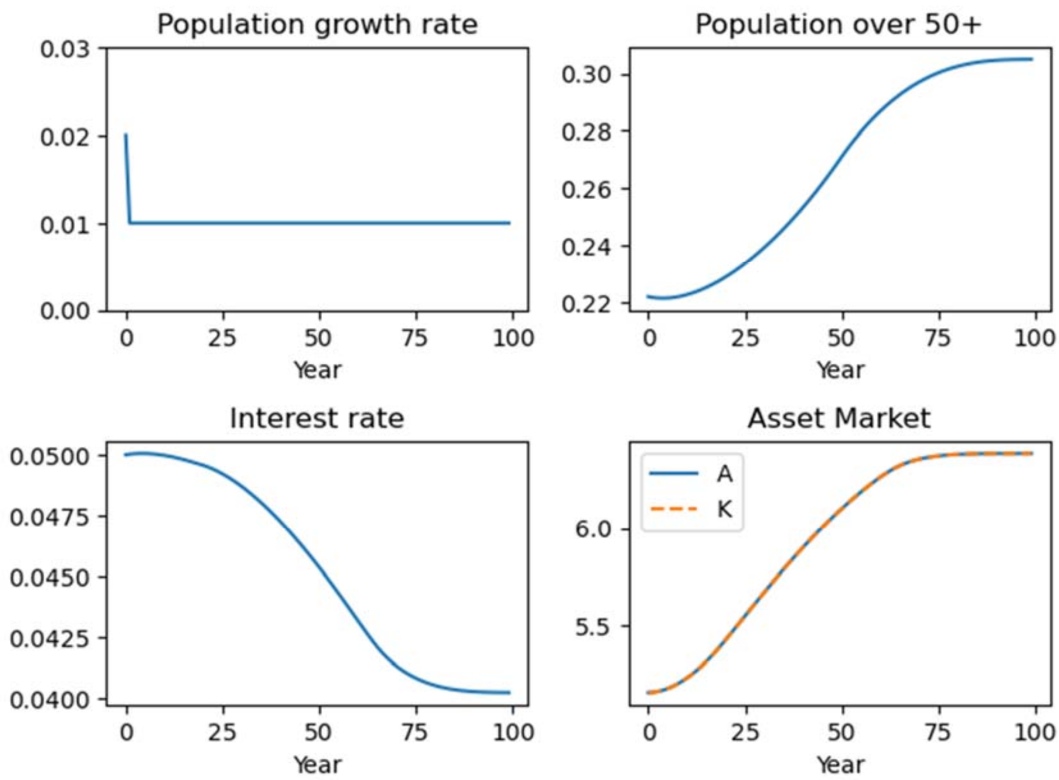
In this economy asset grid and idiosyncratic productivity are :

Figure 2 Asset grid and idiosyncratic productivity in OLGHA model



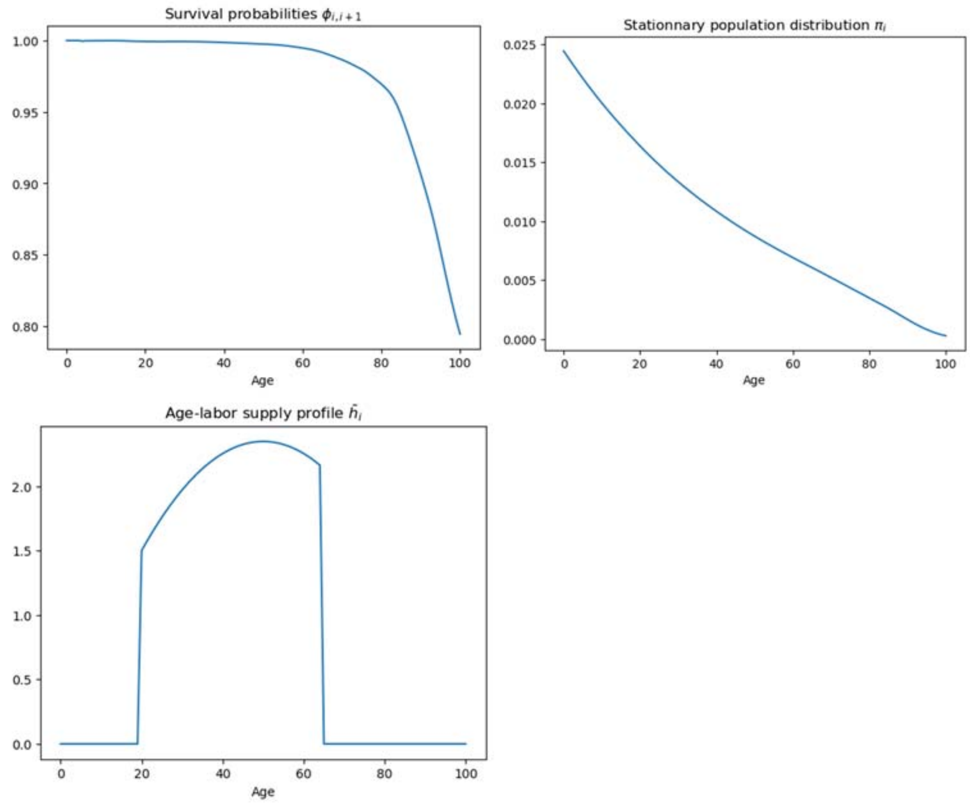
Source: authors own calculation based on code available at: https://github.com/FredericMartenet/OLGHA

Figure 3 Population ,interest rate and asset market in OLGHA model (general equilibrium result)



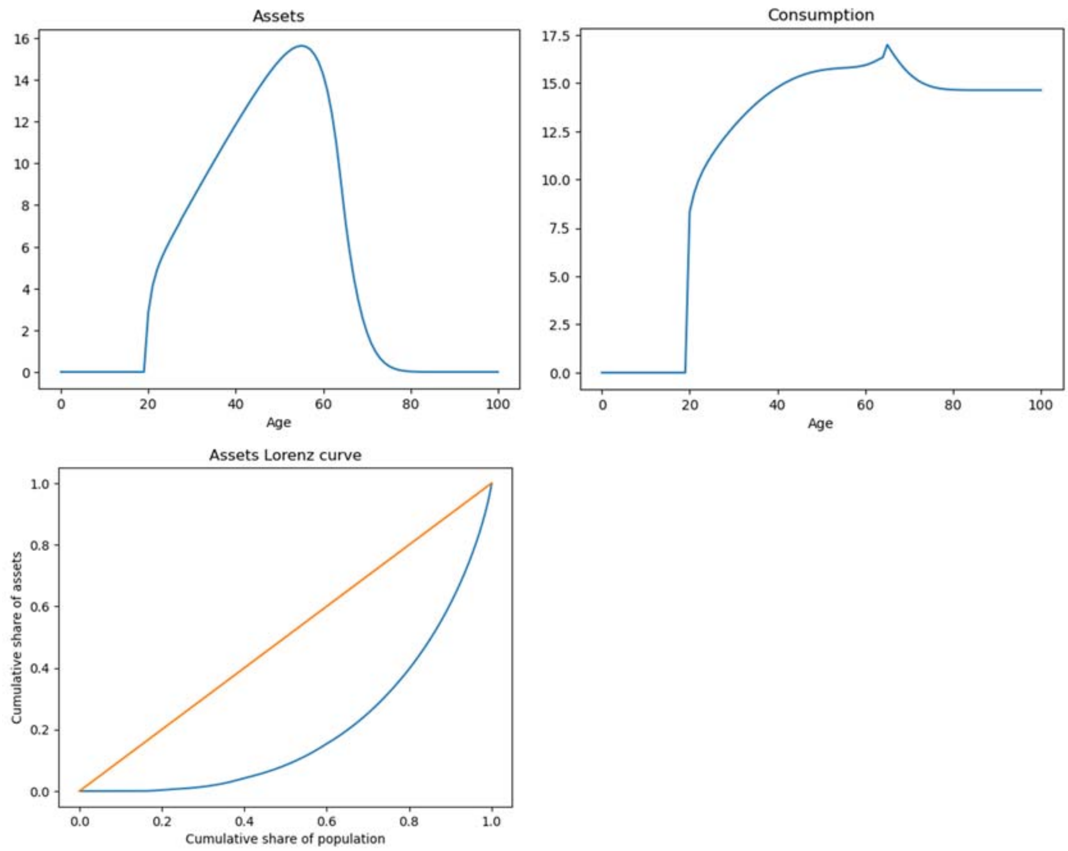
Source: authors own calculation based on code available at: https://github.com/FredericMartenet/OLGHA

Figure 4 Population parameters in OLGHA model



Source: authors own calculation based on code available at: https://github.com/FredericMartenet/OLGHA

Figure 5 Assets, consumption, and Asset Lorenz curve



Source: authors own calculation based on code available at: https://github.com/FredericMartenet/OLGHA

Figure 6 Dynamic inefficient version of A-K OLG model in a Huggett economy with heterogeneous agents and incomplete markets (capital; labor;interest;wage)

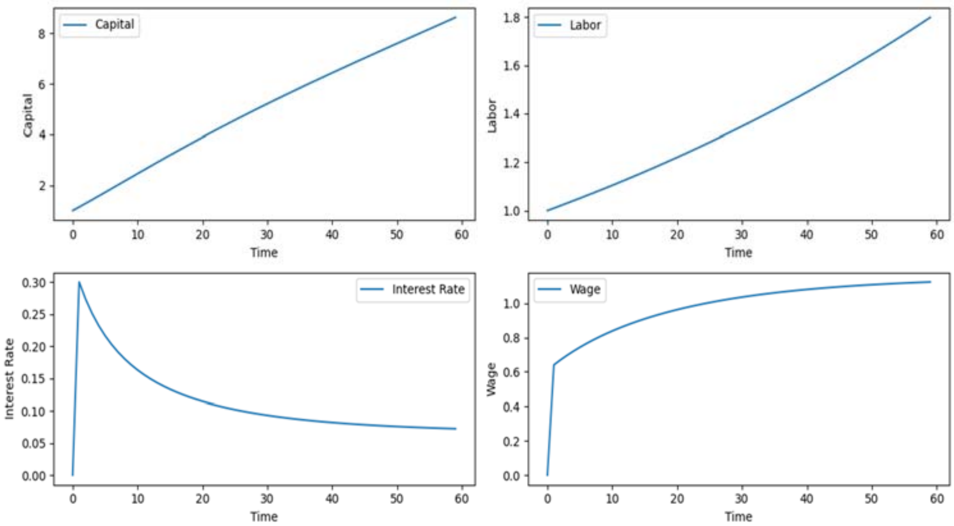


Figure 6a Dynamic inefficient version of A-K OLG model in a Huggett economy with heterogeneous agents and incomplete markets (government taxes; government payroll tax revenue)

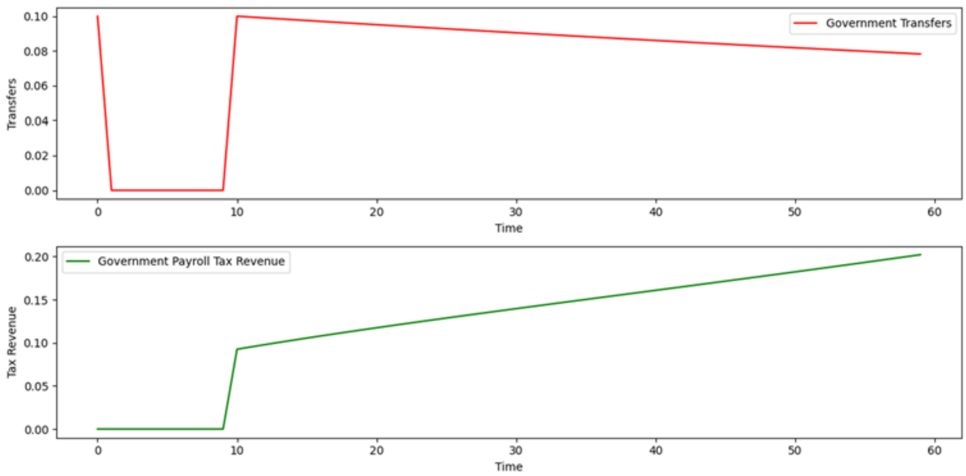
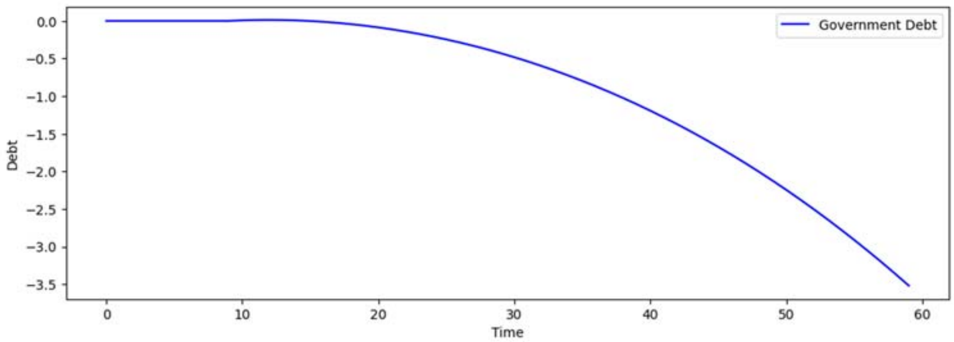
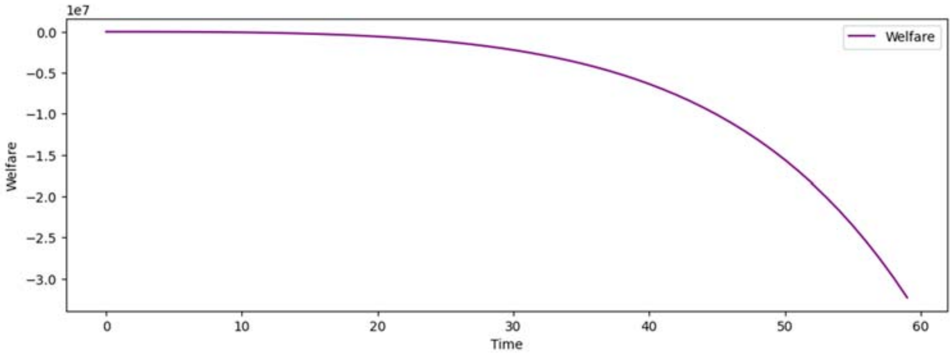


Figure 6b Dynamic inefficient version of A-K OLG model in a Huggett economy with heterogeneous agents and incomplete markets (government debt; welfare)





Source : Authors' own calculation

6. Tax cuts and fiscal policy in incomplete markets economy (RET fails proof due to Divino, Orillo,(2017))

Tax cut is defined as: $\tilde{\tau}_0 - \tau_0 = -d$, and it is financed through debt, $\tilde{\tau} - \tau = (1+r)d = -(1+r)\tilde{\tau}_0 - \tau_0$. Formally Ricardian equivalence to holds following applies:

equation 80

$\Phi(t) = d(t) - \int_t^{\infty}(\tau(\bar{t}) - G(\bar{t}) \cdot e^{-rA(t,\bar{t})}dt = 0$[15].

Government debt is equal to $\dot{d}(t) = r(t)d(t) + G(t) - \tau(t)$, Heijdra, B.J., F.Van Der Ploeg, (2002). In the first period tax cut is financed through debt, but in the second period taxes are increased, by the principal plus interest due on the issued debt. Tax cut should leave present value of government spending unchanged, but the risk free payoff paying $(1+r)$, does not mean that the risk free payoff belongs to the asset span, since $r > 0$ is exogenously determined, and it might or might not belong to the asset span $\mathcal{M}$, Divino, Orillo,(2017). Asset prices are $q\theta \in R_{+}^{j}$, tax obligations are given as: $\tau = (\tau_0, t1) \in R_{++}^{1+S}$ $\forall h \in H$, and $\max_{(x_0,\tilde{x}_s)\in\beta^h(q,\mathcal{M},\tau)} U^h = (x_0, \tilde{x}_s)$, s.t. 2 period constraint:$x_0 + q\theta = \omega_o^h - t_o$; $x_s = \omega_s^h + q_s\theta - t_s$,taxpayer budget set it is defined as: $\beta^h(q,\mathcal{M},\tau) = \{x \in R_{+}^{1+S}: \exists z \in \mathcal{M}\}: x_0 - \omega_0^h + \tilde{\tau}_0 = -q\theta, \tilde{x}_s - \widetilde{\omega}_s^h + \tilde{t}_s 1 = z, q: \mathcal{M} \to R$. If the tax cut previously defined is enacted then we will have: $x_0 - \omega_0^h + \tilde{\tau}_0 = -q\theta - d$, $\tilde{x}_s - \widetilde{\omega}_s^h + \tilde{t}_s = q\theta + (1+r)d$.So , now question here is whether agents can neutralize $(1+r)d$, and not that government bonds are net wealth as in Barro,(1974). For the law of one price to apply here $\beta^h(q,\mathcal{M},\tau) = \beta^h(q,\mathcal{M},\tilde{\tau}), \exists z \in R^s, \exists z(s') = z + (1+r)d1 \in \mathcal{M}$. RET holds if and only if it does not affect the individual demand sets defined as: $\varphi^h(p,q,\tau) = \{x \in \beta^h(p,q,\tau): \neg\exists x' \in \beta^h(p,q,\tau): U^h(x') > U^h(x)\}$. The last expression is in line with the second welfare theorem where if economy is specified by:

equation 81

$\left(\{x \succsim_j\}_{j=1}^{j}, \{x'\}_{s=1}^{S}, \omega_o^h\right), for\ (x,x'), \exists p = (p_1, \dots p_s) \forall q = (q_1, \dots q_s \neq 0, \exists(\omega_1, \dots \omega_s), \sum_s \omega_s = p\omega_s^h + \sum_i p_i x'$,

Previous constitutes pseudo equilibrium with transfers and that is: $\forall s, x', \max p x_s \leq p x', \forall x \in R_{+}^{S}$, and $\forall j, if\ x_j \succ x_j' \parallel q_j x_j \geq \omega_j$ and $\sum_s x'_s = \omega_s^h + \sum_j x_j'$.In such a case $\lambda x + (1-\lambda)x' \in R_{++}^{1+s} \in \mathbb{R}^N$ is convex where $\lambda \in [0,1]$.This is also known as Separating hyperplanes theorem in other words, $R_{++}^{1+s} \subset \mathbb{R}^N$ is convex if it contains two vectors x and x' , and a segment that connects them. Now, law of one price holds if $1 \in \mathcal{M}$, but in the model public debt is not available for consumers to purchase. Only risky assets are available for them to try to replicate risk free payoff. RET does not hold if $1 \notin \mathcal{M}$. Now if we define:

Equation 82

$$\mathrm{Im}\mathcal{M} = \{q(e_1, \dots\dots, e_j) \in (R^S)^j: F(z) = q\ for\ some\ z \in V\}$$
$$\mathrm{Ker}\mathcal{M} = z \in V: F(z) = 0$$

[15] Actuarial revenue is $r^A(t) = r(t) + M(t)$, where $M(t)$ is instant probability for death.

Set of linear mapping function would be given as:

Equation 83

$$\mathcal{F} = \{(e_1, \ldots\ldots, e_j) \in (R^S)^j : q(e_1, \ldots\ldots, e_j) = 0\}$$

We note here that Fourier transform of a common function is given as: $\mathcal{F}_x[1](k) = \int_{-\infty}^{+\infty} e^{-2\pi ikx} dx = \delta(k)$, or Fourier transformation of a delta function is :

equation 84

$$\mathcal{F}_x[\delta(x - x_0)(k)] = \int_{-\infty}^{+\infty} e^{-2\pi ikx} \delta(x - x_0) dx = e^{-2\pi ikx}, \mathcal{F}_x^{-1} = \delta[(x)k] = \int_{-\infty}^{+\infty} \delta(x) e^{2\pi ikx} dx = 1.$$

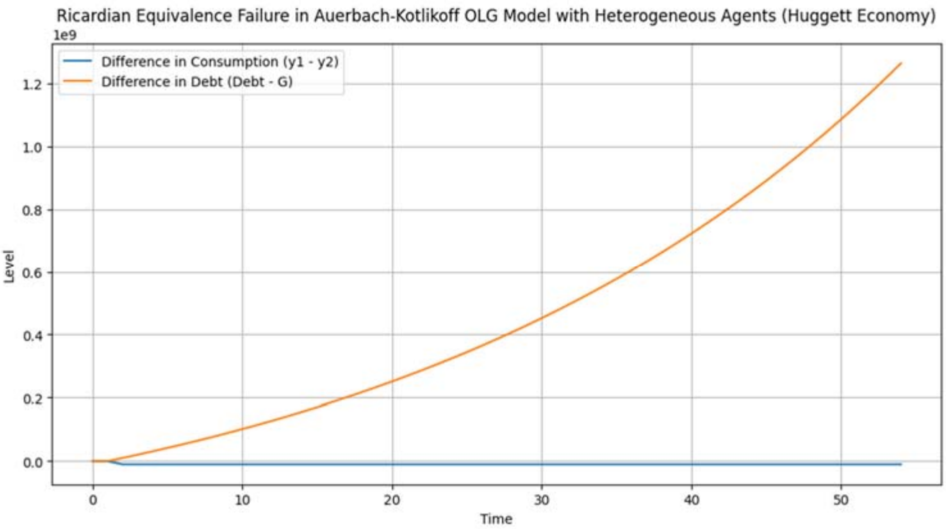
Now since, $(R^S)^j = 1$, in our case Fourier transform of one is 1, since Rank$(R^S) = 1$.If $Ker(\mathcal{M}) = 0$, than its dimension is given as:

equation 85

$$\dim(Ker\mathcal{M}) = \dim(K^N) - \dim(Im\mathcal{M}) = n - rank(\mathcal{M}) = 1 - 1 = 0.$$

This is actually the distance $(1\mathcal{M})$. Now since delta function, is continuous and is close there exists complement of S which is an open set. In this open set one cannot expect to replicate risk free payoffs. This is because the complement set has its own limit points, and has its own set closure, has its own neighborhood, disjoint of S, Croft, Falconer, and Guy, K.(1991) $\mathcal{R} \subset R^{JS}, \forall(e_1, \ldots\ldots, e_j)$, Rank$\mathcal{M} = 0$, RET fails.Since the set of endogenous variables is :$\Theta \coloneqq \{\theta \in R^J : q\theta = 1\}$, since the rank of V is full (vectors are linearly dependent), and it is an injective transformation. Therefore, the Lebesque measure is:$\mu_L(S') = (b - a) - \sum_k (b_k - a_k) = 0$. Hence, agents cannot replicate risk free payoff, see Josheski,D.(2017). . Next this is graphically depicted in A-K OLG model in Huggett economy.

Figure 7 Failure of RET hypotheses in A-K-OLG model in Huggett economy



Source: authors' own calculation

Failure of RET is shown as difference in consumption and government debt does not converge to zero over time,

7. Conclusion

This paper shows that Auerbach-Kotlikoff model has a capacity to study economic policy. In the OLGHA model there were 7 idiosyncratic productivities, population over 50+ was dominant, interest rate was decreasing through time, age labor supply profile peaks around 40-60 years of agents age, assets Lorenz curve shows social inequality in this economy. Dynamic inefficiency i.e., $r < g$ shows that for this model when government debt is decreasing welfare also decreases. Government transfer payments decrease, and payroll tax revenue is increasing, interest rate has is decreasing dramatically, wages rise and capital and labor also rise. Overaccumulation of capital in Auerbach -Kotlikoff model, and crowding out with issuing new debt by government leads to decrease of welfare, see Appendix 1.

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Appendix 1 Golden rules and Ramsey exercise

equation 86

$$\frac{dk}{dt} = sf(k) - nk$$

Or, because $sf(k) > f(k) - c$, then: $\frac{dk}{dt} = f(k) - c - nk$.Thus, we maximize the intertemporal utility stream subject to this equation as a constraint. To solve the problem, we can use the calculus of variations or the maximum principle. Let us use the latter. Thus, setting up the present-value Hamiltonian:

$H = U(ct) + \lambda(f(k) - c - nk)$ where λ is the current-value "costate" variable. The first order conditions for a maximum, then, yield:

equation 87

$$(1)\ dH/dc = U_c - \lambda = 0$$

$$(2)\ -dH/dk = dz/dt - \rho\lambda = -\lambda(f(k) - n)$$

$$(3)\ dH/d\lambda = dk/dt = f(k) - c - nk$$

$$(4)\ \lim\ \lambda e^{-\lambda t} = 0$$

$U_c = \lambda$ (where $U_c = \frac{dU}{dc}$) the marginal utility of consumption at this time period. $\frac{d\lambda}{dt} = U_{cc}(\frac{dc}{dt})$ (where $U_{cc} = d^2U/dc^2$ - the second derivative) $U_{cc}(dc/dt) - pU_c = -U_c(f(k) - n)$ or, rearranging: $dc/dt = -[U_c/U_{cc}][f(k) - n - \rho]$ if we had used a so-called CRRA utility function (i.e. $U(c) = \frac{c^{1-e}}{(1-e)c}$ where $0 < e < 1$), then the entire term $[U_c/U_{cc}]$ would have been merely $1/e$, and our equation reduced to: $\frac{dc}{dt} = \left(\frac{1}{e}\right)[f(k) - n - \rho]$. The "solution" to the optimization program will be a pair of differential equations - $\frac{dc}{dt}$ just derived, and $\frac{dk}{dt}$ derived from our third condition:

$\frac{dk}{dt} = f(k) - c - nk$. Balanced growth or steady state growth is $f(k) - n - \rho = 0$,

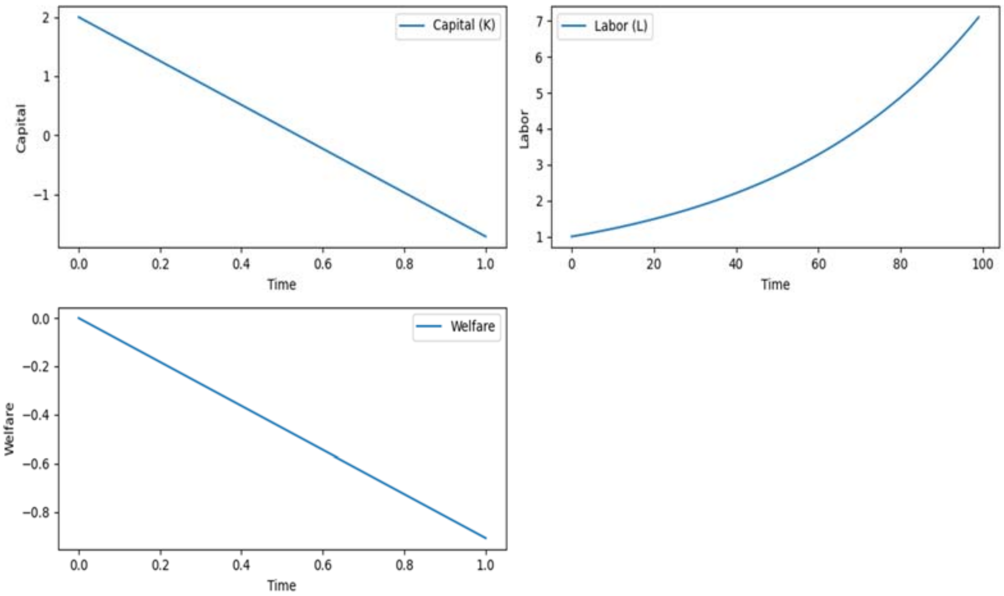
$f(k) - c - nk = 0$, where $c^* = f(k^*) - nk^*$, $f(k) = n + \rho$-Golden Utility growth . he present value of future utility gains from individual consumption at any time period t is then: $U(c_t)e^{-\rho t}$

equation 88

$$U = \int_0^\infty U(c_t)e^{-\rho t}\,dt$$

$f(k) = n$ represents the Golden rule of growth for Allais (1947), Von Neuman (1937) , Robinson (1962). This derivation is from Josheski et al.(2018)

Figure 8 Overaccumulation of capital in Auerbach -Kotlikoff model ,crowding out with issuing new debt by government and decrease of welfare



Source: authors calculations